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PICARD GROUPS OF NORMAL SURFACES

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Abstract. We study the fixed singularities imposed on members of a linear system of surfaces in $\mathbb{P}^3_{\mathbb{C}}$ by its base locus Z. For a 1-dimensional subscheme $Z \subset \mathbb{P}^3$ with finitely many points p_i of embedding dimension three and $d \gg 0$, we determine the nature of the singularities $p_i \in S$ for general $S \in |H^0(\mathbb{P}^3, I_Z(d))|$ and give a method to compute the kernel of the restriction map $\operatorname{Cl} S \to \operatorname{Cl} \mathcal{O}_{S,p_i}$. One tool developed is an algorithm to identify the type of an $\mathbf{A}_n$ singularity via its local equation. We illustrate the method for representative Z and use Noether-Lefschetz theory to compute $\operatorname{Pic} S$.

1. Introduction

The problem of computing Picard groups of surfaces $S \subset \mathbb{P}^3_{\mathbb{C}}$ has a long history. The solution for smooth quadric and cubic surfaces was known in the 1800s in terms of lines on these surfaces. In the 1880s Noether suggested what happens in higher degree, but it was not until the 1920s that Lefschetz proved the famous theorem bearing their names: the very general surface S of degree $d > 3$ has Picard group $\operatorname{Pic} S \cong \mathbb{Z}$, generated by the hyperplane section H. Here *very general* refers to a countable intersection of nonempty Zariski open subsets. To produce typical families of surfaces S with $\operatorname{Pic} S$ not generated by H, Lopez proved that very general surfaces S of high degree containing a smooth connected curve Z have Picard group freely generated by H and Z [15, II, Thm. 3.1], a geometrically pleasing result with many applications [4, 5, 6, 7].

Recently we extended these results, proving that the class group $\operatorname{Cl} S$ of the very general surface S containing an arbitrary 1-dimensional subscheme Z with at most finitely many points of embedding dimension three [1] is freely generated by H and the supports of the irreducible curve components of Z [2, Thm. 1.1]. This allows access to the Picard group via the exact sequence of Jaffe [11, Prop. 3.2] (see also [9, Prop. 2.15])

$$0 \to \operatorname{Pic} S \to \operatorname{Cl} S \to \bigoplus_{p \in \operatorname{Sing} S} \operatorname{Cl} \mathcal{O}_{S,p} \tag{1}$$

provided we can find the kernels of the restriction maps $\operatorname{Cl} S \to \operatorname{Cl} \mathcal{O}_{S,p}$ at the singular points $p \in S$, where $\operatorname{Cl} \mathcal{O}_{S,p}$ is the divisor class group of the local ring. The answer being known at singular points of S where Z has embedding dimension ≤ 2 [2, Prop. 2.2], our motivating question becomes:

Problem 1.1. For $Z \subset \mathbb{P}^3$ and $p \in Z$ a point of embedding dimension three, find the kernel of the restriction map $\operatorname{Cl} S \to \operatorname{Cl} \mathcal{O}_{S,p}$.

A general solution to Problem 1.1 is out of reach because one would need to classify all embedding dimension three points p on curves Z to state an answer. Instead we give a method of attack on the problem:

Method 1.2. The kernel of the restriction $\operatorname{Cl} S \to \operatorname{Cl} \mathcal{O}_{S,p}$ can be computed as follows.

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[1]This is the weakest condition allowing S to be a normal surface, so that $\operatorname{Cl} S$ is defined.

Step 1. The natural map $\operatorname{Cl}\mathcal{O}_{S,p} \to \operatorname{Cl}\widehat{\mathcal{O}}_{S,p}$ being injective, we consider the composite map $\operatorname{Cl} S \to \operatorname{Cl}\mathcal{O}_{S,p} \hookrightarrow \operatorname{Cl}\widehat{\mathcal{O}}_{S,p}$, where power series tools are available.

Step 2. Working in $\widehat{\mathcal{O}}_{S,p}$, use analytic coordinate changes to recognize the form of the singularity and compute the local class group $\operatorname{Cl}\widehat{\mathcal{O}}_{S,p}$ when possible.

Step 3. Since $\operatorname{Cl} S$ is freely generated by H and the supports of the curve components of Z [2, Thm 1.1], it is enough to find the images of those supports for irreducible curve components of Z passing through p in $\operatorname{Cl}\widehat{\mathcal{O}}_{S,p}$ (the rest map to zero).

Remark 1.3. Method 1.2 can *always* be carried out if Z is locally contained in two smooth surfaces meeting transversely at p. This is because the analytic local equation of S at p contains an xy term and we can employ our *recognition theorem*: Theorem 2.6 gives an inductive algorithm that recognizes the type of an $\mathbf{A}_n$ singularity in at most n steps, but finishes in just 1 step with probability 1. While most of this paper is devoted to illustrations of Method 1.2, Theorem 2.6 may be the most useful general result presented here.

Remark 1.4. Regarding Step 3, the images of the supports of the curve components of Z containing p generate $\operatorname{Cl} O_{S,p}$ as a subgroup of $\operatorname{Cl}\widehat{O}_{S,p}$ for very general S [3, Prop. 2.3]. This gives a geometric way to see the class group of a ring in its completion.

(a) In particular, the map $\operatorname{Cl} S \to \operatorname{Cl}\mathcal{O}_{S,p}$ is zero if p is an isolated point of Z, since Z has no curve components passing through p. Therefore only the 1-dimensional part of Z contributes to the answer.

(b) Srinivas has asked [20, Ques. 3.1] which subgroups appear as $\operatorname{Cl} B \subset \operatorname{Cl} A$ where B is a local $\mathbb{C}$-algebra with $A = \widehat{B}$. We proved that for complete local rings A corresponding to the rational double points $\mathbf{A}_n, \mathbf{D}_n, \mathbf{E}_6, \mathbf{E}_7, \mathbf{E}_8$, the answer is *every* subgroup [3, Thm. 1.3]. We constructed the rings B as the geometric local rings $\mathcal{O}_{S,p}$ arising from general surfaces $S \subset \mathbb{P}^3$ containing a fixed base locus forcing the singularity at p. This poses a stark contrast to results of Kumar [13], who showed that if B has fraction field $\mathbb{C}(x,y)$ and the singularity type is $\mathbf{E}_6, \mathbf{E}_7$ or $\mathbf{A}_n$ with $n \neq 7, 8$, then B is determined by A and hence $\operatorname{Cl} B = \operatorname{Cl} A$.

We illustrate Method 1.2 by giving complete answers for the following base loci Z:

(1) Unions of two multiplicity structures near p which are locally contained in smooth surfaces with distinct tangent spaces at p.
(2) Multiplicity structures on a smooth curve of multiplicity ≤ 4 near p.

Regarding organization, we review $\mathbf{A}_n$ singularities and their analytic equations in Section 2, proving the recognition theorem, Theorem 2.6. In Sections 3 - 4 we solve Problem 1.1 in the cases (1) and (2) listed above. Finally in Section 5 we prove Theorem 5.1, which shows how to compute $\operatorname{Pic} S$ and give examples.

2. Analytic expressions for rational double points

In this section we briefly review rational double points of type $\mathbf{A}_n$ and some results about analytic change of coordinates.

2.1. **$\mathbf{A}_n$ singularities.** An $\mathbf{A}_n$ surface singularity has local analytic equation $xy - z^{n+1}$, thus it is analytically isomorphic to $\operatorname{Spec}(R)$ with $R = k[[x,y,z]]/(xy - z^{n+1})$. The resolution of this singularity is well known [10, 5.2]: an $\mathbf{A}_1$ resolves in a single blow-up with one rational exceptional curve having self-intersection -2; an $\mathbf{A}_2$ resolves in one blow-up but with two (-2)-curves meeting at a point. For $n \geq 3$, blowing up with new variables $x_1 = x/z, y_1 = y/z$ gives two exceptional curves, namely E_{x_1} defined by (x_1, z) and E_{y_1} defined by (y_1, z), meeting transversely at an $\mathbf{A}_{n-2}$ at the origin. Blowing up and continuing inductively, the singularity unfolds and we obtain a resolution with exceptional divisors forming a chain of n rational (-2)-curves meeting pairwise transversely. We will adopt the convention, identifying a curve with its strict transform, that $E_1 = E_{x_1}, E_2 = E_{x_2}, \dots, E_n = E_{y_1}$.

To calculate $\operatorname{Cl} R$, identify a curve C with the sequence $(\tilde{C}.E_1, \tilde{C}.E_2 \dots)$ of intersection numbers of its strict transform with the exceptional curves. Then $\operatorname{Cl} R$ is the quotient of the free abelian group on the exceptional curves with relations given by the fact that the exceptional curves themselves correspond to the trivial class [14, §14 and §17]. Let (u_j) be the ordered basis for the free group; then the relations for an $\mathbf{A}_n$ singularity are

$$-2u_1 + u_2, u_1 - 2u_2 + u_3, \dots, u_{n-1} - 2u_n$$

so that $\operatorname{Cl} R \cong \mathbb{Z}/(n+1)\mathbb{Z}$ generated by u_1 and satisfying $u_j = ju_1$ for all j.

Example 2.1. Let $R = k[[x,y,z]]/(xy - z^{n+1})$ be the complete local ring of an $\mathbf{A}_n$ surface singularity. Under the identification $\operatorname{Cl}(R) \cong \mathbb{Z}/(n+1)\mathbb{Z}$, we identify the following classes:

(a) The class of the curve D_1 given by (x,z) is 1.
(b) The class of the curve D_2 given by (y,z) is -1.
(c) For $1 \le r \le n$, the class of the curve $(x - z^{n-r+1}, y - z^r)$ is r.

Parts (a) and (b) are contained in [10, Prop. 5.2] and part (c) is [10, Rem. 5.2.1], where the class considered is $(x - az^{n-r+1}, y - a^{-1}z^r)$ for a unit a. To save a change of coordinates at the end of a calculation, we will often apply part (c) to the curve $(x - uz^{n-r+1}, y - vz^r)$ in the ring $k[[x,y,z]]/(xy - uvz^{n+1})$ with u, v units.

2.2. **Analytic coordinate changes.** We will use coordinate changes in $k[[x,y,z]]$ to recognize the structure of surface singularities. Let $R = k[[x_1, \dots, x_n]]$ be the ring of formal powers series over a field k with maximal ideal $\mathfrak{m}$. A *change of variables* for R is an assignment $x_i \mapsto x_i' \in \mathfrak{m}$ inducing an automorphism of R. An assignment $x_i \mapsto x_i'$ induces an automorphism if and only if induced maps $\mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}/\mathfrak{m}^2$ is an isomorphism if and only if the matrix A of coefficients of linear terms in the x_i' is nonsingular (this is noted by Jaffe [12, Prop. 3.2] when $n = 2$). Examples include multiplication of variables by units and translations of variables by elements in $\mathfrak{m}^2$.

The following lemma allows us to take roots in power series rings.

Lemma 2.2. *Let $(R, \mathfrak{m})$ be a complete local domain, and let n be a positive integer that is a unit in R. If $a_0 \in R$ is a unit and $u \equiv a_0^n \bmod \mathfrak{m}^k$ for some fixed $k > 0$, then there exists $a \in R$ such that $a^n = u$ and $a \equiv a_0 \bmod \mathfrak{m}^k$.*

Proof. Using Hensel's method, we construct a sequence $\{a_i\}$ with $a_i \equiv a_{i+1} \bmod \mathfrak{m}^{(i+1)k}$ and $a_i^n \equiv u \bmod \mathfrak{m}^{(i+1)k}$. Write $u^{(i)} = u - a_i^n \in \mathfrak{m}^{(i+1)k}$ and let $a_{i+1} = a_i + \frac{u^{(i)}}{na_i^{n-1}}$; then we have $a_{i+1} \equiv a_i \bmod \mathrm{m}^{(i+1)k}$ and $a_{i+1}^n = a_i^n + u^{(i)} + \tilde{u}, \tilde{u} \in \mathfrak{m}^{2(i+1)k} \Rightarrow a_{i+1}^n \equiv u \bmod \mathfrak{m}^{(i+2)k}$. □

Lemma 2.3. *Let $R = k[[x,y]]$ with maximal ideal $\mathfrak{m} \subset R$. For $f \in \mathfrak{m}^3$, there is a change of coordinates X, Y such that*

$$xy + f = XY$$

and X, Y may be chosen so that $x \equiv X \bmod \mathfrak{m}^2$ and $y \equiv Y \bmod \mathfrak{m}^2$.

Proof. (*Cf.* [8, I, Ex. 5.6.3]) Since $f \in \mathfrak{m}^3$, we may write $f = xh_1 + yg_1$ with $h_1, g_1 \in \mathfrak{m}^2$. Now write $x_1 = x + g_1, y_1 = y + h_1$ so that $xy + f = x_1y_1 - g_1h_1$, where $g_1h_1 \in \mathfrak{m}^4$. Continue the process inductively, constructing a sequence of coordinate changes that converge to X, Y and for which $f = XY$. □

We frequently encounter the complete local rings considered in the following Proposition.

Proposition 2.4. *Let $R = k[[x,y,z]]/(xy - uyz^t - vx^n)$, where $u, v \in k[[x,y,z]]$ are units and $t \geq 1, n \geq 3$ are integers. Then $\operatorname{Spec} R$ is an $\mathbf{A}_{tn-1}$ singularity and the class of the curve (x,z) (resp. (x,y)) maps to 1 (resp. $tn-t$) via the isomorphism $\operatorname{Cl} R \cong \mathbb{Z}/tn\mathbb{Z}$.*

Proof. In setting $X = x - uz^t$, the expression $xy - uyz^t - vx^n$ becomes

$$(2)\qquad Xy - v(X+uz^t)^n = Xy - vX\underbrace{(X^{n-1} + nX^{n-2}uz^t + \cdots + nu^{n-1}z^{t\cdot(n-1)})}_{\alpha} - vu^n z^{tn}.$$

For α as shown, set $Y = y - v\alpha$ to obtain $XY - vu^n z^{tn}$. Absorbing vu^n into either X or Y brings the expression to the standard form for an $\mathbf{A}_{tn-1}$ singularity, hence we have $\operatorname{Cl} R \cong \mathbb{Z}/tn\mathbb{Z}$ from the previous section. Moreover $(x,z) = (X + uz^t, z) = (X, z)$ gives the canonical generator 1 in $\operatorname{Cl} R = \mathbb{Z}/tn\mathbb{Z}$ by Example 2.1 (a).

For the second curve, write $(x,y) = (X + uz^t, y) = (X + uz^t, Y + v\alpha)$. Then

$$(X+uz^t)^n = X\alpha + (uz^t)^n = (X+uz^t)\alpha - uz^t\alpha + (uz^t)^n$$

by definition of α so that $(uz^t)\alpha \equiv (uz^t)^n \bmod (X + uz^t)$. Since the quotient ring modulo $X + uz^t$ is an integral domain in which uz^t is nonzero, we see that $\alpha \equiv (uz^t)^{n-1} \bmod (X+uz^t)$ and therefore $(x,y) = (X+uz^t, Y + vu^{n-1}z^{tn-t})$ which corresponds to the class $tn - t \in \mathbb{Z}/tn\mathbb{Z}$ by Example 2.1 (c).

□

2.3. **Recognizing $\mathbf{A}_n$ Singularities.** We develop an algorithm to identify the $\mathbf{A}_n$-singularity type defined by a power series F in three variables defining a double point with nondegenerate tangent cone, so that the degree-2 part is not a square. In this case it is known [10, Thm. 4.5] that F defines an $\mathbf{A}_n$ singularity for some n or F factors, which we interpret as $n = \infty$. Our goal is to identify the answer by inspection if possible. Such F can be written

$$F = \sum_{i+j+k>1} c_{i,j,k} x^i y^j z^k \in \mathfrak{m}^2 \subset k[[x,y,z]]$$

with $c_{1,1,0} = 1, c_{2,0,0} = c_{0,2,0} = 0$.

Let A be the sum of all terms satisfying $i, j > 0$ and $i + j + k > 2$. Then $A = xyB$ with $B \in \mathfrak{m}$ and the remaining terms of F fall into three categories: (a) $i = j = 0$, which we write as $h(z) \in k[[z]]$, (b) $i = 0, j > 0$, which we can write as $\sum_{j=1} y^j g_j(z)$ with $g_j \in k[[z]]$ and (c) $j = 0, i > 0$ which can be written as $\sum_{i=1} x^i f_i(z)$ with $f_i \in k[[z]]$. With these choices F becomes

$$(3)\qquad F = xy + h(z) + \sum_{i=1}^{\infty} x^i f_i + \sum_{j=1}^{\infty} y^j g_j + xyB$$

where $h, f_i, g_j \in k[[z]]$, $B \in \mathfrak{m}$ and $\operatorname{ord} g_2 > 0$. Letting $u = 1 + B$ we can drop the last term at the expense of multiplying the xy term by the unit u: now let $X = ux$ and replace f_i with $(u^{-1})^i f_i$ to obtain

$$(4)\qquad F = xy + h(z) + \sum_{i=1}^{\infty} x^i f_i + \sum_{j=1}^{\infty} y^j g_j$$

with $g_j \in k[[z]]$ and $f_i(z) = z^{r_i} u_i$ with u_i a unit. To determine the singularity type, we may assume $r_1 < \infty$ or $\operatorname{ord} g_1 < \infty$, since $f_1 = g_1 = 0$ gives an $\mathbf{A}_{h-1}$ with $h = \operatorname{ord} h(z)$. We make one more simplification. Set $X = x + g_1$ to obtain

$$F = Xy + h(z) + \sum_{i=1}^{\infty} (X - g_1)^i f_i + \sum_{j=2}^{\infty} y^j g_j.$$

Regrouping the f_i by powers of X after expanding the powers of $(X - g_1)$ we arrive at

$$F = xy + h(z) + \sum_{i=1}^{\infty} x^i f_i + \sum_{j=2}^{\infty} y^j g_j. \tag{5}$$

where the f_i, g_j are equal to a power of z times a unit, $0 < \operatorname{ord} f_1 < \infty$ and $0 < \operatorname{ord} g_2$. Since we only make variable changes which fix z, the crux of the matter is to understand the case when $h = 0$.

Lemma 2.5. *Consider the power series*

$$F = xy + \sum_{i=1}^{\infty} x^i f_i + \sum_{j=2}^{\infty} y^j g_j \tag{6}$$

where f_i, g_j are powers of z up to units, (a) $0 < \operatorname{ord} f_1 < \infty$ and (b) $\operatorname{ord} f_2 > 0$ or $\operatorname{ord} g_2 > 0$. Set $m = \min\{\operatorname{ord} f_1^j g_j\}$ and write $\sum_{j=2}^{\infty} (-1)^j f_1^j g_j = z^m \cdot \delta$. Then the change of variables $X = x, Y = y + f_1$ yields

$$F = z^m \cdot \delta + XY + \sum_{i=2}^{\infty} X^i F_i + \sum_{j=1}^{\infty} Y^j G_j \tag{7}$$

where F_i, G_j are powers of z up to units such that
(a) $0 < \operatorname{ord} G_1 < \infty$;
(b) $\operatorname{ord} F_2 > 0$ or $\operatorname{ord} G_2 > 0$;
(c) $M = \min\{\operatorname{ord} G_1^i F_i\} > m$; and
(d) $\operatorname{ord} G_1 \geq m - \operatorname{ord} f_1$.

Proof. Setting $Y = y + f_1$ we have

$$F = xY + \sum_{i=2}^{\infty} x^i f_i + \sum_{j=2}^{\infty} (Y - f_1)^j g_j.$$

The part of the last sum with degree 0 in Y is $\sum_{j=2}^{\infty} (-1)^j f_1^j g_j = z^m \cdot \delta$ by definition of δ.

We take $F_i = f_i$ for $i \geq 2$ and calculate G_j by gathering terms with like powers of Y:

$$G_1 = -2 f_1 g_2 + 3 f_1^2 g_3 - \cdots = \sum_{k=2}^{\infty} (-1)^{k-1} k f_1^{k-1} g_k \tag{8}$$

and for $j \geq 2$,

$$G_j = \sum_{k=j}^{\infty} (-1)^{k-j} \binom{k}{j} f_1^{k-j} g_k.$$

Thus F takes the form of equation (7) and it remains to show that $M = \min\{\operatorname{ord} G_1^i F_i\} > m$. When expanded, each term in $G_1^i F_i = G_1^i f_i$ has the form

$$c f_i f_1^{k_1 + k_2 + \cdots + k_i - i} g_{k_1} g_{k_2} \cdots g_{k_i}$$

where c is a constant and the $k_\ell \geq 2$ are not necessarily distinct. The order of this term is strictly greater than $\operatorname{ord} f_1^{k_1} g_{k_1} \geq m$, unless $i = 2, k_1 = k_2 = 2$. In the case $i = 2, k_1 = k_2 = 2$ we would like to see that $\operatorname{ord} f_2 f_1^2 g_2^2 > \operatorname{ord} f_1^2 g_2$, but this follows from the condition that $\operatorname{ord} f_2 > 0$ or $\operatorname{ord} g_2 > 0$. Thus $\operatorname{ord} G_1^i f_i > m$ for all i and $M > m$.

For (d), the order of the k^{th} term in sum (8) is $(k-1) \operatorname{ord} f_1 + \operatorname{ord} g_k \geq m - \operatorname{ord} f_1$. □

Theorem 2.6. *For F as in equation (5), let $m = \min\{j r_1 + \operatorname{ord} g_j\}$ and set $\mu = \min\{\operatorname{ord} h, m\}$. Let $\delta(\mu)$ be the coefficient of z^μ in $H = h + \sum_{j=2}^{\infty} (-1)^j f_1^j g_j$. Then*

(a) *F defines an $\mathbf{A}_n$ singularity with $n \geq \mu - 1$ ($n = \infty$ is possible).*
(b) *F defines an $\mathbf{A}_{\mu-1}$ singularity if $\delta(\mu) \neq 0$.*

Proof. Apply the lemma to $F - h(z)$ and then add $h(z)$ back in to obtain the form

$$F = H + \sum_{i=2}^{\infty} X^i F_i + \sum_{j=1}^{\infty} Y^j G_j;$$

then relabel and repeat. By Lemma 2.5 (c), m is strictly increasing. Note that after each change of variables $Y = y + f_1($ or $X = x + g_1)$, the new variables (x, Y, z) still form a regular system of parameters at the origin. Now, consider two iterations of the algorithm; start with x, y and f_i, g_j and m-value m; then change to x, Y with f_i, G_j and m-value $M > m$, and next to X, Y with F_i, G_j. Then

$$\operatorname{ord} F_1 \geq M - \operatorname{ord} G_1 > m - (m - \operatorname{ord} f_1) = \operatorname{ord} f_1$$

by Lemma 2.5(d), so $\operatorname{ord} f_1$ increases with every change of x-variable; and similarly for $\operatorname{ord} g_1$. Thus the sequence of variable changes forms a Cauchy sequence and moreover in the limit the terms f_1 and g_1 both vanish. Therefore the expression becomes

$$XY + H(z) + \sum_{i=2}^{\infty} X^i F_i + \sum_{j=2}^{\infty} Y^j G_j,$$

and applying [10, Prop. 4.4] after subtracting $H(z)$ brings us to the form

$$F = XY + H(z).$$

If some $\delta(\mu) \neq 0$, H retains a term of order μ in every subsequent change of variables because each only involves terms of order $\geq m > \mu$, so μ stabilizes. Therefore in this case the form of F is $XY +$ unit $\cdot z^\mu$, an $\mathbf{A}_{\mu-1}$ singularity. Otherwise $\delta(\mu) = 0$ for every μ and $H \to 0$ as $\mu \to \infty$, so F factors. $\square$

Remark 2.7. The inductive procedure given in Theorem 2.6 and Lemma 2.5 recognizes an $\mathbf{A}_n$ singularity in at most n steps. However condition (b) in Theorem 2.6 is an open condition among equations of fixed degree, so the algorithm terminates after only *one* step with probability 1.

Example 2.8. We illustrate the theorem with a few examples.

(a) Applying Theorem 2.6 to $F = xy + xz^2 + y^2 z - z^6$, we have $m = 5, \mu = \min\{5, 6\} = 5$ and $\delta(5) = 1 \neq 0$, so F represents an $\mathbf{A}_4$ singularity. The variable change $Y = y + z^2$ gives $F = xY + (Y - z^2)^2 z - z^6 = xY + Y^2 z - 2Yz^3 + z^5 - z^6$ so that $H(z) = z^5 - z^6$ has order 5. After the sequence of variable changes suggested, the z^5 term survives while the terms involving x, Y eventually factor.

(b) For the singularity given by

$$F = xy + xz^4 + y^2 z^6 + y^3 z^2 + y^4 z^{25} + x^2 z$$

we have $m = \mu = 14$ and $\delta(14) = 0$, so we make the variable change $Y = y + z^4$ suggested by the theorem. Then we have

$$F = xY + (Y - z^4)^2 z^6 + (Y - z^4)^3 z^2 + (Y - z^4)^4 z^{25} + x^2 z$$

When multiplying this out, the z^{14} term drops out (because $\delta(14) = 0$), but that the new incarnation of F has linear Y-terms, namely

$$f = xY + Y(-2z^{10} + 3z^{10}) + \cdots + x^2 z = xY + Yz^{10} + \cdots + x^2 z$$

where the dots represent higher power of Y terms. Continuing with $X = x + z^{10}$ gives

$$f = XY + \cdots + X^2 z - 2Xz^{11} + z^{21}$$

and it becomes clear that we are dealing with an $\mathbf{A}_{20}$.

(c) Proposition 2.4 follows readily from Theorem 2.6 as (with x, y reversed) we have $\mu = mn$ and $\delta(mn) = u^n v \neq 0$, yielding an $\mathbf{A}_{mn-1}$ singularity.

3. Two multiple curves intersect at a point

In this section we give a solution to Problem 1.1 when $Z = Z_1 \cup Z_2$ is a union of two multiple curves of embedding dimension two with respective smooth supports C_1, C_2 meeting transversely at p under the condition that Z_1 and Z_2 do not share the same Zariski tangent space at p. In other words, we consider the following two cases:

(1) No Tangency: C_1 is not tangent to Z_2 and C_2 is not tangent to Z_1.
(2) Mixed Tangency: C_1 is tangent to Z_2 but C_2 is not tangent to Z_1.

For each of these we find canonical forms for the local ideals (Propositions 3.3 and 3.5) and determine the local Picard groups at the corresponding fixed singularity on the very general surface containing the curve (Propositions 3.4, 3.6, 3.7 and 3.8). The following local algebra lemma will facilitate computing the intersection of ideals $I_Z = I_{Z_1} \cap I_{Z_2}$.

Lemma 3.1. *Let R be a regular (local) ring. For $a, b, c, d \in R$, assume that a, c, d form a regular sequence and that $d \in (a, b)$. Then $(a, b) \cap (c, d) = (ac, bc, d)$.*

Proof. Write $d = as + br$ with $s, r \in R$. Since d is a non-zero divisor mod (a), the same is true of r, so that a, r and a, b also form regular sequences in R. Since $(a, d) = (a, br)$, the ideals (a, b) and (a, r) are linked by the complete intersection (a, d). It follows that $(a, b) \cap (c, d)$ is linked to (a, r) by the complete intersection $(ac, d) = (a, d) \cap (c, d)$. The inclusion of ideals $(ac, d) \subset (a, r)$ lifts to a map of the corresponding Koszul complexes

$$\begin{array}{ccccccc} 0 & \to & R & \xrightarrow{(-d,ac)} & R^2 & \xrightarrow{(ac,d)} & (ac,d) \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow \\ 0 & \to & R & \xrightarrow{(-r,a)} & R^2 & \xrightarrow{(a,r)} & (a,r) \end{array}$$

where $\beta(A, B) = (Ac + Bs, Bb)$ and $\alpha(C) = Cbc$. By the mapping cone construction for liaison [19, Prop. 2.6], the ideal $(a, b) \cap (c, d)$ is the image of $R^3 \to R$ given by the direct sum of $\alpha^\vee$ and $(-d, ac)^\vee$, so the ideal is $(bc, -d, ac) = (ac, bc, d)$. $\square$

Example 3.2. Lemma 3.1 fails if a, c, d do not form a regular sequence in R, for example $R = k[x, y, z], a = c = x, b = d = y$ when $(x, y) \cap (x, y) \neq (x^2, yx, y) = (x^2, y)$.

Proposition 3.3. *Let $Z = Z_1 \cup Z_2$ be the union of two multiplicity structures on smooth curves C_1, C_2 meeting at p with respective multiplicities $t \leq n$. Assume Z_i is contained in a local smooth surface $S_i, i = 1, 2$, C_1 is not tangent to S_2 and C_2 is not tangent to S_1. Then there are local coordinates x, y, z at p for which $I_{Z_1} = (x, z^t)$, $I_{Z_2} = (y, z^n)$ and*

$$I_Z = (xy, yz^t, z^n).$$

Proof. Locally we may assume that S_1 is given by equation $x = 0$ and S_2 is given by equation $y = 0$. Letting $z = 0$ be the equation of a smooth surface containing both C_1 and C_2 near p, the lack of tangency conditions imply that x, y, z is a regular system of parameters at p and we obtain $I_{C_1} = (x, z)$ and $I_{C_2} = (y, z)$. Given that $Z_i \subset S_i$ with the multiplicities given, it's clear that $I_{Z_1} = (x, z^t)$ and $I_{Z_2} = (y, z^n)$. Taking $a = x, b = z^t, c = y, d = z^n$, we have $d \in (a, b)$ because $t \leq n$, so the intersection ideal is (xy, yz^t, z^n) by Lemma 3.1. $\square$

Proposition 3.4. *For Z as in Proposition 3.3 above, the general surface S containing Z has an $\mathbf{A}_{n-1}$ singularity at p and C_1 (resp. C_2) maps to 1 (resp. -1) under the isomorphism $\operatorname{Cl} \widehat{\mathcal{O}}_{S,p} \cong \mathbb{Z}/n\mathbb{Z}$.*

Proof. The general element of I_Z has the form $xy + byz^t + cz^n$ with units $b, c \in \mathcal{O}_{\mathbb{P}^3,p}$. In the language of Theorem 2.6, $\mu = n$ and $\delta(n) = c \neq 0$, so the singularity is of type $\mathbf{A}_{n-1}$. The first change of variables $X = x + bz^t$ is the only one necessary, giving us immediately (up to units) the form $XY - z^n$; furthermore, the ideal defining C_1 is is $(x, z) = (X - cz^n, z) = (X, z)$ and $I_{C_2} = (y, z) = (Y, z)$, so these curves correspond to the canonical generators ± 1 for the group $\mathrm{Cl}\,\widehat{\mathcal{O}}_{S,p} \cong \mathbb{Z}/n\mathbb{Z}$ by Examples 2.1 (a) and (b). □

The mixed tangency case is more complicated.

Proposition 3.5. *Let C_1, C_2 be smooth curves meeting transversely at p, $C_i \subset S_i$ local smooth surfaces, and $Z_1 = tC_1 \subset S_1, Z_2 = nC_2 \subset S_2$ multiplicity structures. Assume that C_1 meets S_2 transversely and C_2 is tangent to S_1 of order $q > 1$. Then there are local coordinates x, y, z at p for which*

$$I_{Z_1} = (x - z^q, z^t), \quad I_{Z_2} = (y, x^n)$$

and the intersection ideal $I_{Z_1 \cup Z_2} = I_{Z_1} \cap I_{Z_2}$ takes the form:

(a) If $t \leq q$, then $I_Z = (xy, yz^t, x^n)$.
(b) If $q < t < qn$, then $I_Z = (y(x - z^q), yz^t, x^n)$.
(c) If $t \geq qn$, then $I_Z = (y(x - z^q), yz^t, x^n z^{t-qn})$.

Proof. Let $x = 0$ (resp. $y = 0$) be a local equation for S_1 (resp. S_2). Since C_1 meets S_2 transversely, we can extend x, y to a regular sequence x, y, z with $I_{C_1} = (x, z)$. Locally C_2 meets S_1 tangentially to order $q > 1$, so we may write $I_{C_2} = (x + \alpha, y)$ with $\alpha \in (x, y, z)^q$. Now $I_{C_2 \cap S_1} = (x, y, \alpha)$ defines a scheme of length q, so $\alpha = uz^q$ modulo (x, y) for some unit u; writing $\alpha = uz^q + xf + yg$ we have

$$I_{C_2} = (x + \alpha, y) = (x + uz^q + xf + yg, y) = (x(1 + f) + uz^q, y)$$

where $(1 + f)$ is a unit. Replacing x with $\dfrac{x(1+f)}{u} + z^q$ we have

$$I_{Z_1} = (x - z^q, z^t) \qquad I_{Z_2} = (y, x^n)$$

and it remains to find the intersection $I_Z = I_{Z_1} \cap I_{Z_2}$.

If $t \leq q$ (including the case $q = \infty \Rightarrow \alpha = 0 \Rightarrow C_2 \subset S_1$), then $I_{Z_1} = (x, z^t)$. Lemma 3.1 applies with $a = x, b = z^t, c = y$ and $d = x^n$, showing that $I_Z = (xy, yz^t, x^n)$.

If $q < t < qn$, then $z^{qn} = z^t \cdot z^{qn-t} \in I_{Z_1}$ and also $(x - z^q) | (x^n - z^{qn}) \Rightarrow x^n - z^{qn} \in I_{Z_1}$ so $x^n \in I_{Z_1}$. Application of Lemma 3.1 with $a = x - z^q, b = z^t, c = y, d = x^n$ produces the ideal $I_Z = (y(x - z^q), yz^t, x^n)$.

If $qn \leq t$, then we have the telescoping sum

$$z^t + z^{t-q}(x - z^q) + xz^{t-2q}(x - z^q) + \cdots + x^{n-1}z^{t-qn}(x - z^q) = x^n z^{t-qn} \in I_{Z_1}$$

and so we can again apply Lemma 3.1 with $a = x - z^q, b = z^t, c = y, d = x^n z^{t-qn}$ to obtain $I_Z = (y(x - z^q), yz^t, x^n z^{t-qn})$. □

Proposition 3.6. *For $Z = Z_1 \cup Z_2$ as in Proposition 3.5 (a) with $t \leq q$, the general surface S containing Z has a singularity of type $\mathbf{A}_{tn-1}$ at p and C_1 (resp. C_2) maps to 1 (resp. $tn - t$) under the isomorphism $\mathrm{Cl}\,\widehat{\mathcal{O}}_{S,p} \cong \mathbb{Z}/tn\mathbb{Z}$.*

Proof. In view of Prop. 3.5 (a), the general surface S containing $Z_1 \cup Z_2$ has local equation $xy - uyz^t - vx^n$ with u, v units in $\mathcal{O}_{\mathbb{P}^3,p}$ and x, y, z a regular sequence of parameters. Since C_1 is given by the ideal (x, z) and C_2 is given by (x, y), the result follows from Proposition 2.4. □

Proposition 3.7. *For $Z = Z_1 \cup Z_2$ as in Proposition 3.5 (b) with $q < t < qn$, the general surface S containing Z has a singularity of type $\mathbf{A}_{qn}$ at p and C_1 (resp. C_2) maps to 1 (resp. $qn - q$) under the isomorphism $\mathrm{Cl}\,\widehat{\mathcal{O}}_{S,p} \cong \mathbb{Z}/qn\mathbb{Z}$.*

Proof. The general surface containing Z has local equation $xy - yz^q - uyz^t - vx^n$ for units $u, v \in \mathcal{O}_{\mathbb{P}^3,p}$ and since $w = 1 + uz^{t-q}$ is a unit we may write the equation as $xy - wyz^q - vx^n$ and we can apply Proposition 2.4 to see the $\mathbf{A}_{qn-1}$ singularity and that C_1 with ideal (x, z) corresponds to $1 \in \mathbb{Z}/qn\mathbb{Z} \cong \mathrm{Cl}\,\widehat{\mathcal{O}}_{S,p}$ while C_2 with ideal (x, y) corresponds to $qn - q$. □

Proposition 3.8. *For $Z = Z_1 \cup Z_2$ as in Proposition 3.5 (c) with $qn \leq t$, the very general surface S containing Z has a singularity of type A_{t-1} and C_1 (resp. C_2) maps to 1 (resp. $t-q$) under the isomorphism $\mathrm{Cl}\,\widehat{\mathcal{O}}_{S,p} \cong \mathbb{Z}/t\mathbb{Z}$.*

Proof. By Proposition 3.5 (c), the general surface S containing Z has local equation

$$xy - yz^q + uyz^t + vx^n z^{t-qn}$$

for units u, v in $\mathcal{O}_{\mathbb{P}^3,p}$. Using Theorem 2.6 with the roles of x and y reversed, we see that $\mu = qn + (t - qn) = t$; moreover, $\delta(\mu) = \pm v \neq 0$, so S has a type $\mathbf{A}_{t-1}$ singularity.

In order to determine the classes of C_1 and C_2 in $\mathrm{Cl}\,\widehat{\mathcal{O}}_{S,p} \cong \mathbb{Z}/t\mathbb{Z}$, however, it is expedient to calculate a coordinate change. Therefore set $X = x - z^q + uz^t = x - \beta z^q$ with β a unit in the local ring. With this change of coordinates the defining equation for S becomes

$$Xy + v(X + \beta z^q)^n z^{t-qn}.$$

Write this last expression as

$$X(\underbrace{y + X^{n-1}z^{t-qn} + n\beta X^{n-2}z^{t-q(n-1)} + \cdots + \beta^{n-1}nz^{t-q}}_{Y}) + \beta^n z^t.$$

The ideal of C_1 is $(x, z) = (X, z)$, corresponding to the generator 1 by Example 2.1(a). The ideal of C_2 is $(x, y) = (X + \beta z^q, y)$; evaluating the expression for Y at $X = -\beta z^q$ shows that this ideal is $(X + \beta z, Y - \beta^{n-1}z^{t-q})$, which corresponds to the element $t - q$ by Example 2.1(c). □

4. Multiple structures on a smooth curve

Let Z be a locally Cohen-Macaulay multiplicity structure supported on a smooth connected curve $C \subset \mathbb{P}^3$. The ideal sheaves $\mathcal{I}_Z + \mathcal{I}_C^i$ define one-dimensional subschemes of Z and after removing the embedded points we arrive at the Cohen-Macaulay filtration

$$C = Z_1 \subset Z_2 \subset \cdots \subset Z_m = Z \tag{9}$$

from work of Banica and Forster [1] (see also [16, §2]). If Z has generic embedding dimension two, then the quotient sheaves $\mathcal{I}_{Z_j}/\mathcal{I}_{Z_{j+1}} = L_j$ are line bundles on C and the multiplicity of Z is m (note that this use of the symbol m is different from that in Theorem 2.6 and thereabouts). In this section we solve Problem 1.1 for such a multiplicity structure Z at a point p of embedding dimension three assuming that Z_{m-1} or Z_{m-2} has local embedding dimension two at p. This class of curves contains all multiplicity structures on C with multiplicity $m \leq 4$. First we describe the local ideal of Z at p.

Proposition 4.1. *Let $Z \subset \mathbb{P}^3_{\mathbb{C}}$ be the subscheme with ideal $\mathcal{I}_Z = (x^2, xy, xz^q - y^{m-1}, y^m)$ for $m \geq 3$. Then the very general surface S containing Z has an $\mathbf{A}_{(m-1)q-1}$ singularity at $p = (0, 0, 0, 1)$ and $\mathrm{Picloc}\, p \cong \mathbb{Z}/(m-1)\mathbb{Z}$ is generated by C.*

Proof. For appropriate units $a, b, c \in \mathcal{O}_{\mathbb{P}^3,p}$ the general surface S containing Z has equation

$$xy + ax^2 + by^{m-1} + cy^m - bxz^q = x\underbrace{(y + ax)}_{y_1} + \underbrace{(b + cy)}_{u} y^{m-1} - bxz^q. \tag{10}$$

With $y_1 = y + ax$ and unit $u = b + cy$ as above, we may write

$$xy_1 + u(y_1 - ax)^{m-1} - bxz^q = x_1y_1 + c(y_1 + x_1)^{m-1} - dx_1z^q \tag{11}$$

for new units c, d after setting $x_1 = -ax$ and multiplying by $-1/a$. If $m = 3$, the first two terms are a homogeneous quadratic form: for general a, b, c, this factors into two linear terms and making the corresponding change of variable brings the equation to the form $XY + (AX + BY)z^q$ for units A, B.

For $m \geq 4$, expand the $(m-1)$st power in equation (11) as

$$c(y_1 + x_1)^{m-1} = y_1 \underbrace{[cy_1{}^{m-2}]}_{g_1} + x_1 \underbrace{[c((m-1)y_1{}^{m-2} + \cdots + x_1{}^{m-2})]}_{h_1}$$

and set $x_2 = x_1 + g_1, y_2 = y_1 + h_1$, when equation (11) becomes

$$x_2y_2 - g_1h_1 - dxz^q \tag{12}$$

with $g_1h_1 \in (x, y)^{2m-4} \subset (x, y)^m$. Applying Lemma 2.3, we make another change of variables from x_2, y_2 to X, Y for which $x_2y_2 - g_1h_1 = XY$. Looking at the last term, one observes that $dx = d(x_2 - cy^{m-2}) = d(x_2 - c(y_2 - h_1)^{m-2})$ with $h_1 \in (x, y)^{m-2}$; extracting the multiples of x_2 in the resulting power series this may be written $A_1x_2 + B_1y_2^{m-2}$ with A_1, B_1 units. Switching to the variables X, Y and noting that $x_2 \equiv X \mod (x, y)^{m-1}$ and $y_2 \equiv Y \mod (x, y)^{m-1}$ by the proof of Lemma 2.3 ($g_1h_1 \in (x, y)^m$), this may be written $AX + BY^{m-2}$ with A, B units. Thus S has local equation

$$XY - (AX + BY^{m-2})z^q$$

with units A, B. Setting $Y_2 = Y - Az^q$, we obtain $XY_2 - B(Y_2 + Az^q)^{m-2}z^q$. Multiplying out the $(m-2)$nd power, the second term may be written $Y_2L + BA^{m-2}z^{(m-1)q}$ with

$$L = B(Y_2^{m-3} + A(m-2)Y_2^{m-4}z^q + \cdots + A^{m-3}(m-2)z^{(m-3)q})z^q.$$

Setting $X_2 = X - L$ gives the form

$$X_2Y_2 - A^{m-2}Bz^{(m-1)q},$$

showing that S has an $\mathbf{A}_{(m-1)q-1}$ singularity.

We now follow the ideal (x, y) through its coordinate changes:

$$(x, y) = (x_1, y_1) = (x_2, y_2) = (X, Y) = (X, Y_2 + Az^q) = (X_2 + L, Y_2 - Az^q).$$

Working modulo $Y_2 - Az^q$, replacing Y_2 with Az^q reduces L to

$$BA^{m-3}[1 + (m-2) + \binom{m-2}{2} \cdots + (m-2)]z^{(m-2)q} = (2^{m-2} - 1)BA^{m-3}z^{(m-2)q},$$

so the final form for our ideal is $(X_2 + (2^{m-2} - 1)BA^{m-3}z^{(m-2)q}, Y_2 - Az^q)$, which has class q in $\operatorname{Cl} \widehat{\mathcal{O}}_{S,p} \cong \mathbb{Z}/(m-1)\mathbb{Z}$ by Example 2.1 (c). $\square$

Proposition 4.2. *Let $C \subset \mathbb{P}^3$ be a smooth curve and Let Z be a locally Cohen-Macaulay multiplicity m structure on a C of generic embedding dimension two with filtration (9). Let $p \in Z$ be a point of embedding dimension three at which Z_{m-1} has embedding dimension two. Then*

(a) There are local coordinates x, y, z for which Z has local ideal

$$I_Z = (x^2, xy, xz^q - y^{m-1}).$$

(b) *The general surface S containing Z has an $\mathbf{A}_{(m-1)q-1}$ singularity at p and and the class of C maps to $q \in \operatorname{Cl}\widehat{\mathcal{O}}_{S,p} \cong \mathbb{Z}/q(m-1)\mathbb{Z}$.*

Proof. Consider the exact sequence $0 \to I_Z \to I_{Z_{m-1}} \xrightarrow{\pi} L_{m-1} \to 0$ near p. Since Z_{m-1} has embedding dimension two at p, it locally lies on a smooth surface with equation $x = 0$ and $I_C = (x, y)$ for suitable y, whence $I_{Z_{m-1}} = (x, y^{m-1})$. Now I_Z appears locally as the kernel of a surjection $\pi : (x, y^{m-1}) \to \mathcal{O}/(x, y)$. If $\pi(y^{m-1}) = u$ is a unit and $\pi(x) = \bar{h}$ with $h \in \mathcal{O}$, then $x - u^{-1}hy^{m-1} \in \operatorname{Ker}(\pi) = I_Z$, but then $x - u^{-1}hy^{m-1} \notin \mathfrak{m}_p^2$ implies Z has embedding dimension two at p, contrary to assumption. Therefore we may assume $\pi(y^{m-1}) = \bar{h}$ for $h \in \mathfrak{m}_p$ and $\pi(x) = 1$ in which case $I_Z = \operatorname{Ker}(\pi) = (x^2, xy, hx - y^{m-1})$. If $h \in \mathcal{O}/(x, y)$ vanishes to order $q > 0$ at p, we may write $h = uz^q \bmod (x, y)$ where u is a unit and z is a local parameter for C at p. Absorbing u into x gives part (a).

The local ideal of Z is $(x^2, xy, xz^q - y^{m-1}, y^m)$, the generator y^m being redundant. By Proposition 4.1, then, the associated class group is $\mathbb{Z}/(m-1)\mathbb{Z} \subset \mathbb{Z}/q(m-1)\mathbb{Z}$. ☐

Proposition 4.3. *Let $C \subset \mathbb{P}^3$ be a smooth curve and Let Z be a locally Cohen-Macaulay multiplicity m structure on a C of generic embedding dimension two with filtration (9). Fix a point $p \in Z$ of embedding dimension three. If Z_{m-2} has embedding dimension two at p and Z_{m-1} does not, then there are local coordinates x, y, z for which Z has local ideal*

(a) *$(x^2, xy^2, xyz^q - y^{m-1}, xy - uz^w(z^q x - y^{m-2}))$, u a unit and $w \geq 0$, or*
(b) *$(x^2, xy^2, fxy - (z^q x - y^{m-2}))$ where $f = 0$ or $f = uz^w$ for some $w > 0$ and unit u.*

Proof. Use Prop. 4.2 (a) to write $I = I_{Z_{m-1}} = (x^2, xy, xz^q - y^{m-2})$. At the level of sheaves, the map π factors through $\mathcal{F} = \mathcal{I}_{Z_{m-1}} \otimes \mathcal{O}_C/\{\text{torsion}\}$, a vector bundle on C which has rank two because Z_{m-1} is a generic local complete intersection. Working in the free $\mathcal{O}/(x, y)$-module $F = (I \otimes \mathcal{O}/(x, y))/\{\text{torsion}\}$ near p,

$$z^q(x^2) = x(xz^q - y^{m-1}) + y^{m-2}(xy) = 0$$

shows that x^2 is torsion, hence zero. Therefore $F \cong (\mathcal{O}/(x, y))^2$ is freely generated by xy and $xz^q - y^{m-2}$.

The kernel of the map $I \to F$ is

$$(x^2) + (x, y)I = (x^2, xy^2, xyz^q - y^{m-1}).$$

and we obtain $I_Z = \operatorname{Ker}\pi$ by adding the Koszul relation for the surjection of free modules $F \to \mathcal{O}/(x, y)$. Surjectivity implies that $\pi(xy)$ or $\pi(z^q x - y^{m-2})$ is a unit in $\mathcal{O}/(x, y)$. If $\pi(z^q x - y^{m-2}) = 1$ and $\pi(xy) = f \in \mathcal{O}/(x, y)$, the Koszul relation is $xy - f(z^q x - y^{m-2})$; here $f = uz^w$ for some unit u, since $f = 0$ leads to the ideal (x^2, xy, y^{m-1}) which does not have generic embedding dimension two; this gives the ideal in part (a). Otherwise take $\pi(xy) = 1$ and $\pi(z^q x - y^{m-1}) = f$ where $f = 0$ or $f = uz^w$ for some $w > 0$ and unit u, when the Koszul relation is $fxy - (z^q x - y^{m-2})$, giving ideal in part (b). ☐

Remark 4.4. Propositions 4.2 and 4.3 give a local description of ideals of certain multiplicity structures Z on a smooth curve C. Using the ideals in the Propositions to define multiple curves in $\mathbb{A}^3$, one can obtain global examples by taking closures in $\mathbb{P}^3$. When $C \subset \mathbb{P}^3$ is a line and $m \leq 4$, all such global structures have been classified [16, 17].

Proposition 4.5. *Let Z be a multiplicity-m structure on a smooth curve C with local ideal at p as in Prop. 4.3(a). Then at p the general surface S containing Z has an $\mathbf{A}_{(m-2)(q+w)+w-1}$ singularity and C has class $q + w \in \mathbb{Z}/((m-2)(q+w)+w)\mathbb{Z} \cong \operatorname{Cl}\widehat{\mathcal{O}}_{S,p}$.*

Proof. The very general surface S of sufficiently high degree containing Z has equation

$$xy + ax^2 + bxy^2 + cy^m + dxyz^q - dy^{m-1} - uxz^{q+w} + uy^{m-2}z^w$$

for units a, b, c, d, u in the local ring at the origin p by assumption.

We first apply the preparation steps in the first part of subsection 2.3 to bring the equation into a form recognizable by Theorem 2.6, beginning with changing variables to $Y = y + ax$. Expanding and gathering xY-terms brings the equation to the form

$$\text{unit}\cdot xY + u_1x^3 + u_2Y^m + u_3x^m + u_4x^2z^q + u_5Y^{m-1} + u_6x^{m-1} - uxz^{q+w} + uY^{m-2}z^w + u_7x^{m-2}z^w$$

where $u_1 = a^2b, u_2 = c, u_3 = c(-a)^m, u_4 = -ad, u_5 = -d, u_6 = -d(-a)^m$ and $u_7 = u(-a)^{m-2}$ are units. Applying Theorem 2.6 we have $r_1 = q + w$, $\mu = (m-2)(q+w) + w$, and since $\delta((m-2)(q+w)+w) \neq 0$ for general choice of units, the singularity type is $\mathbf{A}_{(m-2)(q+w)+w-1}$.

To determine the class of C in the completed local ring, we will look at the resolution of the singularity and determine which exceptional curve meets the strict transform of C. On the patch $Z = 1$ on the first blowup, the singularities must lie on the exceptional locus $z = 0$. This gives the equation (recycling the symbols x and y) $xy + ax^2 = 0$; partials similarly give $x = 0$ and $y + 2ax = 0$, so the blown-up surface is singular only at the origin on this patch. On $X = 1$ the exceptional locus has equations $x = 0, y = 0$, which is smooth, and similarly on the other patch. This situation persists until we get to the $(q+w)^{\text{th}}$ blow-up, which on the patch $Z = 1$ has equation

$$xy + ax^2 - ux + bxy^2z^{q+w} + dxyz^q + dy^{m-1}z^{(q+w)(m-3)} + uy^{m-2}z^{(q+w)(m-4)+w}.$$

This surface is smooth at the origin and singular at $(0, u, 0)$. Changing variables to $y' = y + u$ produces an equation of the form

$$xy' + \text{ (terms of order at least 2 in } x \text{ and } y' \text{ times powers of } z) + \text{ unit } \cdot z^{(q+w)(m-4)+w}.$$

As in subsection 2.3, this becomes $XY - Z^{(q+w)(m-4)+w}$ where the variable changes to obtain X and Y do not affect z and then Z is a unit times z.

To determine the class of C, note that its strict transform passes through the origin all the way to the $(q+w)^{\text{th}}$ blowup, at which point it still passes through the origin but misses the singular point. This gives C the class $q + w$ in the complete local Picard group $\mathbb{Z}/((q+w)(m-2)+w)\mathbb{Z}$. As C generates the class group of the original singular point, the order of this group depends on the greatest common divisor of $q + w$ and $(q+w)(m-2)+w$. $\square$

Proposition 4.6. *For $p \in Z$ as in Prop. 4.3(b) with $C = \operatorname{Supp} Z$, let S be the general surface containing Z. Then locally the equation of S at p has the form*

$$F = ax^2 + bxy^2 + e(fxy - (z^qx - y^{m-2}))$$

for local parameters x, y, z general units $a, b, e \in \mathcal{O}_{\mathbb{P}^3,p}$ and $f = uz^w$ for some $w > 0$ (interpret $w = \infty$ as $f = 0$) and C has order $m-2$ in $\operatorname{Cl}\widehat{\mathcal{O}}_{S,p}$. Furthermore

1. *(1) If $m = 4$, then S has an $\mathbf{A}_{2q-1}$ singularity at p and $C \mapsto q \in \mathbb{Z}/2q\mathbb{Z} \cong \operatorname{Cl}\widehat{\mathcal{O}}_{S,p}$.*
2. *(2) If $m > 4, q = 1$, then S has an $\mathbf{A}_{m-3}$ singularity at p and $C \mapsto 1 \in \mathbb{Z}/(m-2)\mathbb{Z}$.*
3. *(3) If $m = 5, q = 2$, then S has an $\mathbf{E}_6$ singularity at p and $C \mapsto 1 \in \mathbb{Z}/3\mathbb{Z} \cong \operatorname{Cl}\widehat{\mathcal{O}}_{S,p}$.*
4. *(4) For $m = 5$ and $q \geq 3$ or $m \geq 6$ and $q \geq 2$, the singularity of S at p is not a rational double point.*

Proof. The local equation for S follows immediately from Prop. 4.3 (b). To see that C has order $m-2$, first observe that $(m-2)C$ is Cartier on S at p simply because

$$(x, F) = (x, ax^2 + bxy^2 + fxy - (z^qx - y^{m-2})) = (x, y^{m-2}).$$

This shows that the order of C divides $m-2$ and it remains to show the order cannot be less. For this, recall that by construction dC has local ideal (x, y^d) for all $d \le m-2$, so we must show that (x, y^d) is not Cartier on S at p for $d < m-2$. By Nakayama's lemma, this is equivalent to showing that the $\mathcal{O}/(x,y,z)$-vector space

$$\frac{(x, y^d)}{(x,y,z)(x,y^d) + (F)}$$

has dimension > 1, but this is clear because $F \in (x,y,z)(x,y^d)$ for $d < m-2$, so the dimension is 2.

First assume $m = 4$. Take $e = -1$ so that the local equation for S at p is

$$F = ax^2 - y^2 - dy^3 + cy^4 + bxy^2 + dxyz^q - fxy + xz^q.$$

By Lemma 2.2 a has a square root $\sqrt{a}$ in the complete local ring; set $x_1 = \sqrt{a}x + y$ and $y_1 = \sqrt{a}x - y$ so that the equation takes the form

$$x_1 y_1 + G + \frac{(x_1 + y_1)}{2\sqrt{a}} z^q$$

with $G \in (x_1, y_1)^2 \mathfrak{m}$. By Lemma 2.3 there is a coordinate change X, Y for which

$$F = XY + (AX + BY) z^q$$

where A, B are units for general choices of a, b, c, d. Making the elementary transformation $X_1 = X + Bz^q$ and $Y_1 = Y + Az^q$ brings the equation to the form $F = X_1 Y_1 - ABz^{2q}$ displaying the $\mathbf{A}_{2q-1}$ singularity. Tracing the class of the supporting curve we have

$$(x,y) = (x_1, y_1) = (X, Y) = (X_1 - Bz^q, Y_1 - Az^q)$$

which has class $q \in \mathbb{Z}/(2q)\mathbb{Z} \cong \operatorname{Cl} \widehat{\mathcal{O}}_{S,p}$ by Example 2.1 (c).

Now assume $m > 4$ and $q = 1$; again take $e = -1$ so that the local equation for S at p is

$$xz - y^{m-2} - fxy + ax^2 + bxy^2 + cy^m + d(xyz - y^{m-1})$$

for units $a, b, c, d \in \mathcal{O}_{\mathbb{P}^3, p}$. For $Z = z + fy + ax + by^2 + dyz^q$ and unit $u = 1 + dy - cy^2$ this takes the form $xZ - uy^{m-2}$, defining an $\mathbf{A}_{m-3}$ singularity. The class of the curve with ideal (x, y) is $1 \in \mathbb{Z}/(m-2)\mathbb{Z}$ by Example 2.1 (a).

The case $m = 5$ and $q = 2$ has a different flavor: Write the equation for S as

$$x^2 + 2axy^2 + by^5 + 2cxyz^2 - cy^4 + \underbrace{2uxyz^w}_{\text{or } 0} - 2vxz^2 + 2vy^3;$$

let $x_1 = x + y^2 + cyz^2 + uyz^w - vz^2$, and note that the expression takes on the form

$$x_1^2 + \alpha y^3 + \beta y^2 z^2 + \gamma z^4,$$

where α, β, γ are units. We may assume $\alpha = 1$; rewrite this expression as

$$x_1^2 + \left(y + \frac{\beta}{3} z^2 \right)^3 + \gamma' z^4$$

where γ' is another unit. Taking $y_1 = y + \frac{\beta}{3} z^2$ shows that S has an $\mathbf{E}_6$ singularity at the origin with $\operatorname{Cl} \widehat{\mathcal{O}}_{S,p} \cong \mathbb{Z}/3\mathbb{Z}$. (The ideal for the curve C has become $(x_1 - \gamma z^4, y)$.)

For the case $m = 5, q \ge 3$ in part (4), a calculation entirely analogous to the previous one gives the form $X^2 + Y^3 + aYZ^4 + bZ^6$, a, b units. Lemma 2.2 shows that b has a square root, so the ideal of C can be written $(X + i\sqrt{b}Z^3, Y)$. The first blow-up of this surface on the patch (recycling variables as usual) has equation $X^2 + Y^3 z + aYz^3 + bz^4$, which is not the equation for a rational double point, since it is congruent to a square mod $\mathfrak{m}^4$ (see the classification given in

[14, §24]). Therefore the original singularity is not a rational double point, since the resolution of a rational double point only involves other rational double points.

Finally, for $m \geq 6, q \geq 2$ in part (4), after the first algebraic step of completing the square, we see that the equation for S is congruent to a square mod $\mathfrak{m}^4$, so again the singularity is not a rational double point. □

5. Global Picard groups of normal surfaces

In this section we give a formula for the Picard group of very general high degree surfaces containing a fixed curve Z with at most finitely many points of embedding dimension three. The solution to Problem 1.1 is required to apply the formula and we illustrate this with the examples worked out in the previous two sections.

Theorem 5.1. *Let $Z \subset \mathbb{P}^3$ be a closed one-dimensional subscheme with curve components $Z_1, \dots Z_r$ having respective supports C_i and suppose that the set T of points where Z has embedding dimension three is finite. If S is a very general surface of degree $d \gg 0$ containing Z with plane section H, then*

(1) S is normal and $\operatorname{Cl} S$ is freely generated by H and the C_i.

(2) The Picard group of S is

$$\operatorname{Pic} S = \bigcap_{p \in T} \operatorname{Ker}(\operatorname{Cl} S \to \operatorname{Cl} \mathcal{O}_{S,p}) \cap \langle Z_1, Z_2, \dots, Z_r, H\rangle \subset \operatorname{Cl} S. \tag{13}$$

Remark 5.2. The condition $d \gg 0$ can be expressed effectively, namely that $\mathcal{I}_Z(d-1)$ is generated by global sections and either (1) Z is reduced of embedding dimension $le2$ at each point or (2) $h^0(\mathcal{I}_Z(d-2)) \neq 0$ [2, Sections 1 and 2].

Proof. Part (1) is [2, Thm. 1.1]. It follows from sequence (1) in the introduction that

$$\operatorname{Pic} S = \bigcap_{p \in \operatorname{Sing} S} \operatorname{Ker}(\operatorname{Cl} S \to \operatorname{Cl} \mathcal{O}_{S,p}). \tag{14}$$

Along with the fixed singularities T, which forcibly lie on every surface S containing Z, there are *moving singularities* p, which vary with the surface and lie on exactly one component Z_i of multiplicity $m_i > 1$ [2, Prop. 2.2]: these are $\mathbf{A}_{m_i - 1}$ singularities and the corresponding map $\operatorname{Cl} S \to \operatorname{Cl} \widehat{\mathcal{O}}_{S,p} \cong \mathbb{Z}/m_i\mathbb{Z}$ sends C_i to 1 and the remaining C_i to 0, therefore the corresponding kernel is $\langle C_1, C_2, \dots, m_i C_i = Z_i, C_{i+1}, \dots, C_r, H\rangle$. Intersecting these subgroups for $1 \leq i \leq r$ yields $\langle Z_1, Z_2, \dots, Z_r, H\rangle$, which gives equation (13) provided there is at least one component Z_i of multiplicity $m_i > 1$. If there are no components of multiplicity $m_i > 1$, then Z is reduced and there are no moving singularities: here formula (13) still works because $Z_i = C_i$ for each $1 \leq i \leq r$ and hence $\langle Z_1, \dots, Z_r, H\rangle = \langle C_1, \dots, C_r, H\rangle = \operatorname{Cl} S$. □

We first note some easy special cases.

Corollary 5.3. *Let Z and S be as in Theorem 5.1. Then*

(a) If Z is reduced of embedding dimension at most two, then $\operatorname{Pic} S = \operatorname{Cl} S$.

(b) If Z is reduced, then $\operatorname{Pic} S = \bigcap_{p \in F} \operatorname{Ker}(\operatorname{Cl} S \to \operatorname{Cl} \mathcal{O}_{S,p})$.

(c) If Z has embedding dimension ≤ 2, then $\operatorname{Pic} S = \langle Z_1, \dots, Z_r, H\rangle$.

Proof. (a) Here T is empty and $Z_i = C_i$ for each i, so part (2) of the theorem says that $\operatorname{Pic} S$ is generated by H and the C_i, which is exactly $\operatorname{Cl} S$ by part (1). (b) Here $Z_i = C_i$ again, so $\langle Z_i, H\rangle = \operatorname{Cl} S$. (c) Here again T is empty. □

Remarks 5.4. We make a few observations about Corollary 5.3.

(1) [2, Cor. 1.3] highlights some other special cases of the Theorem.
(2) In part (a), S is in fact globally smooth by [18, Thm. 1.2]; see also [2, Cor. 2.3].
(3) We see that the Picard groups aren't very interesting for nicely behaved curves, which explains why we have focused on non-reduced base locus Z. For example, if Z has at worst nodes, then by Corollary 5.3 (a) the Picard group of S is freely generated by H and the components of Z, extending Lopez' theorem [15, II, Thm. 3.1].

Example 5.5. To see what can happen to smooth curves intersecting at a point, consider the simplest case when $Z = \bigcup_{i=1}^{r} L_i$ is the union of r lines passing through p.

(a) If $r = 2$, then S is smooth at p and $\operatorname{Pic} S$ is freely generated by the two lines and H by Corollary 5.3 (a).
(b) If $3 \leq r \leq 5$ and the lines are not coplanar, then p is a fixed singularity, but a mild one. Even when the lines are in general position with respect to containing p, S has an $\mathbf{A}_1$ singularity at p and the map $\operatorname{Cl} S \to \operatorname{Cl} \mathcal{O}_{S,p}$ takes each line to $1 \in \mathbb{Z}/2\mathbb{Z}$. Therefore the Picard group is

$$\operatorname{Pic} S = \{\sum a_i L_i + bH : 2 | \sum a_i\}$$

in this case. We had worked out the case $r = 4$ in [2, Ex. 1.4].
(c) If $r > 5$ and the lines are in general position, then by [3, Cor. 5.2] p is a non-rational singularity and the local class group $\mathcal{O}_{S,p}$ contains an Abelian variety. Moreover the images of the lines are involved in no relations in $\mathcal{O}_{S,p}$, so that $\operatorname{Pic} S = \langle H \rangle$.
(d) There are many ways that the lines can lie in special position. We have not explored all of them, but we did work out the case of r planar lines $L_1, \dots, L_r$ through p union a line L_0 not in the plane, this configuration resembles a pinwheel [3, Ex. 5.3 (b)]. Here the point p is an $\mathbf{A}_{r-1}$-singularity on S and the map $\operatorname{Cl} S \to \widehat{\operatorname{Cl}} \mathcal{O}_{S,p} \cong \mathbb{Z}/r\mathbb{Z}$ sends L_0 to 1 and the other lines L_i to -1. Therefore $\operatorname{Pic} S = \langle rL_0, L_0 + L_1, L_0 + L_2, \dots L_0 + L_r, H \rangle$.

Now we consider examples in which Z is non-reduced, but the set T of embedding dimension three points is non-empty.

Example 5.6. Consider the very general high-degree surface S containing a locally Cohen-Macaulay m-structure Z of generic embedding dimension two supported on a line L.

(a) If Z has embedding dimension two at each point (note that this always holds if $m = 2$), then $\operatorname{Pic} S = \langle H, Z \rangle$ by Cor. 5.3 (c).
(b) If the underlying $(m-1)$-structure has embedding dimension two but Z itself does not, then S has an $\mathbf{A}_{(m-1)q-1}$-singularity at p for some $q > 0$ and the restriction map $\operatorname{Cl} S \to \widehat{\operatorname{Cl}} \mathcal{O}_{S,p} \cong \mathbb{Z}/q(m-1)\mathbb{Z}$ sends L to $q \in \mathbb{Z}/q(m-1)\mathbb{Z}$ by Prop. 4.2. Applying Theorem 5.1 (c) we have

$$\operatorname{Pic} S = \langle mL, H \rangle \cap \langle (m-1)L, H \rangle = \langle m(m-1)L, H \rangle.$$

For example, the very general surface S containing a typical triple line Z supported on L has Picard group $\operatorname{Pic} S = \langle 6L, H \rangle$.
(c) The story is more complicated if the underlying $(m-2)$-structure has embedding dimension two and the underlying $(m-1)$-structure does not because there are two possibilities for the local ideal of Z at p by Prop. 4.3. For the form of the local ideal given in Proposition 4.6 (b), L has order $m-2$ in $\operatorname{Cl} \mathcal{O}_{S,p}$ and Theorem 5.1 gives

$$\operatorname{Pic} S = \langle mL, H \rangle \cap \langle (m-2)L, \mathcal{O}_S(1) \rangle = \langle LCM(m, m-2)L, H \rangle.$$

The actual singularity may be an $\mathbf{A}_n$, an $\mathbf{E}_6$ or even irrational.

Example 5.7. In section 3 consider the very general high degree surface S containing a union of two multiple lines Z_1, Z_2 supported on L_1, L_2.

(a) If $Z_1 \cap Z_2 = \emptyset$, then $\operatorname{Pic} S = \langle Z_1, Z_2, H\rangle$ by Corollary 5.3 (c).

(b) Now suppose $I_{Z_1} = (x, z^m)$ and $I_{Z_2} = (y, z^n)$ with $m \le n$ so that $Z_1 \cap Z_2$ is a length n subscheme supported at $p = (0,0,0,1)$. By Props. 3.3 and 3.4, p is an $\mathbf{A}_{n-1}$ singularity of S and the restriction map $\operatorname{Cl} S \to \operatorname{Cl} \widehat{\mathcal{O}}_{S,p} \cong \mathbb{Z}/n\mathbb{Z}$ takes L_1, L_2 to $1, -1$. Taking the kernel of this map we find that $\operatorname{Pic} S = \langle nL_1, L_1 + L_2, H\rangle$.

(c) Now replace Z_1 with the multiple line with having (x^m, z). The support of $Z_1 \cup Z_2$ is the same as the last example, but now L_2 is contained in the plane $S_1 : \{z = 0\}$ containing Z_1, so L_2 has order of tangency $q = \infty$ to S_1. According to Proposition 3.6, S has an $\mathbf{A}_{mn-1}$ singularity at p and the restriction $\operatorname{Cl} S \to \operatorname{Cl} \widehat{\mathcal{O}}_{S,p} \cong \mathbb{Z}/mn\mathbb{Z}$ takes L_1 to 1 and L_2 to $mn - m$. Therefore $\operatorname{Pic} S = \langle mnL_1, L_2 - (mn - m)L_1, H\rangle$.

Example 5.8. These results can be used in combination, so we close with an example illustrating several behaviors at once. Start with three non-planar lines L_1, L_2, L_3 meeting at p_1. Let Z_4 be a 4-structure on a line L_4 intersecting L_1 at $p_2 \neq p_1$, and assume that Z_4 is contained in a smooth quadric surface Q which is tangent to L_1. Let Z_5 be a 3-structure on a line L_5 which intersects L_2 in a reduced point $p_3 \neq p_1$, and suppose that Z_5 has at least one point $p_4 \neq p_3$ of embedding dimension three. Finally, let Z_6 be a double line supported on L_6 which intersects Z_5 at a point $p_5 \neq p_4, p_3$ and assume that L_6 intersects a local surface S_5 defining Z_5 in a double point. Finally let $Z = L_1 \cup L_2 \cup L_3 \cup Z_4 \cup Z_5 \cup Z_6$ and consider the very general surface S of high degree containing Z.

By Theorem 5.1 (a), $\operatorname{Cl} S$ is freely generated by H and $L_1, L_2, \dots, L_6$ and to find $\operatorname{Pic} S$ we must compute the kernels of the maps $\operatorname{Cl} S \to \operatorname{Cl} \mathcal{O}_{S,p_i}$ for $1 \le i \le 5$:

(1) By Ex. 5.5 (b) the kernel at p_1 is $\langle 2L_1, L_2 - L_1, L_3 - L_1, L_4, L_5, L_6, H\rangle$ and S has an $\mathbf{A}_1$ singularity at p_1.

(2) By Prop. 3.8 with $m = 4, n = 1, q = 2$, the natural restriction map is given by $L_4 \mapsto 1, L_1 \mapsto 2 \in \mathbb{Z}/4\mathbb{Z}$, so the kernel at p_2 is $\langle L_1 - 2L_4, L_2, L_3, 4L_4, L_5, L_6, H\rangle$ and S has an $\mathbf{A}_3$ singularity at p_2.

(3) By Prop. 4.2, the kernel at p_3 is $\langle L_1, L_2 + L_3, 3L_3, L_4, L_5, L_6, H\rangle$.

(4) By Ex. 5.6 (b) the kernel at p_4 is $\langle L_1, L_2, L_3, L_4, 6L_5, L_6, H\rangle$.

(5) By Prop. 3.8 with $n = q = 2, m = 3$, the kernel at p_5 is $\langle L_1, L_2, L_3, L_4, L_6 - 2L_5, 4L_5, H\rangle$.

Using Hermite Normal Form and Mathematica, we compute the intersection of

$$\langle L_1, L_2, L_3, 4L_4, 3L_5, 2L_6, H\rangle$$

and the above kernels to be $\operatorname{Pic} S = \langle 2L_1, 6L_2, L_3 + L_2, 4L_4, 12L_5, 2L_6, H\rangle$.

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