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Subextremal Curves

Scott Nollet

University of Barcelona
08007 Barecelona, Spain
e-mail: nollet@cerber.mat.ub.es

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In this paper we improve known bounds on the Rao function for non-extremal curves, as well extending these results to nonzero characteristic. These improved bounds are shown to be sharp, and the curves which yield sharpness are classified. Schlesinger's notion of the spectrum of a curve is introduced and used to streamline proofs.

0. Introduction

In [8], Martin-Deschamps and Perrin obtain sharp bounds for the Rao function $h^1(\mathcal{I}_C(n))$ of a curve $C \subset \mathbb{P}^3$ (see [8], theorem 2.5 and corollary 2.6). The non-arithmetically Cohen-Macaulay curves which achieve equality for their bound are called *extremal* curves. In a later paper, they characterize the extremal curves as the minimal curves associated to certain Koszul modules ([9], theorem 2.4). Further, they show that these curves form an irreducible component of the Hilbert scheme $H(d,g)$, which is nonreduced when $d > 2$ and $g < \frac{1}{2}(d-2)(d-3)$ ([9], theorems 4.2 and 4.3).

Using a different method, Ellia recovers the bounds of Martin-Deschamps and Perrin ([2], §2, corollary 6) and geometrically characterizes the extremal curves (assuming that char $k = 0$; see [2], §2, theorem 8). Ellia further proves that if a curve is neither arithmetically Cohen-Macaulay nor extremal, then its Rao function satisfies even stronger bounds ([2], §2, proposition 9).

The goal of this note is twofold. Firstly, we improve the stronger bounds of Ellia, and show that the improved bounds are sharp. Secondly, we classify the curves (called *subextremal* curves) which achieve these bounds. While previous work in this direction has assumed that the ground field has characteristic zero, extra care is taken here to make the results valid in all characteristics. We also give examples which suggest that there are not natural stronger bounds on the Rao function for curves which are neither arithmetically Cohen-Macaulay, extremal, nor subextremal.

The paper is organized as follows. The first section gives the definition and basic properties of the spectrum of a curve in $\mathbb{P}^3$; these results can be found in Enrico Schlesinger's Ph.D. thesis [13]. In the second section, the sharp bounds for the Rao function of non-extremal curves are given, and the curves achieving these bounds are classified. At the end of the second section, we give examples to show that things get complicated if one tries to find stronger bounds for curves which are not subextremal.

We work over an algebraically closed field k of arbitrary characteristic. $S = k[x, y, z, w]$ denotes the homogeneous coordinate ring of $\mathbb{P}^3_k$. All curves considered here are locally Cohen-Macaulay and have no zero-dimensional components. We use the abbreviation ACM to denote arithmetically Cohen-Macaulay curves. I would like to thank Enrico Schlesinger for the use of some of his thesis results, as these made proofs much easier. Finally, I am grateful for the careful reading of the referee, who found some problems with the first draft.

1. The Spectrum of a Curve

In this section we review some results from Schlesinger's thesis [13]. In particular, we recall the notion of the *spectrum* of a curve (this is analogous to the spectrum of torsion free sheaves studied in [12]) and give a few of its properties. In order to define the spectrum, we note the following proposition.

Proposition 1.1. *Let $C \subset \mathbb{P}^3$ be a curve. If L is a line which does not meet C, then the Cohen-Macaulay ring $A_C = H^0_*(\mathcal{O}_C)$ is a free graded S_L-module, where S_L is the polynomial subring of S generated by the linear forms vanishing on L.*

Proof. See [13], proposition 1.2.4.

Definition 1.2. Let $C \subset \mathbb{P}^3$ be a curve. Let $L \subset \mathbb{P}^3$ be a line which does not meet C. By proposition 1.1 above, we have an isomorphism of graded S_L-modules

$$H^0_*(\mathcal{O}_C) \cong \oplus_{n\in\mathbb{Z}} S_L(-n)^{h_C(n)}.$$

The *spectrum* of C is the nonnegative function $h_C : \mathbb{Z} \to \mathbb{Z}$. This definition does not depend on the line L chosen, for the formula above shows that $h_C(n) = \Delta^2 h^0(\mathcal{O}_C(n))$.

Proposition 1.3. *Let $C \subset \mathbb{P}^3$ be a curve with spectrum h_C. Then*
(a) $P_C(n) = \sum_k (n-k+1)h_C(k)$.
(b) $\deg C = \sum_n h_C(n)$.
(c) $\operatorname{genus} C = 1 + \sum_n (n-1)h_C(n)$.
(d) $h^0(\omega_C(n)) = \sum_{k\leq n}(n-k+1)h_C(2-k)$.
(e) $e(C) + 2 = \max\{n : h_C(n) \neq 0\}$.

Proof. See [13], propositions 1.4.8 and 1.4.9.

Notation 1.4. By part (b) above, the spectrum $h : \mathbb{Z} \to \mathbb{N}$ of a curve is a finitely supported function. It is convenient to describe such functions with the *exponent notation.* Specifically, the spectrum h can be represented by the tuple of integers with exponents $\{n^{h_C(n)}\}$. The exponent is suppressed when it is equal to 1.

If $k : \mathbb{Z} \to \mathbb{N}$ is another finitely supported function described in exponent notation as $\{n^{k(n)}\}$, then we use the notation

$$\{n^{h(n)}\} \cup \{n^{k(n)}\}$$

to denote the finitely supported function $h + k$. We use similar notation for any number of such functions.

Example 1.5. An elementary calculation shows that a plane curve C of degree d has spectrum $h_C(n) = \binom{n}{0} - \binom{n-d}{0}$. In the exponent notation, this is denoted

$$\{0, 1, 2, \ldots, d-1\}.$$

Conversely, if C has the spectrum above, the definition of spectrum shows that $h^0(\mathcal{O}_C(1)) = 3$. It follows that the map $H^0(\mathcal{O}_{\mathbb{P}^3}(1)) \to H^0(\mathcal{O}_C(1))$ is not surjective and hence C is planar.

Remark 1. If $C \subset \mathbb{P}^3$ is an Arithmetically Cohen-Macaulay curve, then the spectrum of C can be interpreted easily in terms of the numerical character used by Gruson and Peskine [3] or the gamma character employed by Martin-Deschamps and Perrin [7]. Specifically, if $s = s(C)$ and C has numerical character $(n_0, n_1, \ldots, n_{s-1})$, then the gamma character of C is given by the formula

$$\gamma_C(n) = \begin{cases} -1 & \text{if } 0 \le n < s \\ \#\{k : n = n_k\} & \text{otherwise.} \end{cases}$$

Given the gamma character γ_C of an ACM curve C, the spectrum is given by

$$h_C(n) = \sum_{k>n} \gamma_C(k).$$

For example, let C be a twisted cubic curve. Then C has numerical character $(2,2)$ and γ-character $\gamma_C(n) = -\binom{n}{0} + 3\binom{n-2}{0} - 2\binom{n-3}{0}$. The spectrum of C is the function $h_C(n) = \binom{n}{0} + \binom{n-1}{0} - 2\binom{n-2}{0}$. In exponent notation, this is written simply $\{0, 1^2\}$.

Following along the lines of Okonek and Spindler's work [12], Schlesinger proves the following necessary conditions on the spectrum. For convenience we give only the statement for space curves.

Theorem 1.6. *Let $C \subset \mathbb{P}^3$ be a curve with Rao module M_C. Let $\mu_C(n) = \dim(M_C \otimes_S k)_n$ and $e = e(C)$. Then $e + 2 \ge 0$ and*
(a) $h_C(n) \ge 1 + \mu_C(n)$ for $0 \le n \le e+2$.
(b) $h_C(n) = \mu_C(n) = 0$ for $n > e+2$.
(c) If $h_C(l) = \mu_C(l) + 1$ for some $1 \le l \le e+2$, then $h_C(n) = 1$ for $l < n \le e+2$. Further, if $l \le e+1$, then C contains a plane curve of degree $e+3$.

Proof. See [13], theorem 1.7.1.

Definition 1.7. We say that a function $h : \mathbb{Z} \to \mathbb{N}$ is *1-admissible* if
(a) h is has finite support
(b) $h(0) > 0$
(c) if $h(k) = 0$ for some $k > 0$, then $h(l) = 0$ for all $l \geq k$
(d) if $h(k) = 1$ for some $k > 0$, then $h(l) \leq 1$ for all $l \geq k$.

From proposition 1.3 and theorem 1.6 it is immediate that the spectrum of a curve in $\mathbb{P}^3$ is 1-admissible. Now we prove a combinatorial proposition which will be used in the next section.

Proposition 1.8. *Let $h : \mathbb{Z} \to \mathbb{N}$ be a 1-admissible function with $\sum_{n\in\mathbb{Z}} h(n) = d$. Let $g(h) = 1 + \sum_{n\in\mathbb{Z}}(n-1)h(n)$ denote the genus of h.*
(a) We have $g(h) \leq \frac{1}{2}(d-1)(d-2)$ with equality if and only if h is given by

$$\{0, 1, \ldots, d-1\}.$$

(b) Let $a \leq 1$ and suppose that the function h' given by $h'(n) = h(n) - \binom{n-a}{0} + \binom{n-a-1}{0}$ is 1-admissible. Then $g(h) \leq a - 1 + \frac{1}{2}(d-2)(d-3)$ with equality if and only if h is given by

$$\{a\} \cup \{0, 1, \ldots, d-2\}.$$

(c) Suppose that $h(d-1) = 0$ and $d \geq 3$. Then $g(h) \leq \frac{1}{2}(d-2)(d-3)$ with equality if and only if h is given by

$$\{1\} \cup \{0, 1, \ldots, d-2\}.$$

(d) Let $a \leq 1$ and suppose that the function h' given by $h'(n) = h(n) - \binom{n-a}{0} + \binom{n-a-1}{0}$ is 1-admissible. Further assume that $h'(d-2) = 0$. Then $g(h) \leq a - 1 + \frac{1}{2}(d-3)(d-4)$ with equality if and only if h is given by

$$\{a\} \cup \{1\} \cup \{0, 1, \ldots, d-3\}.$$

(e) Suppose that $d \geq 5$ and $h(d-2) = 0$. Then $\sum_{n\geq 1}(n-1)h(n) \leq 1 + \frac{1}{2}(d-3)(d-4)$ with equality if and only if h is given by

$$\{1, 2\} \cup \{0, 1, \ldots, d-3\}.$$

Proof. Noting the easy implications $(a) \Rightarrow (b)$ and $(c) \Rightarrow (d)$, it suffices to prove (a), (c), and (e). The proofs of these are quite similar, so we only prove (e).

We induct on d. For the induction base $d = 5$, we have $h(3) = 0$ and hence $h(l) = 0$ for all $l \geq 3$ by condition (c) of 1-admissibility. It follows that $\sum_{n\geq 1}(n-1)h(n) = h(2)$. If $h(2) > 2$, then since $\sum h(n) = 5$ and $h(0) > 0$, we must have $h(1) \leq 1$, which contradicts either condition (c) or (d) of 1-admissibility, depending on whether $h(1) = 0$ or 1. Thus $h(2) \leq 2$ and we deduce the bound. Moreover, if $h(2) = 2$, then $h(1) > 1$ from condition (d). Since $h(0) > 0$ by condition (a) and h sums to 5, we must have h is given by $\{0, 1^2, 2^2\}$.

For the induction step, suppose that $d > 5$. Let $p = \max\{l : h(l) > 0\}$. If $p = 0$, then $\sum_{n\geq 1}(n-1)h(n) = 0 < 1 + \frac{1}{2}(d-3)(d-4)$. If $p > 0$, then we note that the function h' defined by $h'(l) = h(l) - \binom{l-p}{0} + \binom{l-p-1}{0}$ is 1-admissible and sums to $d-1$. It follows from induction hypothesis that

$$\sum_{n\geq 1}(n-1)h(n) = \sum_{n\geq 1}(n-1)h'(n) + (p-1) \leq 1 + \frac{1}{2}(d-4)(d-5) + (p-1)$$

and since $p \leq d-3$ we deduce the bound of part (e). Moreover, if this is an equality then $p = d-3$ and h' is given by

$$\{1,2\} \cup \{0,1,\ldots,d-4\},$$

finishing the proof of part (e).

We close this section by giving Schlesinger's criterion for when a curve has a biliaison of negative height on a quadric surface.

Proposition 1.9. (Schlesinger) *Let $C \subset \mathbb{P}^3$ be a non-complete intersection curve lying on a quadric surface Q. Then there exists a curve $C_0 \subset Q$ obtained from C by an elementary biliason of height $h < 0$ if and only if $h_C(1) \geq 2$.*

Proof. See [13], proposition 2.2.6.

2. Bounds on the Rao Function

First we state the absolute bounds on the Rao module of a curve.

Proposition 2.1. (Martin-Deschamps and Perrin) *Let $C \subset \mathbb{P}^3$ be a non-degenerate curve. Then*

$$h^1(\mathcal{I}_C(n)) \leq \begin{cases} 0 & \text{if } n \leq g - \frac{1}{2}(d-2)(d-3) \\ n + \frac{1}{2}(d-2)(d-3) - g & \text{if } g - \frac{1}{2}(d-2)(d-3) < n < 0 \\ \frac{1}{2}(d-2)(d-3) - g & \text{if } 0 \leq n \leq d-2 \\ \frac{1}{2}(d-1)(d-2) - g - n & \text{if } d-2 < n \leq \frac{1}{2}d(d-3) - g \\ 0 & \text{if } n > \frac{1}{2}d(d-3) - g \end{cases}$$

Proof. If char $k = 0$, this result can be read from [8], theorem 2.5 and corollary 2.6. If char $k = p > 0$, their proof still holds for curves whose general plane section is not contained in a line. For curves whose general plane section *is* contained in a line, see proposition 2.6 below for even stronger bounds.

A curve C is called *extremal* if it achieves the bounds of proposition 2.1 and is not ACM. The following result (which is inspired by [2], §2, theorem 8) gives several characterizations of the extremal curves.

Proposition 2.2. *Let $C \subset \mathbb{P}^3$ be a curve of degree $d \geq 2$ and genus g. Then the following conditions are equivalent:*
(i) C is an extremal curve.
(ii) There exists $a \leq 0$ such that C has spectrum

$$\{a\} \cup \{0,1,2,\ldots,d-2\} \tag{1}$$

(iii) C is not planar and is the scheme-theoretic union of a plane curve P of degree $d-1$ (contained in the plane H) and a line L such that there is a residual exact sequence

$$0 \to \mathcal{I}_L(-1) \to \mathcal{I}_C \to \mathcal{I}_{C\cap H,H} \to 0.$$

Proof. Suppose that C is extremal. Proposition 2.1 shows that $h^1(\mathcal{I}_C(n)) \leq \frac{1}{2}(d-2)(d-3) - g$ for all $n \in \mathbb{Z}$. Since C is not ACM, this implies that $g < \frac{1}{2}(d-2)(d-3)$ and $h^1(\mathcal{I}_C(a)) = 1$, where $a = g - \frac{1}{2}(d-2)(d-3) + 1 \leq 0$. It follows that $h_C(a) = 1$ when $a < 0$ and $h_C(a) = 2$ if $a = 0$. In either case, we see that the function $h'(n) = h_C(n) - \binom{n-a}{0} + \binom{n-a-1}{0}$ is 1-admissible and that $g = a - 1 + \frac{1}{2}(d-2)(d-3)$ by choice of a. Proposition 1.8(b) shows that h_C is given by

$$\{a\} \cup \{0, 1, 2, \ldots, d-2\},$$

proving $(i) \Rightarrow (ii)$.

Now we prove the implication $(ii) \Rightarrow (iii)$. To prove condition (iii), it suffices to show that C is not planar but that C contains a plane curve P of degree $d-1$. Indeed, in this case the kernel of the surjection $\mathcal{I}_C \to \mathcal{I}_{C\cap H,H}$ is contained in $\mathcal{I}_H = \mathcal{O}(-1)$, and hence can be written $\mathcal{I}_Z(-1)$ for some subscheme $Z \subset \mathbb{P}^3$. The snake lemma gives an exact sequence

$$0 \to \mathcal{O}_Z(-1) \to \mathcal{O}_C \to \mathcal{O}_{C\cap H,H} \to 0$$

which shows that Z is a line (Z has degree one and $\mathcal{O}_Z$ has no associated points of dimension zero).

Assume that C satisfies condition (ii). In this case C is not planar by example 1.5, so it suffices to show that C contains a plane curve P of degree $d-1$. If $d \geq 4$, this follows immediately from theorem 1.6(c), so we may assume that $d = 2$ or $d = 3$. If $d = 2$, then C is either a union of two skew lines or a double line (because C is not planar; see [11], corollary 1.6), hence contains a line.

For the case $d = 3$, we have $g = a - 1 < 0$. Every curve in $H(3,-1)$ contains a (planar) subcurve of degree two since $H(3,-1)$ is irreducible of dimension 12 ([11], proposition 3.1) and the general member is the disjoint union of a plane conic and a line. Now suppose that $g \leq -2$. The spectrum shows that $h^1(\mathcal{I}_C(g+1)) \neq 0$ while $h^1(\mathcal{I}_C(g)) = 0$, hence the Rao module of C has a generator in degree $g+1$. On the other hand, [11] propositions 3.2, 3.4 and 3.5 show that if C is a curve in $H(3,g)$ whose Rao module has a generator in degree $g+1$, then C is either (a) a union of a double line meeting a reduced line in a double point, (b) a union of a double line meeting a reduced line in a reduced point or (c) a triple line which contains a planar double line; in each of these cases C contains a plane curve of degree two.

In [2], §2, theorem 8, Ellia proves the implication $(iii) \Rightarrow (i)$ under t additional assumptions that $d \geq 5$ and char $k = 0$, however he makes no use these extra conditions. Thus $(iii) \Rightarrow (i)$ and the proof is complete.

Remark 2.3. Ellia proves a similar equivalence in [2], §2, theorem 8. Let $\chi($ denote the numerical character of the general hyperplane section of C and $\theta_{d,2}$ be the numerical character of a length d subscheme of $\mathbb{P}^2$ which meets so line in length $d-1$ but which is not contained in a line. Ellia's result states that if $d \geq 5$ and char $k = 0$, then statements (i) and (iii) above are equivalent to the statement that $\chi(C) = \theta_{d,2}$.

If char $k = p > 0$, then the condition that $\chi(C) = \theta_{d,2}$ is not equivalent to the other conditions. Hartshorne has shown that when char $k = p > 0$, there exist multiple lines Z of any degree $d \geq 2$ which are not planar, but for which $H \cap Z$ is contained in a line for the general plane H (see [5], example 2.3). If Z is such a multiple line and L is a line disjoint from Z, then $C = Z \cup L$ clearly satisfies $\chi(C) = \theta_{d,2}$ but fails condition (iii).

Corollary 2.4. *Let C be a curve of degree d and genus g. If $h_C(d-2) \neq 0$, then C is ACM or C is extremal.*

Proof. We may assume that $d \geq 3$, since all curves of degree $d \leq 2$ are planar or extremal. Since h_C is 1-admissible, we see that $h_C(l) > 0$ for $0 \leq l \leq d-2$ and hence h_C is given by

$$\{a\} \cup \{0, 1, \ldots, d-2\}$$

for some $a \in \mathbb{Z}$. Note that $a \leq d-1$, as otherwise $h_C(d-1) = 0$ and $h_C(a) = 1$ contradicts condition (b) of 1-admissibility. If $a = d-1$, then C is planar by example 1.5, hence ACM.

Now suppose that $a \leq d-2$. Then in fact $a \leq 1$ (if $a > 1$, then $h_C(1) = 1$ and $h_C(a) > 1$ contradict condition (c) of 1-admissibility). If $a \leq 0$, then C is extremal by proposition 2.2. If $a = 1$, then $g = \frac{1}{2}(d-2)(d-3)$ and hence C is ACM by [5], proposition 3.5.

Now we come to the bounds on the Rao function for curves which are not extremal. The bounds in degrees $n > 0$ are merely a restatement of [2], §2, corollary 9, but there is a slight improvement for degrees $n \leq 0$. We start with the lemma which gives the improvement.

Lemma 2.5. *Let $C \subset \mathbb{P}^3$ be a curve of degree d and genus g which is neither ACM nor extremal. Then $h^1(\mathcal{I}_C) \leq \frac{1}{2}(d-3)(d-4) - g$.*

Proof. Let h_C be the spectrum of C. Since all curves of degree $d \leq 2$ are ACM or extremal, we may assume $d \geq 3$. Corollary 2.4 shows that $h_C(d-2) = 0$.

Now we show that $\sum_{n \geq 1}(n-1)h_C(n) \leq \frac{1}{2}(d-3)(d-4)$. If $d = 3$ or $d = 4$, then the condition $h_C(d-2) = 0$ and condition (c) of 1-admissibility shows that this sum is zero, so we may assume $d \geq 5$. Applying proposition 1.8(e), we see that $\sum_{n \geq 1}(n-1)h_C(n) \leq \frac{1}{2}(d-3)(d-4) + 1$ with equality if and only if h_C is given by

$$\{1, 2\} \cup \{0, 1, \ldots, d-3\}.$$

However, this spectrum only occurs for an ACM curve: The definition of h_C gives $h^0(\mathcal{O}_C(2)) = 9$, hence C lies on a quadric surface Q. By proposition 1.9 there is a height -1 biliaison from C to a curve C_0 on Q. Biliaison formulas (see [7]) show that C_0 has degree $d-2$ and genus $\frac{1}{2}(d-4)(d-5)$, hence C_0 is ACM by [5], proposition 3.5. It follows that C is ACM. Thus we conclude that equality does not occur, hence $\sum_{n \geq 1}(n-1)h_C(n) \leq \frac{1}{2}(d-3)(d-4)$. To finish, we use this inequality and the genus formula (1.3(c)) to obtain

$$h^1(\mathcal{I}_C) = h^0(\mathcal{O}_C) - 1 = -1 + \sum_{k \leq 0} h_C(k)(1-k) \leq \frac{1}{2}(d-3)(d-4) - g.$$

Proposition 2.6. *Let C be a non-planar curve of degree $d \geq 3$ whose general plane section $H \cap C$ is contained in a line. Then char $k = p > 0$ and C is a multiplicity structure on a line of embedding dimension two. Further, we have*
(a) $h^1(\mathcal{I}_C(n)) \leq \frac{1}{2}(d-3)(d-4) + 1 - g$ for $n \in [1, d-3]$.
(b) $h^1(\mathcal{I}_C(d-2)) \leq \frac{1}{2}(d-3)(d-4) - g$.
(c) $h^1(\mathcal{I}_C(d-1)) \leq \frac{1}{2}(d-3)(d-4) + 1 - g - d$.
(d) $h^2(\mathcal{I}_C(n)) = 0$ for $n \geq -1$.

Proof. The first part is due to Hartshorne ([5], theorem 2.1). Let $L = \{x = y = 0\}$ be the line of support of C. Since C has embedding dimension two, it is a primitive extension (see [1]), hence the Cohen-Macaulay filtration $L = C_1 \subset C_2 \subset \ldots \subset C_d = C$ gives rise to exact sequences

$$0 \to \mathcal{I}_{C_{j+1}} \to \mathcal{I}_{C_j} \xrightarrow{u_j} \mathcal{O}_L(ja) \to 0 \tag{2}$$

for some $a \geq -1$. Since C is not planar, we must have $a \geq 0$ (see [11], lemma 1.3). From the sequences we see that C_j has genus $\frac{1}{2}a(j-1)(j) + (j-1)$ and we immediately deduce the vanishings of part (d), since for $n \geq -1$ and $j \geq 1$ we have $H^1(\mathcal{O}_L(ja+n)) = H^2(\mathcal{O}_L(ja+n)) = 0$. We will use the sequences to inductively prove the bounds of parts (a),(b) and (c), but first we must make an observation about the maps u_j.

Since C_j is a multiplicity j-structure on L, it is clear that $(x,y)^j \subset I_{C_j}$ for $1 \leq j \leq d$. I claim further that $((x,y)^{j-1} \cap I_{C_j})_{j-1} = (0)$. Since C has embedding dimension two, C is contained in a surface S which is generically smooth along C. Assume that $0 \neq \beta \in ((x,y)^{j-1} \cap I_{C_j})_{j-1}$. Since k is algebraically closed, β can be written as a product of linear factors, say $\beta = \Pi z_i$. Let H_i be the plane $\{z_i = 0\}$ and let D_i be the irreducible component of $H_i \cap S$ supported on L. If D_i is reduced for each i, then on an open set of S z_i restricts to a local equation for L, hence the restriction of β to S cuts out a line of multiplicity $j-1$ containing C_j, a contradiction. On the other hand, if D_i is not reduced for some i, then the planar double line $Z \subset H_i$ supported on L is contained in C_j, when $a = -1$, contradicting our assumption. Thus $\operatorname{rank} H^0(u_j(j)) \geq j+1$ for $j \geq 2$.

Now we inductively prove the bounds. The exact sequence 2 gives the inequality

$$h^1(\mathcal{I}_{C_{j+1}}(n)) \leq h^1(\mathcal{I}_{C_j}(n)) + ja + 1 + n - \operatorname{rank} H^0(u_j(n))$$

for $n \geq -a$. For the induction base $d = 3$ we consider this inequality with $j = 2$. Since C_2 has genus $-a-1$, proposition 2.1 shows that $h^1(\mathcal{I}_{C_2}(1)) \leq a$ and hence $h^1(\mathcal{I}_{C_3}(1)) \leq 3a + 2 = -g(C_{j+1})$, proving bound (b). For bound (c), one uses the fact that $\operatorname{rank} H^0(u_2(2)) \geq 3$. The induction step is similar and left to the reader.

Remark 2.7. The proof above shows that the bounds of proposition 2.6 apply to any quasiprimitive multiplicity structure on a line which does not contain a planar double line.

Definition 2.8. A multiplicity structure on a line which satisfies the hypotheses of proposition 2.6 will be called a *Hartshorne multiple line*. The existence of such lines was established by Hartshorne ([5], example 2.3).

Proposition 2.9. *Let C be a curve of degree $d \geq 5$ which is neither planar nor extremal. Assume that the general plane section $H \cap C$ is not contained in a line, but intersects a line in a scheme of length $d-1$. Then either*
(1) C is a multiple line of generic embedding dimension three or
(2) C is the union of a Hartshorne multiple line and a reduced line.
In either case, conditions (a) and (b) of proposition 2.6 hold.

Proof. Let $\overline{C} = C_{red}$ and $\overline{d} = \deg \overline{C}$. First we show that $\overline{C} \cap H$ is contained in a line for the general plane H. We may assume that $\overline{d} \geq 3$. If $\overline{C} \cap H$ is not linear for general H, we can find H_1 such that $\overline{C} \cap H_1$ is reduced and not contained in a line. Let L_1 be the $(d-1)$-secant line to $C \cap H_1$. Since $\overline{C} \cap H_1$ is reduced, we can find $Q \in (\overline{C} \cap H_1) - L_1$ and $P \in (\overline{C} \cap H_1) \cap L_1$. If H_2 is a general plane which contains P and Q, then $C \cap H_2$ has a $(d-1)$-secant line L_2. If $P \in L_2$, then the plane spanned by L_1 and L_2 meets C in a scheme of length $\geq \ell((L_1 \cup L_2) \cap C) \geq \ell(L_1 \cap C) + \ell(L_2 \cap C) - \ell(L_1 \cap L_2 \cap C) \geq 2d-3$, a contradiction. If $P \notin L_2$, then $Q \in L_2$. In this case we can find another general plane H_3 (through P and Q) such that the $(d-1)$-secant line L_3 to $C \cap H_3$ contains Q. In this case the plane spanned by L_2 and L_3 meets C in a scheme of length $\geq 2d-3$, contradiction. Thus $\overline{C} \cap H$ is contained in a line for general H.

If $\overline{d} \geq 3$, we can apply Hartshorne's restriction theorem ([5], theorem 2.1) to see that $\overline{C}$ is a multiple line, contradicting the fact that $\overline{C}$ is reduced, hence $\overline{d} = 1$ or $\overline{d} = 2$. If $\overline{d} = 2$ and $\overline{C}$ is an irreducible conic in the plane H, then $C \subset H$: if not, then $\deg C \cap H \leq d-2$ and hence the general line $L \subset H$ is a $(d-2)$-secant line, a contradiction. Thus $\overline{C}$ is a line or a pair of lines.

Case 1: $\overline{C}$ is a pair of lines. Write $\overline{C} = L_0 \cup L_1$ and let C_0, C_1 be the corresponding irreducible components of C, having respective degrees d_0, d_1. We will show that for the general plane H, the corresponding $(d-1)$-secant line L does not intersect one of the lines L_i. First suppose that $L_0 \cap L_1 \neq \emptyset$ and that these lines span the plane K. Let H be a general plane and consider the $(d-1)$-secant line L to $C \cap H$. If L meets both lines, then $L = H \cap K$ and $\deg C \cap K = d-1$. In this case C is extremal by 2.2(iii), contradicting hypothesis.

Now suppose that $L_0 \cap L_1 = \emptyset$. Let H be a general plane with corresponding $(d-1)$-secant line L. Suppose that L meets both L_0 and L_1 for general H. Then for $P \in L_0$ and $Q \in L_1$, the general line L_{PQ} through P and Q is a $(d-1)$-secant to C. The family of all such lines L_{PQ} is irreducible and the condition that L be a d_i-secant line to C_i is a closed condition; it follows that L_{PQ} is a d_i-secant line to C_i for all pairs (P,Q) for some i, say $i=0$. Fixing $P \in L_0$ and choosing $R \neq Q \in L_1$, both L_{PQ} and L_{PR} are d_0-secants to $C_0 \cap K$ in the plane K spanned by P and L_1. Thus $C_0 \cap K$ is contained in both these lines, $d_0 = 1$ and C_0 is a reduced line. We can choose $P \neq P' \in L_0$ and $Q \in L_1$ such that L_{PQ} and $L_{P'Q}$ are both (d_1-1)-secants to C_1 at Q. If K' is the plane spanned by Q and L_0, then these lines are in fact (d_1-1)-secants to $C_1 \cap K'$. This is a contradiction, since a zero-dimensional projective scheme Z of degree $d \geq 4$ can have at most one $(d-1)$-secant line (if Z has two distinct $(d-1)$-secant lines R_1 and R_2, then $\ell((R_1 \cup R_2) \cap Z) \geq \ell(R_1 \cap Z) + \ell(R_2 \cap Z) - \ell(R_1 \cap R_2 \cap Z) \geq 2d-3 > d$, a contradiction).

From the two preceding paragraphs, we conclude that if H is a general plane containing the corresponding $(d-1)$-secant line L to C, then L does not meet both L_0 and L_1. If the general such $(d-1)$-secant does not meet L_i, then C_i is reduced and $H \cap C_{i-1}$ is contained in a line. Since $d_{i-1} \geq 4$, C_{i-1} is a Hartshorne multiple line.

Above we have seen that if $\overline{C}$ consists of two lines, then C is the union of a reduced line L and a Hartshorne multiple line Z. Letting $r = \operatorname{length} Z \cap L$, we have an exact sequence

$$0 \to \mathcal{I}_C \to \mathcal{I}_Z \oplus \mathcal{I}_L \xrightarrow{\pi} \mathcal{I}_{Z \cap L} \to 0 \qquad (3)$$

which shows that

$$h^1(\mathcal{I}_C(n)) \leq h^1(\mathcal{I}_Z(n)) + h^0(\mathcal{O}_L(n-r))$$

and $g(C) = g(Z) + r - 1$. If $r \leq 1 < d-3$, the inequalities from proposition 2.6 yield inequalities (a) and (b) of proposition 2.6 for C.

Now suppose that $r \geq 2$. Let H be a general plane containing L. Since Z is a Hartshorne multiple line of degree $d-1$, $H \cap Z$ is contained in a line L' which must equal L. Indeed, we have the inequality

$$\ell(H \cap Z) \geq \ell((L' \cup L) \cap Z) \geq \ell(L' \cap Z) + \ell(L \cap Z) - \ell(L' \cap L \cap Z).$$

If $L' \neq L$, then this last expression is $\geq d$, a contradiction since $\deg Z = d-1$. Thus $L' = L$ and we see that $r = d-1$.

Now we estimate $h^1(\mathcal{I}_C(n))$. First we show that $H^1(\pi(n))$ is surjective for $n \geq 0$. Since $H^1_*(\mathcal{I}_L) = 0$, it suffices to show this for the natural map $H^1_*(\mathcal{I}_Z) \to H^1_*(\mathcal{I}_{Z \cap L})$. Since $Z \cap L = Z \cap H$, we may use the exact sequence

$$0 \to \mathcal{I}_Z(-1) \to \mathcal{I}_Z \xrightarrow{\psi} \mathcal{I}_{Z \cap H, H} \to 0$$

to analyze this map. The exact sequence

$$0 \to \mathcal{O}(-1) \to \mathcal{I}_{Z \cap H} \to \mathcal{I}_{Z \cap H, H} \to 0$$

shows that $H^1_*(\pi)$ can be identified with $H^1_*(\psi)$, and these maps are surjective in degrees ≥ 0 by proposition 2.6(d).

Since $Z \cap H$ is an effective divisor on the line L, it is a complete intersection of L and a curve of degree $d-1$ on the plane H. It follows that $H^0_*(\pi(n))$ is surjective in degrees $\leq d-2$ and an easy calculation shows that rank $H^1(\psi(n)) = h^1(\mathcal{I}_{Z \cap H, H}(n)) = d - n - 2$ for $0 \leq n \leq d-2$. Thus we find that

$$h^1(\mathcal{I}_C(n)) = h^1(\mathcal{I}_Z(n)) - d + n + 2$$

for $0 \leq n \leq d-2$. For $n \leq d-4$, we obtain the desired bound from proposition 2.6(a). For $n = d-3$ we use 2.6(b), and the bound for $n = d-2$ follows from 2.6(c).

Case 2: $\overline{C}$ is a line. Let L be the support of C. Let $\{C_i\}$ be the Cohen-Macaulay filtration for C. Let H be a general plane with corresponding $(d-1)$-secant line L to $C \cap H$. The residual exact sequence with respect to L in H can be written

$$0 \to \mathcal{I}_P(-1) \to \mathcal{I}_{C \cap H} \to \mathcal{I}_{C \cap L, L} \to 0$$

where P is the support of $C \cap H$. If z is the equation for L in H and w is the equation of another line through P, then the sequence shows that we may write $I_{C \cap H, H} = (z^2, zw, w^{d-1})$. Restricting the Cohen-Macaulay filtration to H gives the sequence of ideals

$$(z, w) \supset (z, w)^2 \supset (z^2, zw, w^3) \supset \ldots \supset (z^2, zw, w^{d-1})$$

and we conclude that $C_2 = L^{(2)}$ and $\deg C_j = j + 1$ for $j \geq 2$. For $j \geq 2$, we have exact sequences

$$0 \to \mathcal{I}_{C_{j+1}} \to \mathcal{I}_{C_j} \xrightarrow{u_j} \mathcal{O}_L(a_j) \to 0$$

and hence inequalities

$$h^1(\mathcal{I}_{C_{j+1}}(n)) \le h^1(\mathcal{I}_{C_j}(n)) + h^0(\mathcal{O}_L(a_j+n)) - \operatorname{rank} H^0(u_j(n)). \qquad (4)$$

It is evident that $(x,y)^j \subset I_{C_j}$. I claim that $((x,y)^{j-1} \cap I_{C_j})_{j-1} = (0)$ for $j \ge 3$. Letting $0 \ne \beta \in ((x,y)^{j-1})_{j-1}$, we may write $\beta = \Pi_1^{j-1} z_i$, where the z_i are linear factors which give rise to corresponding planes H_i. Since C is neither planar nor extremal, we have $\deg H_i \cap C_j < j$ for each i. We can now choose a general plane K meeting C properly in a point P such that $\deg(H_i \cap C \cap K) < j$ for each i. Write $J = I_{C_j \cap K, K} = (x^2, xy, y^j)$ as above (x is the equation of the unique j-secant line). By construction, the image $\overline{z}_i$ of z_i in J is not a multiple of x, so we may write $\overline{z}_i = u_i(y+\alpha_i x)$ where u_i is a unit. Since $\overline{\beta} \in J$, we can add multiples of x^2 and xy to see that $y^{j-1} \in J$. This gives a contradiction, since the ideal (x^2, xy, y^{j-1}) defines a scheme of length j, while $C \cap K$ has length $j+1$. It follows that $\operatorname{rank} H^0(u_j(j)) \ge j+1$ for $j \ge 2$.

Letting g_j denote the genus of C_j, the exact sequence above show that $g_{j+1} = g_j - a_j - 1$. Combining the fact that $\operatorname{rank} H^0(u_j(j)) \ge j+1$ for $j \ge 2$ with the inequalities 4 above, one can show by induction on $j \ge 3$ that

$$h^1(\mathcal{I}_{C_j}(n)) \le \frac{1}{2}(j-2)(j-3) + 1 - g_j$$

for $1 \le n < j-1$ and

$$h^1(\mathcal{I}_{C_j}(j-1)) \le \frac{1}{2}(j-2)(j-3) + 1 - g_j - j.$$

Taking $j = d-1$ proves inequalities (a) and (b) of proposition 2.6 for C, finishing the proof.

Example 2.10. Examples of both cases (1) and (2) of proposition 2.9 exist. Let $L \subset \mathbb{P}^3$ be a line and let Z be a Hartshorne multiple line supported on L. Letting C be the scheme-theoretic union $Z \cup L^{(2)}$ gives a curve satisfying the hypotheses of proposition 2.9 which falls into case (1). For case (2), we can simply take the disjoint union of Z and another line L'.

Theorem 2.11. *Let $C \subset \mathbb{P}^3$ be a curve of degree $d \ge 4$ and genus g which is neither ACM nor extremal. Then*

$$h^1(\mathcal{I}_C(n)) \le \begin{cases} 0 & \text{if } n < g - \frac{1}{2}(d-3)(d-4)+1 \\ \frac{1}{2}(d-3)(d-4) - g + n & \text{if } g - \frac{1}{2}(d-3)(d-4) + 1 \le n < 1 \\ \frac{1}{2}(d-3)(d-4) + 1 - g & \text{if } 1 \le n \le d-3 \\ \frac{1}{2}(d-2)(d-3) + 1 - g - n & \text{if } d-3 < n \le \frac{1}{2}(d-2)(d-3) - g \\ 0 & \text{if } n > \frac{1}{2}(d-2)(d-3) - g \end{cases}$$

Proof. Migliore's lemma (see [8], lemma 0.1 and proposition 2.3) shows that the Rao function is strictly increasing on $[r_a, 0]$ and strictly decreasing on $[d-2, r_o]$, hence it suffices to prove these bounds for $n \in [0, d-2]$. In particular, the bounds in degrees $n \le 0$ follow from lemma 2.5, so we reduce to proving the bound for $n \in [1, d-2]$. If $d \ge 5$ and the general plane section $C \cap H$ meets no line in a scheme of length $\ge d-1$, then Ellia's proof of [2], §2, proposition 9 gives the bounds above in degrees $n \ge 1$. If $d \ge 5$ and the general plane section $C \cap H$

meets some line in a scheme of length $\geq d-1$, then propositions 2.6 and 2.9 give the bounds on $[1, d-2]$.

To finish, we need to prove the bounds for $n \in [1, d-2]$ when $d = 4$. The bound for $n = 1$ follows from proposition 2.1, so it suffices that $h^1(\mathcal{I}_C(2)) \leq -g$. Theorem 1.6(a) forces $h_C(4) = 0$, hence proposition 1.3(e) shows that $h^2(\mathcal{I}_C(2)) = 0$. Computing the Euler characteristic of $\mathcal{I}_C(2)$, we reduce to showing that $h^0(\mathcal{I}_C(2)) \leq 1$. If this is not the case, then C lies on two quadric surfaces which share a common plane H. (otherwise the quadrics are independent, when C must equal their complete intersection by reason of degree). In considering the residual exact sequence

$$0 \to \mathcal{I}_Z(-1) \to \mathcal{I}_C \to \mathcal{I}_{C\cap H,H} \to 0$$

with respect to the plane H, we see that Z is contained in two distinct planes, hence has degree ≤ 1. It follows that either C is contained in H (when C is ACM) or contains a plane curve of degree 3 (when C is extremal by proposition 2.2(iii)), a contradiction. This finishes the proof.

Definition 2.12. We say that a curve $C \subset \mathbb{P}^3$ of degree d and genus g is *subextremal* if C is neither ACM nor extremal and the inequalities of proposition 2.11 above are all equalities.

Corollary 2.13. *Let C be a subextremal curve of degree d and genus g. Then $d \geq 4$ and $g \leq \frac{1}{2}(d-3)(d-4)$.*

Proof. The bounds of theorem 2.11 and theorem 2.1 are identical for curves of degree $d = 2$ or $d = 3$, hence the curves achieving equality are already extremal. It follows that $d \geq 4$. Since $h^1(\mathcal{I}_C(n)) \leq \frac{1}{2}(d-3)(d-4) + 1 - g$ for all n, the fact that a subextremal curve is not ACM shows that $g \leq \frac{1}{2}(d-3)(d-4)$.

Theorem 2.14. *Let $C \subset \mathbb{P}^3$ be a curve of degree d and genus g. Then C is subextremal if and only if C is obtained from an extremal curve by an elementary biliaison of height 1 on a quadric surface Q. If one (hence both) of these conditions hold, then C has spectrum*

$$\{g - \frac{1}{2}(d-3)(d-4) + 1\} \cup \{0, 1^2, 2, \ldots, d-3\}.$$

Proof. If C is obtained from an extremal curve by an elementary biliaison of height 1 on a quadric surface, it is easy to calculate that C is subextremal. Conversely, let C be a subextremal curve and set $a = g - \frac{1}{2}(d-3)(d-4) + 1$. Corollary 2.13 shows that $a \leq 1$. Since C is subextremal, $h^1(\mathcal{I}_C(l)) = l - a + 1$ for $a \leq l \leq 1$. Since C is not planar, the exact sequence

$$0 \to R_C \to A_C \to H^1_*(\mathcal{I}_C) \to 0$$

shows that $h^0(\mathcal{O}_C(l)) = l - a + 1$ for $a \leq l < 0$, $h^0(\mathcal{O}_C(0)) = -a + 2$, and $h^0(\mathcal{O}_C(1)) = -a + 6$. Using its definition, we see that the spectrum of C takes the form

$$\{a\} \cup \{0, 1^2, \ldots\}.$$

Applying proposition 1.8(d), we see that the spectrum of C is given by

$$\{a\} \cup \{0, 1^2, 2, 3, \ldots, d-3\}.$$

Having computed the spectrum, proposition 1.3(d) and duality show that $h^2(\mathcal{I}_C(2)) = \frac{1}{2}(d-5)(d-6)$ when $d \geq 5$ and $h^2(\mathcal{I}_C(2)) = 0$ when $d = 4$. The definition of subextremal shows that $h^1(\mathcal{I}_C(2)) = \frac{1}{2}(d-3)(d-4) + 1 - g$ when $d \geq 5$ and $h^1(\mathcal{I}_C(2)) = -g$ when $d = 4$. From the degree and genus we compute that $\chi(\mathcal{I}_C(2)) = 9 - 2d + g$. Putting these three pieces of information together, we find that $h^0(\mathcal{I}_C(2)) = 1$ and hence C lies on a unique quadric surface Q. Since $h_C(1) \geq 2$, proposition 1.9 applies to show that C is obtained from a curve C_0 by an elementary biliaison of height $h > 0$ on Q. If C_1 is obtained from C_0 by an elementary biliaison of height $h-1$ on Q, then C is obtained from C_1 by an elementary biliaison of height 1. Calculating the degree, genus, and Rao function for C_1 shows that C_1 is extremal.

Example 2.15. Unlike the case of extremal curves, subextremal curves are not determined by their spectrum. For $d \geq 5$ and $g \leq \frac{1}{2}(d-3)(d-4)$, let Z be a double line of genus $g' = g - d - \frac{1}{2}(d-2)(d-5)$. If C is obtained from Z by an elementary biliaison of height 1 on a surface of degree $d-2$, then C has degree d and genus g. If C' is a subextremal curve of degree d and genus g, then the Rao functions of C and C' are identical in degrees ≤ 1 and both have spectrum of theorem 2.14.

Remark 2.16. The example above (and the following example) show that the statement of [2] §2 proposition 10 does not hold as stated. The stronger bounds on the Rao module suggested there hold for curves C of degree $d \geq 7$ whose general plane section $C \cap H$ does not meet a line in a scheme of length $\geq d-2$. The problem is that there are curves C whose general plane section meets a line in length $d-2$ but which are not subextremal.

Example 2.17. In the previous example we found curves which were not subextremal, but which achieved the bound of theorem 2.11 in degree 1 and had the same spectrum as the subextremal curves. In this example we give curves which are not extremal, achieve the bound of theorem 2.11 in degree 1, and give a different spectrum. Specifically, we will show that for any $-1 \leq b \leq a$ and $d \geq 5$ there exists a quasiprimitive multiple lines Z with spectrum

$$\{-a, -b, 0, 1, 2, \ldots, d-3\}.$$

When $b = -1$, we get the subextremal spectrum of theorem 2.14.

We construct the curve Z in two steps. Let X be a planar multiplicity $(d-2)$-line with ideal $I_X = (x, y^{d-2})$. The support of X is the line L given by $\{x = y = 0\}$. Let h, k be a two homogeneous polynomials in S_L of degrees $1+b, b+d-2$ which have no common zeros along L. These define a map $\phi : I_X \to S_L(b)$ by $x \mapsto h, y^{d-2} \mapsto k$. This map sheafifies to a surjection $u : \mathcal{I}_X \to \mathcal{O}_L(b)$ whose kernel is the ideal sheaf of a quasiprimitive multiplicity $(d-1)$ structure Y on L. It is easy to check that $I_Y = (x^2, xy, y^{d-1}, xk - y^{d-2}h)$.

I_Y has an S-presentation

$$S(-3) \oplus S(-d) \oplus S(-b-d)^2 \xrightarrow{\psi} S(-2)^2 \oplus S(-d+1) \oplus S(-b-d+1) \to I_Y \to 0 \quad (5)$$

where ψ is given by the matrix

$$\begin{pmatrix} y & 0 & -k & 0 \\ -x & y^{d-2} & y^{d-3}h & -k \\ 0 & -x & 0 & h \\ 0 & 0 & x & y \end{pmatrix} \tag{6}$$

Tensoring ψ with S_L shows that

$$I_Y \otimes S_L \cong S_L/(k)(-2) \oplus (h,k)(b-1) \oplus S_L(-b-d+1)$$

and hence $\mathcal{I}_Y \otimes \mathcal{O}_L \cong \mathcal{O}_{Z(k)\cap L}(-2) \oplus \mathcal{O}_L(b-1) \oplus \mathcal{O}_L(-b-d+1)$ (the injection $\theta : (h,k) \hookrightarrow S_L$ has cokernel of finite length, hence $\widetilde{\theta}$ is an isomorphism).

Now let p, q be homogeneous polynomials in S_L of degrees $a+1-b, a+b+d-1$ which have no common zeros along L. Using the identification above, (p,q) defines a map $I_Y \to S_L(a)$ whose cokernel has finite length. Sheafifying this map gives a surjection $w : \mathcal{I}_Y \to \mathcal{O}_L(a)$ whose kernel is the ideal sheaf of a quasiprimitive multiplicity d-structure Z on L.

To check that Z has the spectrum claimed in the proposition, note that the construction above gives exact sequences

$$0 \to \mathcal{I}_Y \to \mathcal{I}_X \xrightarrow{u} \mathcal{O}_L(b) \to 0$$

$$0 \to \mathcal{I}_Z \to \mathcal{I}_Y \xrightarrow{w} \mathcal{O}_L(a) \to 0.$$

Since X is planar, $H^1_*(\mathcal{I}_X) = 0$ and $H^1_*(u)$ is the zero map. The first sequence shows that $H^1_*(\mathcal{I}_Y)$ is either 0 or generated by an element of degree $-b$. Since $b \leq a$, we have that $H^1(\mathcal{O}_L(a-b)) = 0$, hence $H^1_*(w)$ is the zero map. In particular, we now obtain short exact sequences

$$0 \to H^1_*(\mathcal{O}_L(b)) \to H^1_*(\mathcal{O}_Y) \to H^1_*(\mathcal{O}_X) \to 0$$

$$0 \to H^1_*(\mathcal{O}_L(a)) \to H^1_*(\mathcal{O}_Z) \to H^1_*(\mathcal{O}_Y) \to 0.$$

Since X is a plane curve of degree $d-2$, it has spectrum

$$\{0, 1, 2, \ldots, d-3\}.$$

Taking second difference functions of the two exact sequences gives the spectrum claimed. The genus of Z is $\frac{1}{2}(d-3)(d-4)-a-b-2$, so the bound on $h^1(\mathcal{I}_Z(1))$ given by theorem 2.11 is $a+b+3$. Since $H^0(u(x)) \neq 0$, the exact sequences above show that $h^1(\mathcal{I}_Z(1)) = a+b+3$.

Example 2.18. For a more complete example, consider curves of degree 5 and genus 0. In his PhD thesis, Rich Liebling uses generic initial ideals to find all possible Rao functions for these curves (see [6], §5.2). Here is the list:

(a) Extremal curves in $H(5,0)$ have Rao module of type $\{1,2,3,3,3,3,2,1\}$ starting in degree -2. This family forms an irreducible component of dimension 24.

(b) Subextremal curves have Rao module of type $\{1,2,2,1\}$ starting in degree 0. The closure of this family is an irreducible component of dimension 20. The general subextremal curve is a disjoint union of a conic and a plane cubic.

(c) The disjoint unions of elliptic quartic curves and lines forms an irreducible family of dimension 20 whose closure is an irreducible component of $H(5,0)$. These curves have Rao module of type $\{1,2,1\}$ starting in degree 0. Curves

from example 2.15 above also have this Rao function.
(d) The general smooth rational quintic curves give an irreducible family of dimension 20 whose closure is an irreducible component. The corresponding Rao modules have type $\{2,1\}$ starting in degree 1.
(e) The curves of type $(4,1)$ on a smooth quadric surface form an irreducible family of dimension 18. Their Rao modules have type $\{2,2\}$ starting in degree 1. This family lies in the closure of family (d) above.

Remark 2.19. It seems reasonable to expect that the closure of the subextremal curves form an irreducible component in $H(d,g)$ for $d \geq 4$ and $g \leq \frac{1}{2}(d-3)(d-4)$, as happens in the example above.

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