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DEFORMATIONS OF SPACE CURVES: CONNECTEDNESS OF HILBERT SCHEMES

Warmly dedicated to Paolo Valabrega on his 60th birthday

Abstract. We survey the Hilbert schemes $H_{d,g}$ of Cohen-Macaulay space curves having degree d and genus g, giving their geography and the current state of the connectedness problem. Focusing on a specific example, we then describe the irreducible families of curves in $H_{4,-99}$ and explain the connectedness, paying special attention to certain deformations on the double quadric surface. We close with some new results, determining which families of degree four curves are subcanonical and showing how some examples of Chiantini and Valabrega fit into this classification.

1. Introduction

Early in the development of scheme theory in algebraic geometry, Grothendieck constructed the fine moduli space for flat families of subschemes in $\mathbb{P}^n$, known as the Hilbert scheme [15]. Since the Hilbert polynomial is constant for flat families over a connected base, the Hilbert scheme $\mathrm{Hilb}^{\mathrm{n}}$ can be written as a disjoint union of pieces $\mathrm{Hilb}^{\mathrm{n}}_{\mathrm{p(z)}}$ indexed by the corresponding Hilbert polynomials. As a fine moduli space, these schemes come equipped with universal flat family

$$\begin{array}{ccc} X & \subset & \mathrm{Hilb}^{\mathrm{n}}_{\mathrm{p(z)}} \times \mathbb{P}^n \\ & & \downarrow \\ & & \mathrm{Hilb}^{\mathrm{n}}_{\mathrm{p(z)}} \end{array} \tag{1}$$

having fibres with Hilbert polynomial $p(z)$ such that for any flat family

$$\begin{array}{ccc} Y & \subset & T \times \mathbb{P}^n \\ & & \downarrow \\ & & T \end{array} \tag{2}$$

with fibres of Hilbert polynomial $p(z)$, there is a unique map $T \to \mathrm{Hilb}^{\mathrm{n}}_{\mathrm{p(z)}}$ such that diagram (2) is obtained from diagram (1) by pull-back. Thus one studies the Hilbert scheme by producing flat families. As Grothendieck showed that $\mathrm{Hilb}^{\mathrm{n}}_{\mathrm{p(z)}}$ is projective over $\mathrm{Spec}\,\mathbb{Z}$, the set of all projective subschemes is encoded by equations with integer coefficients.

Since flat families over a connected base have constant Hilbert polynomial, it's natural to ask whether the converse is true: given two subschemes in $\mathbb{P}^n$ with the same Hilbert polynomial, is there a connected flat family of which both are a member? Equivalently, is the Hilbert scheme connected? This was answered by Hartshorne in his PhD thesis [19].

THEOREM 1. (Hartshorne, 1962) *For any $p(z) \in \mathbb{Q}[z]$ and any field k, the Hilbert scheme $\mathrm{Hilb}^{\mathrm{n}}_{\mathrm{p(z)}}$ for closed subschemes $X \subset \mathbb{P}^n_k$ with Hilbert polynomial $p(z)$ is connected whenever it is non-empty.*

The *geography* is an important aspect of any moduli problem: for which natural invariants of the problem is the moduli space non-empty? There are at least three characterizations of the polynomials $p(z) \in \mathbb{Q}[z]$ for which there is a subscheme $V \subset \mathbb{P}^n$ having Hilbert polynomial $p(z)$. One follows from Macaulay's theorem on the growth of the Hilbert function of a standard k-algebra [26], another is a consequence of Hartshorne's thesis [19] and a third occurs naturally from Green's interpretation of Macaulay's bound in terms of restricted linear series [14]: a summary and comparison is given in [3].

We now specialize to space curves: take $n = 3$ and let $\mathrm{Hilb}_{d,g}$ denote the Hilbert scheme of subschemes in $\mathbb{P}^3$ with Hilbert polynomial $p(z) = dz + 1 - g$, the curves of degree d and arithmetic genus g. Classically one is interested in the open subscheme

$$H^0_{d,g} \subset \mathrm{Hilb}_{d,g}$$

corresponding to smooth connected curves. The geography for this problem (the pairs (d, g) for which $H^0_{d,g}$ is non-empty) was known to Halphen and completely proved by Gruson and Peskine a hundred years later [16]. As to connectedness, we have the following results of Harris [18] and Ein [9].

THEOREM 2. (Harris, 1982) *$H^0_{d,g}$ is irreducible if $d \geq \frac{5}{4}g + 1$.*

THEOREM 3. (Ein, 1986) *$H^0_{d,g}$ is irreducible if $d \geq g + 3$.*

EXAMPLE 1. The Hilbert schemes $H^0_{d,g}$ are not connected in general: the smallest example is $H^0_{9,10}$ [20, IV, Ex. 6.4.3], which has two connected components, the curves of type $(3, 6)$ on a smooth quadric and complete intersections of two cubics. More generally, $H^0_{d,g}$ is not connected for $d \geq 9$ and $g = 2d - 8$. Indeed, the curves C of type $(3, d-3)$ on a smooth quadric satisfy $h^0\mathcal{O}_C(2) = 9$ and $h^0\mathcal{I}_C(2) = 1$ while curves D not lying on a quadric satisfy $h^0\mathcal{O}_D(2) \geq 10$ and $h^0\mathcal{I}_D(2) = 0$. By semicontinuity, it follows that the curves of type $(3, d-3)$ form a connected component of $H^0_{d,2d-8}$. Note that there exist other components, as such curves exist on a cubic or quartic surface. Guffroy conjectures that $H^0_{d,g}$ is irreducible for $g < 2d - 8$ (i.e. $d > \frac{1}{2}g + 4$) and proves it for $d \leq 11$ [17]. If true, the conjecture would strongly improve the results above.

The subject of this survey is yet a third moduli space, namely the Hilbert scheme of locally Cohen-Macaulay curves without isolated points, the pure one-dimensional subschemes of $\mathbb{P}^3$ of degree d and genus g. Following Martin-Deschamps and Perrin [27, 28], we denote these Hilbert schemes by $H_{d,g}$, which sit between the two extremes considered above:

$$H^0_{d,g} \subset H_{d,g} \subset \mathrm{Hilb}_{d,g}.$$

The Hilbert schemes $H_{d,g}$ are natural from the perspective of liaison theory, which has seen a great deal of activity over the last 25 years: Migliore's book [31] provides an excellent survey of this work. The point is that liaison preserves the property of being locally Cohen-Macaulay [31, Cor. 5.2.12] but does not preserve geometric properties such as smoothness, irreduciblity, or reducedness. On the other hand, even the most general locally Cohen-Macaulay curves can be brought to the classical curves through a sequence of liaisons, as proved by Rao [38, Thm. 2.6].

Theorem 4. (Rao, 1979) *Every liaison class contains a smooth connected curve.*

Thus the schemes $H_{d,g}$ are the result of starting with the smooth connected curves and closing off under the equivalence relation of liaison. In view of the connectivity results above, the following question is natural:

Problem 1. *For which is pairs (d, g) is $H_{d,g}$ connected?*

Remark 1. This does not follow in any easy way from the proof of Theorem 1, as Hartshorne constructs deformations which typically pass through (non-reduced) subschemes having embedded points. The real question here is whether curves with embedded points can be avoided.

In addressing the status of Problem 1, we begin with the geography of locally Cohen-Macaulay space curves in §2. This includes (a) the determination of the pairs (d, g) for which $H_{d,g}$ is non-empty and (b) the cohomological bounds leading to the special families of extremal and subextremal curves. The extremal curves become prominent in §3 when we give connectedness results for the Hilbert schemes. We follow this up with an example in §4, describing all the irreducible components of the Hilbert scheme $H_{4,-99}$ and explaining why this scheme is connected. In §5 we discuss deformations of curves on a double surface and show how a disjoint union of two double lines can be deformed to a multiplicity four line without adding embedded points, a crucial part of the proof that $H_{4,-99}$ is connected. Finally, in §6 we determine which families of degree four curves are sub-canonical. In particular, we show how examples of Chiantini and Valabrega [5, Ex. 3.1 and 3.2] fit into our classification.

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2. The Geography of Cohen-Macaulay Curves

In this section we describe the pairs (d, g) for which our Hilbert schemes $H_{d,g}$ are non-empty. As a byproduct of the proof, we will encounter the extremal curves, which play an important role in the following section. The starting point is the following theorem [29, Thm. 2.5 and Cor. 2.6].

THEOREM 5. (Martin-Deschamps and Perrin, 1993) *Assume char $k = 0$. If $C \in H_{d,g}$ is non-planar, then the Rao function $h^1\mathcal{I}_C(n)$ is bounded by the function depicted in Figure 1.1. In particular, $g \leq \binom{d-2}{2}$.*

Figure 1.1: Bound of Theorem 5 on $h^1\mathcal{I}_C(n)$ for non-planar curves

This generalizes to curves in higher dimensional projective space [7], though the bounding function is more complicated. The characteristic zero hypotheses is used to prove that if C is a curve of degree $d \geq 3$ not contained in a plane, then the general hyperplane section $H \cap C$ is not contained in a line. While this fails in characteristic $p > 0$ [21, Ex. 2.3], the bound on cohomology still holds [32, Prop. 2.1], as does the bound on the genus [21, Cor. 3.6]:

THEOREM 6. (Hartshorne 1994) *The Hilbert scheme $H_{d,g}$ is nonempty if and only if either*

(a) $d \geq 1$ and $g = \binom{d-1}{2}$ or

(b) $d \geq 2$ and $g \leq \binom{d-2}{2}$.

One way to prove that $H_{d,g}$ is non-empty for $g \leq \binom{d-2}{2}$ is to observe that there are curves which achieve equality in Theorem 5 [28, Prop. 0.5]:

THEOREM 7. (Martin-Deschamps and Perrin, 1996) *For all $d \geq 2$ and $g \leq \binom{d-2}{2}$, there are curves $C \in H_{d,g}$ giving equality in Theorem 5 for all n.*

The curves of Theorem 7 are called *extremal curves* and have some interesting properties. For example, the subset of extremal curves forms an irreducible component $E \subset H_{d,g}$ [28, Thm. 3.7], which is non-reduced except when $d = 2$ (double lines), $g = \binom{d-2}{2}$ (ACM extremal curves) or $d = 3$ and $g = -1$ [28, Thm. 5.3].

Remark 2. The following are equivalent:

1. C is an extremal curve.

2. C is a minimal curve for a complete intersection module annihilated by two linear forms (this allows one to write the total ideal and minimal resolutions for extremal curves [28, Prop. 0.5, 0.6 and Thm. 1.1]).

3. C is non-planar of degree d and contains a planar subcurve of degree $d-1$ ([11, §2, Thm. 8] or [32, Prop. 2.2]).

Assuming char $k = 0$, Ellia observed [11, §2, Prop. 9] that a curve which is neither planar nor extremal satisfies even stronger bounds on the Rao function. Using Schlesinger's *spectrum* of a curve [40], this bound was refined while removing the characteristic zero hypothesis [32, Thm. 2.11]:

Theorem 8. (Ellia and Nollet, 1997) *If $C \in H_{d,g}$ is a non-planar and non-extremal, then the Rao function $h^1\mathcal{I}_C(n)$ is bounded by the function depicted in Figure 1.2. In particular, $g \leq \binom{d-3}{2} + 1$.*

Figure 1.2: The bound of Theorem 8 on $h^1\mathcal{I}_C(n)$ for non-extremal curves

A curve $C \in H_{d,g}$ is *subextremal* if it achieves the bound of Theorem 8 for all n. A curve $C \in H_{d.g}$ is subextremal if and only if it is a height one elementary biliaison of an extremal curves $C' \in H_{d-2,g+3-d}$ on a quadric surface [32, Thm. 2.14] and hence exist for all $d \geq 4$ and $g \leq \binom{d-3}{2} + 1$:

letting $S \subset H_{d,g}$ denote the family of subextremal curves, the universal biliaison scheme of Martin-Deschamps and Perrin shows that S is irreducible. Indeed, if $E \subset H_{d-2,g+3-d}$ is the extremal component, we can consider the set B of triples (C, C', Q) for which C is a height one biliaison of C' on the quadric surface Q. The natural projections

$$\begin{array}{ccc} B & \xrightarrow{p_1} & S \\ p_2 \downarrow & & \\ E & & \end{array}$$

are smooth and irreducible [27, VII, §4], hence irreducibility of E implies irreducibility of S.

Remark 3. Given Theorem 5 and Theorem 8, one might expect that curves which are neither planar nor extremal nor subextremal should satisfy even stronger bounds. This fails, however: there are curves which give equality in Theorem 8 for some values of n, but not others [32, Ex. 2.15 and 2.17].

Remark 4. As the extremal curves form an irreducible component, one might expect that the closure of the subextremal curves $S \subset H_{d,g}$ to form an irreducible component as well (though S itself is not closed: its closure contains extremal curves [34]). Uwe Nagel has informed me that this is indeed true and is current joint work between he, Nadia Chiarli and Silvio Greco.

Figure 1.3: The geography for locally Cohen-Macaulay curves

3. Connectedness Results

In this section we summarize the current state of Problem 1. We will begin with some general results about families of curve that can be deformed to extremal curves and then proceed to particular ranges. In terms of the geography of Cohen-Macaulay curves (Figure 1.3), we will see in Theorems 10 and 11 that $H_{d,g}$ is connected for pairs (d, g) near the boundaries at the top and to the left.

Many families of curves can be deformed to extremal curves (without passing through curves with embedded points).

THEOREM 9. *The following families of curves can be deformed in $H_{d,g}$ to extremal curves.*

(1) Disjoint unions of lines

(2) Smooth rational curves

(3) Smooth connected curves with $d \geq g + 3$

(4) ACM curves

(5) The disjoint union of an extremal curve and a line

(6) The union of an extremal curve and a line meeting at a point

(7) Any curve in the liaison class of an extremal curve

Proof. (1)-(6) are results of Hartshorne [22] and (7) is due to Perrin [37]. □

When the arithmetic genus g is large relative to the degree d, the Hilbert scheme $H_{d,g}$ has few irreducible components, making it relatively easy to check connectedness. The following result is the work of several authors.

THEOREM 10. *If $g \geq \binom{d-3}{2} - 1$, then $H_{d,g}$ is connected.*

Proof. According to Theorem 6, either $g = \binom{d-1}{2}$ (in which case $H_{d,g}$ is the irreducible family of plane curves) or $g \leq \binom{d-2}{2}$. In the range $\binom{d-3}{2} + 1 < g \leq \binom{d-2}{2}$, Theorem 8 shows that $H_{d,g} = E$ is the family of extremal curves, which is irreducible by the work of Martin-Deschamps and Perrin [28].

There are three more arithmetic genre to check, but things become more delicate, as $H_{d,g}$ is not irreducible.

If $g = \binom{d-3}{2} + 1$, then Theorem 8 shows that each curve $C \in H_{d,g}$ is extremal or ACM, since the bound on $h^1\mathcal{I}_C(n)$ is zero. Conversely each ACM curve in $H_{d,g}$ is subextremal by definition, hence $H_{d,g} = E \cup S$ consists only of extremal and subextremal curves. Finally $E \cap \overline{S} \neq \emptyset$ by [34] and $H_{d,g}$ is connected.

If $g = \binom{d-3}{2}$, then the non-extremal curves C satisfy $h^1\mathcal{I}_C(n) \leq 1$. Samir Aït-Amrane showed [1] that $H_{d,g}$ has three irreducible components for large d: (a) extremal curves, (b) subextremal curves and (c) bilinks of height one from a double line of genus -1 on a surface of degree $d-2$. Both families (b) and (c) specialize to family (a) by Theorem 9 (7), but Samir's method was to use the triads developed by Hartshorne, Martin-Deschamps and Perrin [23].

If $g = \binom{d-3}{2} - 1$, then the non-extremal curves C satisfy $h^1\mathcal{I}_C(n) \leq 2$. Irene Sabadini showed [39] that $H_{d,g}$ has 4 irreducible components for $d \geq 9$: (a) extremal curves, (b) subextremal curves, (c) bilinks of height one from a double line of genus -2 on a surface of degree $d-2$ and (d) disjoint unions of an ACM extremal curve of degree $d-1$ and a line. Families (b) and (c) specialize to (a) by Theorem 9(7) and family (d) specializes to (a) by Theorem 9(5). □

THEOREM 11. *For $d \leq 4$, $H_{d,g}$ is connected whenever it is non-empty.*

Proof. Since $H_{d,g}$ is irreducible for $g = \binom{d-1}{2}$, we may assume that $d \geq 2$ and $g \leq \binom{d-2}{2}$ by Theorem 6. There are just three cases to consider.

If $d = 2$, then $H_{2,g}$ consists only of double lines, which were classified by Migliore [30]. These form an irreducible family.

If $d = 3$, then $H_{3,g}$ has exactly $\lceil \frac{4-g}{3} \rceil$ irreducible components, most consisting only of triple lines. In this case there are curves which lie in the intersection of all the irreducible components [33, Prop. 3.6 and Remark 3.9], hence $H_{3,g}$ is connected.

Finally if $d = 4$, then $H_{4,g}$ has roughly $\frac{g^2}{24}$ irreducible components, most of the families consisting of 4-lines (there are roughly $\frac{-3g}{2}$ families whose general member is not supported on a line). In work of the author and Enrico Schlesinger [36], these components were classified and connectedness was established through a variety of methods (see next two sections). One new feature to this example is the existence of an irreducible component which does *not* intersection the extremal component: the general curve is a multiplicity four structure on a line which has generic embedding dimension three. □

Looking at the number of irreducible components of the Hilbert schemes, one might guess that $H_{d,g}$ has on the order of g^{d-2} irreducible components, at least for $g << 0$. For degrees $d = 2$ and $d = 3$, the reason for the large number of components is the number of different families of multiplicity structures on a line. Will this behavior persist for larger d? At the other edge, it there are few components for $g \sim \binom{d-3}{2}$. Can one find an upper bound on the number?

PROBLEM 2. *How many irreducible components does $H_{d,g}$ have?*

(a) For $g << 0$? Is it of order g^{d-2}? Can one show this is a lower asymptotic bound?

(b) For g near $\binom{d-3}{2}$? Can one find an upper bound?

4. The Hilbert scheme $H_{4,-99}$

In this section we fully describe an example, the Hilbert scheme $H_{4,-99}$. We list the irreducible components and their dimensions, as well as describing the general curve in the corresponding family. Complete proofs for general arithmetic genus g can be found in [36].

Table 1.1: The 529 Irreducible Components of $H_{4,-99}$

Label	General Curve	Dimension
G_1 Extremal curves	$D \cup Z$ D smooth conic $p_a(Z) = -102$, length$(D \cap Z) = 4$	213
G_2 Subextremal curves	$L_1 \cup_{2P} Z \cup_{2Q} L_2$ $L_1 \cap L_2 = \emptyset$ $p_a(Z) = -101$	211
G_3	$D \cup_{2P} Z$ D smooth conic $p_a(Z) = -100$	211
G_4	thick 4-line	306
G_5	double conic	211
G_6	$Z \cup_{2P} L_1 \dot{\cup} L_2$ $p_a(Z) = -99$	209
$G_{7,a}$ $1 \leq a \leq 33$	$W \cup_{3P} L$ W quasiprimitive 3-line of type $(a, 99 - 3a)$	$209 - a$
$G_{8,a}$ $1 \leq a \leq 32$	$W \cup_{2P} L$ W quasiprimitive 3-line of type $(a, 98 - 3a)$	$208 - a$
$G_{9,a}$ $1 \leq a \leq 32$	$W \dot{\cup} L$ W quasiprimitive 3-line of type $(a, 96 - 3a)$	$206 - a$
$G_{10,m}$ $0 \leq m \leq 49$	$Z_1 \dot{\cup} Z_2$ $p_a(Z_1) = -m$ $p_a(Z_2) = m - 98$	$206 + \epsilon(m)$
$G_{11,a,b}$ $1 \leq a \leq 16$ $0 \leq b \leq 48 - 3a$	Quasiprimitive 4-line of type $(a, b, c = 96 - 6a - b)$	$205 - 3a$

Remark 5. The following refer to Table 1.1.

(a) Notation: L always denotes a line, D a smooth conic, Z a curve of degree two with given genus, and W a triple line.

(b) In family $G_{10,m}$, we set $\epsilon(m) = 0$ for $m > 1$, $\epsilon(1) = 1$ and $\epsilon(0) = 3$.

(c) Most of the families consist of multiplicity structures on a line.

(1) The *thick* 4-lines that occur in family G_4 are curves C with linear support L such that $\mathcal{I}_C \subset \mathcal{I}_L^2$ (they contain $L^{(1)}$).

(2) A multiplicity structure C of degree k on a line L which is not thick is called *quasi-primitive* [2] and has a Cohen-Macaulay filtration

$$L \subset Z_2 \subset Z_3 \subset \cdots \subset Z_k = C$$

with quotients $\mathcal{I}_L/\mathcal{I}_{Z_2} \cong \mathcal{O}_L(a)$, $\mathcal{I}_{Z_2}/\mathcal{I}_{Z_3} \cong \mathcal{O}_L(2a+b)$ and (if necessary) $\mathcal{I}_{Z_4}/\mathcal{I}_{Z_3} \cong \mathcal{O}_L(3a+c)$ with $b \leq c$. The numbers a, b and c give the *type* of the multiple line: thus a double line has type a, a triple line has type (a, b) and a quadruple line has type (a, b, c). We do not give the type for double lines, because the type is determined by the genus.

(d) The last five families listed come with parameters, meaning that there are several irreducible components. For example, there are actually 32 irreducible families of curves of $G_{9,a}$ (each consists of a disjoint union of a triple line and a reduced line), one for each $1 \leq a \leq 32$. Similarly there are 33 families of type $G_{7,a}$, 32 of type $G_{8,a}$, 50 of type $G_{10,m}$ and 376 of type $G_{11,a,b}$, for a total of 529 irreducible components.

(e) We prove connectedness by the following plan:

$$\begin{array}{ccccccccc}
G_{9,a} & & G_{10,0}/G_{10,1} & & G_6 & & G_3 & & \\
 & \searrow & \downarrow & \swarrow & & & \downarrow & & \\
G_{8,a} & \rightarrow & E = G_1 & \leftarrow & G_2 & \rightarrow & G_4 & \leftarrow & G_{11,a,b} \\
 & \nearrow & \uparrow & & & & \uparrow & & \\
G_{7,a} & & G_5 & & & & G_{11,0,m-1} & \subset & G_{10,m>1}
\end{array}$$

Each arrow represents a specialization of curves. The extremal component G_1 draws several arrows. The arrows $G_6 \to G_1, G_{8,a} \to G_1, G_{9,a} \to G_1$ and $G_{10,0}/G_{10,1} \to G_1$ follow from Theorem 9, parts (5) and (6) and results in [33]. The arrows $G_2 \to G_1$ and $G_5 \to G_1$ can be found in [25], as the relevant curves lie on a double plane. The arrow $G_{7,a} \to G_1$ is obtained by actually writing down equations of the deformation. The arrows $G_2 \to G_4$ and $G_3 \to G_4$ arise by varying a resolution for the Rao module [36, Prop. 4.2 and 4.3], while the arrow $G_{11,a,b} \to G_4$ arises by a tricky deformation of a resolution for the ideals, using the Buchsbaum-Eisenbud criterion [4] to check exactness [36, Prop. 2.4].

Finally, the curves in $G_{10,m}$ with $m > 1$ consist of disjoint unions of double lines of genus < -1. As the support of these curves lies on a smooth quadric, the curves themselves lie on a double quadric. On this surface we were able to deform these curves to a quasi-primitive 4-line in $G_{11,0,m-1}$ on a fixed double quadric: we explain this in the next section. The quasi-primitive 4-lines deform to G_4 as in arrow $G_{11,a,b} \to G_4$.

5. Curves on the double quadric

Hartshorne and Schlesinger gave a satisfying classification of curves lying on the double plane [25], describing all the irreducible components and showing connectedness. Their primary tool was a certain triple associated to such a curve (Definition 1 below). In this section we describe joint work of Enrico Schlesinger and the author [35], which uses these triples on a double surface to give a criterion for when the underlying triple of a curve can be spread out in a flat family. As an application we obtain in Example 3 (a) the inclusion

$$G_{11,0,m-1} \subset G_{10,m} \tag{3}$$

needed to show connectedness of $H_{4,-99}$ (see Remark 5 (e)).

To set the scene, let F be a smooth surface on a smooth threefold T with doubling $X = 2F$. More generally one can take X to be a *ribbon* over F in the sense of Eisenbud and Bayer [10].

Definition 1. *For each curve $C \subset X$, the triple $T(C) = \{Z, R, P\}$ is defined as follows:*

1. *P is the support of C, the one dimensional part of $C \cap F$.*
2. *R is the curve part of C residual to P.*
3. *Z is the zero-dimensional part of $C \cap F$, so $\mathcal{I}_{C\cap F,F} \cong \mathcal{I}_{Z,F}(-P)$*

Remark 6. If $T(C) = \{Z, R, P\}$, then $Z \subset R$ is zero-dimensional and Gorenstein [35, Prop. 2.1] and $R \subset P$ are divisors on F. The arithmetic genus is given by

$$p_a(C) = p_a(P) + p_a(R) + \deg_R \mathcal{O}_R(F) - \deg Z - 1 \tag{4}$$

Example 2. We show below that both families of curves involved in inclusion (3) lie on a double quadric in $\mathbb{P}^3$ and compute their triples.

(a) A curve C in the family $G_{10,m}$ is a disjoint union $C = D_1 \cup D_2$ of double lines of genera $-m$ and $m - 98$. The support $L_1 \cup L_2$ being contained in a 3-dimensional family of smooth quadrics, we can choose such a quadric Q containing neither D_1 nor D_2. Then C lies on the double quadric $X = 2Q$ and

$$T(C) = \{Z_1 \cup Z_2, L_1 \cup L_2, L_1 \cup L_2\}$$

where $Z_1 \subset L_1$ has length $m + 1$ and $Z_2 \subset L_2$ as length $99 - m \geq m + 1$ by formula (4). For C general, Z_i can be taken to be reduced sets of points.

(b) A curve C in the family $G_{11,0,m-1}$ is a quasi-primitive 4-line supported on L of type $(0, m-1, 97-m)$ (see Remark 5 (c)) and has underlying double line of type 0 and hence genus -1. Such a double line necessarily lies on a smooth quadric surface Q [33, Remark 1.5], hence C itself lies on the double quadric $X = 2Q$. It takes some work [36, Prop. 3.1], but one finds that

$$T(C) = \{Z, 2L, 2L\},$$

where $2L$ is the double line on Q and Z consists of $98 - 2m$ reduced points and $m + 1$ double points on $2L$, none of which are contained in L .

Remark 7. Looking at the triples in Example 2, we note that triple in part (b) is a limit of the triples in part (a): The two lines L_1 and L_2 come together on Q to form the double line $2L$, and the sets of reduced points Z_1 and Z_2 can be brought together in this limit to form $m+1$ double points and $98-2m$ reduced points. If we could lift this flat family of triples to a flat family of curves on $X = 2Q$, we would have proved the inclusion (3).

Thus we consider the map $C \mapsto T(C) = \{Z, R, P\}$, which yields a natural transformation of functors

$$H \xrightarrow{t} D$$

where H is the set of flat families of curves on $X = 2F$ and D is the set of triples $\{Z, R, P\}$. The functor D is represented by a disjoint union of locally closed subschemes $D_{z,r,p}$, where $\{z, r, p\}$ are the respective Hilbert polynomials of the entries in the triple $\{Z, R, P\}$. The pre-images under t stratify the Hilbert scheme H into locally closed subschemes $H_{z,r,p}$. The map t has a nice structure over the locus of the triples in D given by a vanishing [35, Thm. 3.2]:

Theorem 12. (Nollet and Schlesinger, 2003) *Let $V \subset D_{z,r,p}$ be the open subscheme corresponding to triples $\{Z, R, P\}$ satisfying $H^1(\mathcal{O}_R(Z+P-F)) = 0$. Then the map $t^{-1}(V) \to V$ is the composition of an open immersion and an affine bundle projection. In particular, if $Y \subset V$ is irreducible, then $t^{-1}(Y)$ is also irreducible (hence connected).*

Example 3. Here are two applications of Theorem 12.

(a) In view of Remark 7, Theorem 12 will prove the inclusion (3) if the vanishing $H^1(\mathcal{O}_R(Z+P-Q)) = 0$ holds for both the triples in Example 2. This is easy for the triples in (a): writing $R = L_1 \cup L_2$ the vanishing boils down to $H^1(\mathcal{O}_{L_i}(Z_i + 1 - 2)) = 0$ for $i = 1, 2$, which is immediate because $\deg Z_i \geq 0$. The vanishing for family (b) uses the Cohen-Macaulay filtration (Remark 5 (c)) for the 4-line C [36, Prop. 3.1].

(b) Some of the deformations used in showing the connectedness of $H_{3,g}$ follow from Theorem 12, for example [33, Prop. 3.3].

We close this section with some open questions involving the fibres of the map $t : H \to D$. Given a triple $T = \{Z, R, P\} \in D$ on F, the fibre $t^{-1}(T)$ is the set of locally Cohen-Macaulay curves $C \subset X$ with $T(C) = T$ (there may be none). There is a bijection between such curves C and surjections

$\phi : \mathcal{I}_P \otimes \mathcal{O}_R \to \mathcal{O}_R(Z - F)$ such that $\phi \circ \tau = \sigma$, where

$$\tau = (\mathcal{O}(-F) \hookrightarrow \mathcal{I}_P) \otimes \mathcal{O}_R \qquad \sigma = (\mathcal{O}_R(-Z) \hookrightarrow \mathcal{O}_R) \otimes \mathcal{O}_R(Z - F)$$

are the natural maps [35, Prop. 2.2], hence these maps can be identified with an open subset

$$U \subset \hom_R(\mathcal{O}_R(-P), \mathcal{O}_R(Z - F)) \cong H^0(\mathcal{O}_R(Z + P - F)).$$

PROBLEM 3. *Under what conditions is the open set U non-empty? When does a given triple $T = \{Z, R, P\}$ arise from a curve $C \subset X$?*

REMARK 8. Obviously a solution to Problem 3 will have applications to classifying non-reduced curves of low degree. Here are some partial results.

(a) For triple $T = \{Z, R, P\}$, the open subset U is non-empty if any of the following conditions hold [35, Remark 2.7 and Prop. 2.5]:

  (1) $H^1(\mathcal{O}_R(Z + P - F)) = 0$ and $\mathcal{O}_R(Z + P - F)$ is generated by global sections.
  (2) $H^1(\mathcal{O}_R(Z + P - F - H)) = 0$ for a very ample divisor H on R.
  (3) $H^1(\mathcal{O}_R(P - F)) = 0$.

(b) For the double plane $X = 2H \subset \mathbb{P}^3$, the subset U is non-empty for any triple, because condition (3) above holds. Chiarli, Greco and Nagel have described the curves with fixed triple using a matrix of homogeneous polynomials over H, giving a certain "normal form" to such curves C [8].

(c) The double quadric $X = 2Q \subset \mathbb{P}^3$ is more interesting [35, Ex.2.8]. Let $T = \{Z, R, P\}$ be a triple with Z Gorenstein of dimension zero.

  (1) If $R = P$ is a smooth rational curve, then T arises from a curve with one exception: $R = P$ is a conic and Z is a reduced point.

  (2) If P is ample on Q and $R \neq P$, then T arises from a curve.

  (3) If $R \subset P$ are disjoint unions of rulings on Q, then T arises from a curve if and only if $Z \cap L \neq \emptyset$ for each ruling $L \subset R$.

PROBLEM 4. *Answer the question implicit in part (c) above: Which triples on a smooth quadric in $\mathbb{P}^3$ come from a curve on the double quadric? Describe the Hilbert schemes $H_{d,g}(2Q)$.*

PROBLEM 5. (Hartshorne) *Which curves on a double surface $2F \subset \mathbb{P}^3$ are flat limits of curves on smooth surfaces? For example, the thick triple line $L^{(2)}$ on the double plane $2H$ is a flat limit of twisted cubic curves lying on smooth quadric surfaces. What is special about the curve $L^{(2)}$ or its triple $\{\emptyset, L, 2L\}$ that allow it to be such a limit?*

6. Subcanonical curves

In view of Paolo Valabrega's research interests [5, 6, 13, 41], we thought it would be interesting to determine which families of curves in $H_{4,-99}$ are subcanonical. A local complete intersection curve C is α-subcanonical if $\omega_C \cong \mathcal{O}_C(\alpha)$. The following restricts our attention to just a few families in $H_{4,-99}$.

PROPOSITION 1. *Suppose that $C \in H_{4,-99}$ is subcanonical. Then*

(1) $\omega_C \cong \mathcal{O}_C(-50)$.

(2) C has no smooth rational irreducible components.

(3) C is one of the following:

(a) A double conic.
(b) A union of two double lines.
(c) A quasi-primitive 4-line.

Proof. An α-subcanonical of degree d and genus g satisfies $d\alpha = 2g - 2$ in general, hence $\alpha = -50$ in our case.

Suppose that C has a smooth rational component R. Then $\deg R \neq 4$ because then $C = R$ has genus $0 \neq -99$. Also $\deg R \neq 3$ because then $C = R \cup L$ (L a line) forces $p_a(C) = \deg(R \cap L) - 1 \geq -1$ is not equal to -99. Thus R is a line or a conic. We write $C = S \cup R$ and restrict the exact sequence

$$0 \to \omega_S \oplus \omega_R \to \omega_C \to \omega_{S \cap R} \to 0$$

to R. Using $\omega_C = \mathcal{O}_C(-50)$ we obtain

$$\omega_S|_R \oplus \omega_R \xrightarrow{\phi} \mathcal{O}_R(-50) \to \omega_{S \cap R} \to 0.$$

Now the sheaf $\omega_S|_R$ is torsion and ω_R is either isomorphic $\omega_R = \mathcal{O}_R(-2)$ (if R is a line) or $\mathcal{O}_R(-1)$ (if R is a conic), hence ϕ is the zero map. This proves (2) by contradiction, since the cokernel of ϕ is finitely supported.

Let $B = \operatorname{Supp} C$. Then $\deg B < 4$ (since $g < -3$) and $\deg B \neq 3$ (since then C consists of a double line and a reduced curve of degree two). Thus $\deg B = 2$ or 1 and C is either (a) a double conic, (b) a union of two double lines or (c) a multiple line by part (2). If C were a thick 4-line supported on L, then it contains the triple line with ideal $\mathcal{I}_L^2$, which has degree 3 and genus 0 (a degenerate twisted cubic curve). According to [36, Lem. 4.1], C has spectrum

$$\{-98, 0, 1^2\},$$

which is a shorthand way of saying that the function $h_C(n) = \Delta^2 h^0 \mathcal{O}_C(n)$ satisfies $h_C(-98) = 1$, $h_C(0) = 1$, $h_C(1) = 2$ and $h_C(n) = 0$ otherwise. Such a curve C cannot satisfy $\omega_C = \mathcal{O}_C(-50)$, for in this case it would not satisfy the symmetry $h_C(n) = h_C(-50 + 2 - n))$ [40, Prop. 2.15]. $\square$

Proposition 2. *There are* 18 *irreducible components of* $H_{4,-99}$ *whose general member is* (-50)*-subcanonical, as listed in Table 1.2.*

Table 1.2: Irreducible families of subcanonical curves in $H_{4,-99}$

Label from Table 1.1	Dimension	Spectrum
G_5 Double conics	211	$\{-49, -48, 0, 1\}$
$G_{10,49}$ Disjoint union of two double lines of genus -49	206	$\{-48^2, 0^2\}$
$G_{11,a,48-3a}$ for $0 < a \leq 16$ Quasi-primitive 4-line	$205 - 3a$	$\{-48, -48 + a, -a, 0\}$

Proof. By Proposition 1 we need only consider (a) double conics, (b) unions of double lines, and (c) quasi-primitive 4-lines. The double conics are automatically subcanonical, for if D is the support of a double conic C, then the Cohen-Macaulay filtration is

$$0 \to \mathcal{I}_C \to \mathcal{I}_D \to \mathcal{O}_D(49) \to 0.$$

Noting that $\mathcal{O}_D(49) = \omega_D(50)$, we see that C arises by the Ferrand construction [12] and hence is subcanonical.

Next consider a union $C = D_1 \cup D_2$ of double lines. If C is connected, then the support is planar and C is contained in the double plane. It follows that C is a limit of double conics by Theorem 12 or [25, Thm. 5.1], so we need only consider disjoint unions of double lines. Since a double line of genus g is $(g-1)$-subcanonical, a disjoint union of such can only be subcanonical if the double lines have the same genus, which in this case must be -49.

Now let C be a quasi-primitive 4-lines of type (a, b, c) with $0 < a \leq 16$, $0 \leq b \leq 48 - 3a$ and $c = 96 - 6a - b$. This means that there are locally Cohen-Macaulay curves $L \subset D \subset W \subset C$ with quotients $\mathcal{I}_L/\mathcal{I}_D \cong \mathcal{O}_L(a)$, $\mathcal{I}_D/\mathcal{I}_W \cong \mathcal{O}_L(2a + b)$ and $\mathcal{I}_W/\mathcal{I}_C \cong \mathcal{O}_L(3a + c)$ (see Remark 5 (c)). Piecing together the exact sequences and using $a > 0$, the spectrum of C is

$$\{-3a - c, -2a - b, -a, 0\}.$$

To be (-50)-subcanonical, this sequence of integers must be symmetric about -24 [40, Prop. 2.15], which forces $b = 48 - 3a$ and $c = 48 - 3a$. It now suffices to show that the general 4-line C of type $(a, 48 - 3a, 48 - 3a)$ is subcanonical.

The exact sequence

$$0 \to \mathcal{I}_D \to \mathcal{I}_L \to \mathcal{O}_L(a) \to 0 \tag{5}$$

shows that the underlying double line $D \subset C$ arises from the Ferrand construction and is $(-a-2)$-subcanonical, since $\mathcal{O}_L(a) \cong \omega_L(a + 2)$. In fact, D is a divisor on a smooth surface $S \subset \mathbb{P}^3$ of degree $a+2$ by [33, Rmk. 1.5]. In view of the isomorphisms $\mathcal{I}_S \cong \mathcal{O}_{\mathbb{P}^3}(-a-2)$ and $\mathcal{I}_{D,S} \otimes \mathcal{O}_D = \mathcal{O}_S(-D) \otimes \mathcal{O}_D \cong \omega_S \otimes \omega_D^{-1}$ with $\omega_S \cong \mathcal{O}_S(a-4)$ and $\omega_D \cong \mathcal{O}_D(-a-2)$, restricting the exact sequence

$$0 \to \mathcal{I}_S \to \mathcal{I}_D \to \mathcal{I}_{D,S} \to 0$$

to D yields

$$\mathcal{O}_D(-a-2) \xrightarrow{\tau} \mathcal{N}_D^\vee \xrightarrow{\pi} \mathcal{O}_D(2a-2) \to 0. \tag{6}$$

Since π is a surjection of bundles on D, the kernel is a line bundle on D. Since any surjection of line bundles is an isomorphism, τ is injective and sequence (6) is short exact.

Exact sequence (5) shows that $h^0\mathcal{O}_D(m) = h^1\mathcal{I}_D(m) = 0$ for $m < -a$, hence sequence (6) yields the vanishing $H^1(\mathcal{N}_D \otimes \omega_D(m)) \perp H^0\mathcal{N}_D^\vee(-m) = 0$ for $m > 3a-2$. Therefore $\mathcal{N}_D \otimes \omega_D$ is $(3a)$-regular and so $\mathcal{N}_D \otimes \omega_D(n)$ is generated by global sections for $n \geq 3a$ by the Castelnuovo-Mumford theorem. Since $a \leq 16$, we have in particular that $\mathcal{N}_D \otimes \omega_D(50)$ is generated by global sections and we obtain a nowhere vanishing section yielding a surjection $\mathcal{I}_D \to \mathcal{N}_D^\vee \to \omega_D(50)$ whose kernel $\mathcal{I}_C$ is the ideal sheaf for a (-50)-subcanonical curve C by Ferrand's construction. Clearly C is supported on L and the sequence

$$0 \to \omega_D(50) \to \mathcal{O}_C \to \mathcal{O}_D \to 0$$

shows that the spectrum of C is $\{-48, -48+a, -a, 0\}$, so C is quasi-primitive of type $(a, 48-3a, 48-3a)$. For curves with fixed spectrum, the property of being subcanonical is open and we conclude. □

Remark 9. Chiantini and Valabrega have given equations of such curves [5, Examples 3.1 and 3.2]. For $m, n, u > 0$ and $p \geq \max\{m, n\}$, they observe that the curve V with homogeneous ideal

$$I_V = ((x^n, y^m)^u, z^{p-n}x^n - w^{p-m}y^m = \varphi)$$

is $((1-u)p + (m+n)u - 4)$-subcanonical. Setting $4 = \deg V = mnu$, we find just a few possibilities. When $u = 1$ we obtain plane curves $(m = 4, n = 1)$ and complete intersections of two quadrics $(m = n = 2)$. More interesting are these:

(a) $m = 2$, $n = 1$ and $u = 2$. To obtain a (-50)-subcanonical curve we take $p = \deg \varphi = 52$. This is a quasi-primitive 4-line of type $(-1, 50, 52)$. It does not appear in Table 1.1 because such 4-lines are limits of double conics. This one is a Ferrand doubling of the plane curve with ideal (x, y^2).

(b) $m = n = 1$ and $u = 4$. To obtain a (-50)-subcanonical curve we take $p = \deg \varphi = 54$. This curve is a quasi-primitive 4-line of type $(16, 0, 0)$.

Remark 10. Here we make a list of the families of subcanonical curves of degree four. There are none when g is even. For $g = 3$ there are plane curves and for $g = 1$ there are complete intersections of two quadrics. For odd $g < 0$ we have:

1. Double conics.
2. Disjoint unions of two lines of genus $\frac{g+1}{2}$
3. Quasi-primitive 4-lines of type $(a, \frac{-g+3-6a}{2}, \frac{-g+3-6a}{2})$ for $0 < a \leq \frac{-g-3}{6}$ (this last family is empty for $g > -9$, as no such a exist).

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