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Bounds on the Rao function

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Abstract

In this article we find upper bounds on the Rao function for space curves in terms of the degree, genus and the minimal degree s of a surface which contains the curve. These bounds are shown to be sharp for $s \leq 4$. This paper is dedicated to David Buchsbaum on the occasion of his 70th birthday. © 2000 Elsevier Science B.V. All rights reserved.

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0. Introduction

In their paper [9], Martin-Deschamps and Perrin find an upper bound on the Rao function of a locally Cohen–Macaulay curve in terms of the degree and genus. Moreover, for each pair (d, g) there exist curves achieving the bound, called extremal curves. Ellia showed that for curves which are not extremal, there is a stronger bound on the Rao function. This was improved slightly by the second author [13] and the improved bounds are sharp (the sharp examples are classified and called subextremal curves). As it turns out, both extremal curves and subextremal curves lie on (degenerate) quadric surfaces. This prompted us to look for a sharp upper bound on the Rao function for curves in terms of the degree, genus, and minimal surface degree s.

The bounds that we obtain hold for curves of degree $d \geq 2s$ and generalize those of Martin-Deschamps and Perrin [9]. The hyperplane arguments of Martin-Deschamps and Perrin [9] and Ph. Ellia [3] alone break down when $s > 2$ is fixed, so we combined these with a study of two combinatorial invariants associated to a curve, namely

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the spectrum (introduced in [15]) and the triangle diagram (introduced in [8]). As a corollary we recover the genus bound of Beorchia and Schlesinger [2,15].

The paper is organized as follows. In the first section, we review some basic properties of the spectrum of a space curve, and use these to deduce bounds on the Rao function on the left (in degrees $< s$). In the second section, we review some properties of triangle diagrams associated to a curve, which lead to bounds for the Rao function on the right (in degrees > 0). Finally, in the third section we give examples of sharpness for $s \leq 4$. In the case $s = 3$, we are able to give sharp upper bounds on the Rao function for all $d \geq 3$.

We work over an algebraically closed field k of characteristic zero. A curve C in $\mathbb{P}^3$ is a locally Cohen–Macaulay closed subscheme of codimension two. Unless otherwise stated, d is the degree of C, g is the arithmetic genus of C, e is the speciality of C, and s is the minimal degree of a surface containing C. The first author would like to thank Uwe Nagel for interesting discussions.

1. The spectrum

In [14] the notion of the spectrum for vector bundles was extended to torsion free sheaves. Expanding on this, Schlesinger defined the spectrum for a space curve in his Ph.D. Thesis [15]. We will recall some of the basic results there and use them to give some bounds on the Rao function of a curve in terms of the degree, genus and minimal surface degree.

Definition 1.1. Let $C \subset \mathbb{P}^3$ be a curve. The *spectrum of* C is the function $h_C(n) = \Delta^2 h^0(\mathcal{O}_C(n))$.

The following proposition shows how the spectrum of a curve, which is a nonnegative finitely supported function, is related to other invariants.

Proposition 1.2. *Let $C \subset \mathbb{P}^3$ be a curve with invariants d, g, e and spectrum h_C. Then $h_C(l) \geq 0$ for all $l \in \mathbb{Z}$ and*
(a) $d = \sum_l h_C(l)$.
(b) $g = 1 + \sum_l (l-1)h_C(l)$.
(c) $h^0(\omega_C(l)) = \sum_{k \leq l} (l-k+1)h_C(2-k)$.
(d) $e + 2 = \max\{l\colon h_C(l) \neq 0\}$.

The following theorem puts certain restraints on the spectrum. We will use these in the sequel to bound the Rao function on the left.

Theorem 1.3. *Let $C \subset \mathbb{P}^3$ be a curve with Rao module M_C. Let $\mu_C(n) = \dim(M_C \otimes_S k)_n$ and $e = e(C)$. Then $e + 2 \geq 0$ and*
(a) $h_C(n) \geq 1 + \mu_C(n)$ *for* $0 \leq n \leq e + 2$.

(b) $h_C(n) = \mu_C(n) = 0$ *for* $n > e + 2$.
(c) *If* $h_C(l) = \mu_C(l) + 1$ *for some* $1 \le l \le e + 2$, *then* $h_C(n) = 1$ *for* $l < n \le e + 2$. *Further, if* $l \le e + 1$, *then* C *contains a plane curve of degree* $e + 3$.

Proof. See [15, Theorem 1.7.1]. □

Proposition 1.4. *Let* $C \subset \mathbb{P}^3$ *be a curve having invariants* d, g, s *and spectrum* h_C. *Then*
(a) $h_C(l) = 0$ *for all* $l > d - s$.
(b) *If* $h_C(d - s) \neq 0$, *then* C *contains a planar subcurve* P *of degree* $d - s + 1$. *If* H *is the plane containing* P *and* $Y = \mathrm{Res}_H(C)$ *is the residual curve, there is an exact sequence*

$$0 \to \mathscr{I}_Y(-1) \to \mathscr{I}_C \to \mathscr{I}_{C \cap H, H} \to 0. \tag{1}$$

Moreover, $s_0(Y) = s - 1$ *and* $h_C(l) = 1$ *for* $2 \le l \le d - s$.
(c) *With the hypotheses of part* (b), *the Rao function is bounded* (*on the left*) *by*

$$h^1(\mathscr{I}_C(l)) \le \begin{cases} 0, & l < g + 1 - \binom{d-s}{2}, \\ l - g + \binom{d-s}{2}, & g + 1 - \binom{d-s}{2} \le l \le 0, \\ \binom{d-s-l}{2} - \binom{l+3}{3} + dl + 1 - g, & 0 \le l < s. \end{cases}$$

Proof. Let $e = e(C)$. Then $\max\{l\colon h_C(l) \neq 0\} = e + 2$ by Proposition 1.2(d). Thus, part (a) becomes the statement $s \le d - e - 2$, proven in [15, Proposition 1.8.1], while part (b) is [15, Proposition 2.8.9]. For the bounds on the Rao function, we note that $h^1(\mathscr{I}_C(l))$ is completely determined by h_C in the range $l < s$. Using arguments similar to those of [13, Proposition 1.8], it is not hard to see that with respect to the restraints of Theorem 1.3, the spectrum

$$\{a\} \cup \{0, 1^{s-1}, 2, 3, \ldots, d - s\} \tag{2}$$

maximizes the Rao function in degrees $< s$, where $a = g + 1 - \binom{d-s}{2}$. Computing the Rao function gives the bounds of part (c). □

Lemma 1.5. *Let* $Z \subset \mathbb{P}^2$ *be a zero-dimensional closed subscheme of length* d *and suppose that* $r = \max\{l\colon h^1(\mathscr{I}_Z(l)) \neq 0\} > d/2 - 1$. *Then there is a unique line* $\{y = 0\} = L \subset \mathbb{P}^2$ *such that length*$(Z \cap L) = r + 2$ *and the greatest common divisor of elements in* $H^0(\mathscr{I}_Z(r + 1))$ *is* y.

Proof. The existence of the line is well known when Z is reduced [6, Theorem 6.1], however, the nonreduced case requires a different proof. Letting $x = 0$ be the equation of a hyperplane H (a line in this case) which misses Z, there is a standard short exact sequence

$$0 \to \mathscr{I}_Z(-1) \xrightarrow{\cdot x} \mathscr{I}_Z \xrightarrow{\pi} \mathscr{O}_H \to 0.$$

Letting $N = H^0_*(\mathscr{O}_H)/(\operatorname{Im} H^0_*(\pi))$, we obtain an exact sequence

$$0 \to N \to H^1_{\geq 0}\mathscr{I}_Z(-1) \xrightarrow{\cdot x} H^1_{\geq 0}\mathscr{I}_Z \to 0.$$

Since N is a principal graded module of finite length, we immediately read off the standard fact that the function $f(n) = h^1\mathscr{I}_Z(n)$ is strictly decreasing for $n \geq 0$ until it becomes zero (since $\Delta f(n) = \dim N_n$). Moreover, the hypothesis $r > d/2 - 1$ shows that $\dim N_n = 1$ for some $0 < n < r+1$ (otherwise $\dim N_n > 1$ for $0 < n < r+1$ which implies that $2r + 1 \leq h^1\mathscr{I}_Z = d - 1$, a contradiction). Finally, since N is a standard k-algebra, Macaulay's growth bound (see [5, Theorem 1]) shows that if $\dim N_n = 1$ for some $n > 0$, then the same holds until it becomes zero. In particular, we deduce that $h^1\mathscr{I}_Z(r)=1$ and $h^1\mathscr{I}_Z(r-1)=2$. It follows that there is a linear form $0 \neq y \in H^0\mathscr{O}_{\mathbb{P}^2}(1)$ such that the map $H^1\mathscr{I}_Z(r-1) \xrightarrow{\cdot y} H^1\mathscr{I}_Z(r)$ is zero.

Let L be the line $\{y = 0\}$ and let $Y = \operatorname{Res}_L(Z)$ be the residual scheme of Z with respect to L. Then we obtain a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{I}_Z(-1) & \xrightarrow{\cdot y} & \mathscr{I}_Z & \longrightarrow & K & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathscr{I}_Y(-1) & \xrightarrow{\cdot y} & \mathscr{I}_Z & \longrightarrow & \mathscr{I}_{Z\cap L,L} & \longrightarrow & 0,
\end{array}$$

and the snake lemma shows that the kernel of the map $K \to \mathscr{I}_{Z\cap L,L}$ is $\mathscr{I}_{Y,Z}(-1)$, hence a sheaf of finite length. In particular, $H^1_*(K) \cong H^1_*(\mathscr{I}_{Z\cap L,L})$ and hence the facts that $H^1(K(r)) \neq 0$ and $H^1(K(r+1)) = 0$ show that $\mathscr{I}_{Z\cap L,L} \cong \mathscr{O}_L(-r-2)$, whence $\operatorname{length}(Z\cap L) = r+2$. Clearly, the line L is unique, for if L' were another such, then the length of $Z \cap (L \cup L')$ is at least $2r + 3 > d$, a contradiction. Finally, since $Y \subset \mathbb{P}^2$ is a scheme of length $d - r - 2 < r + 2$, $\mathscr{I}_Y$ is $(r+1)$-regular and hence the GCD of elements in $H^0(\mathscr{I}_Y(r+1))$ is 1. It follows that the GCD of elements in $H^0(\mathscr{I}_Z(r+1))$ is y. $\square$

Corollary 1.6. *Let $C \subset \mathbb{P}^3$ be a curve of degree d, genus g and $s_0(C)=s > 2$. Assume that $d \geq 2s$. If $r_0(C) = \max\{n\colon h^1(\mathscr{I}_C(n)) \neq 0\} > d - s$, then the Rao function $h^1(\mathscr{I}_C(l))$ is strictly decreasing on $[d-s, r_0(C)]$.*

Proof. We first claim that $h^1(\mathscr{I}_{C\cap H,H}(d-s)) = 0$ for the general hyperplane $H \subset \mathbb{P}^3$. Supposing that this is not the case, let $r \geq d-s$ be the integer such that $h^1(\mathscr{I}_{C\cap H,H}(r)) \neq 0$ for general H while $h^1(\mathscr{I}_{C\cap H,H}(r+1))=0$. Since $r > d/2-1$, we may apply Lemma 1.5 to see that there is a unique line $L_H \subset H$ such that length $(C\cap L)=r+2$. Moreover, all curves in H of degree $r+1$ containing $C \cap H$ contain the line L as a fixed curve. In this situation we can apply a result of Strano [16, Lemma 2] to see that there is a subcurve $C' \subset C$ whose general hyperplane section is $C' \cap H = C \cap L$. The curve C' has degree $r+2$ and has general restriction hyperplane section $C' \cap H$ contained in a line L. Since $r+2 \geq 3$, it follows from Hartshorne's restriction theorem [7, Theorem 2.1] that C' in planar. In particular, $e(C')=r-1$ and hence $e(C) \geq r-1 \geq d-s-1 > d-s-2$. In view of Proposition 1.4, this contradicts the hypothesis that $s_0(C)=s$.

Given the vanishing above, we see that for general $h \in H^0\mathcal{O}_{\mathbb{P}^3}(1)$ yielding the hyperplane H, the ideal sheaf $\mathcal{I}_{C\cap H,H}$ is $(d-s+1)$-regular. It follows by a standard Castelnuovo–Mumford regularity argument (see [11, p. 102]) that $h^1(\mathcal{I}_C(l))$ is strictly decreasing for $l \geq d-s$. $\square$

2. Triangles

In this section we introduce the triangle diagrams used by Liebling in his Ph.D. Thesis [8]. The triangle diagram is a discrete invariant of a curve which captures quite a lot of information, including all the dimensions of associated cohomology groups. Using various restrictions on these triangle diagrams, we will give a bound on the Rao function in the range $[s, d-s]$ for curves C of degree d with $s_0(C)=s$. Combined with the results of the previous section, this will give an upper bound on the Rao function.

We begin by describing the triangle diagrams that we will use (see [8, Chapters 2 and 3] for a more complete account). The homogeneous coordinate ring of $\mathbb{P}^3$ is $S=k[x_0,x_1,x_2,x_3]$. For multi-indexes $I=(i_0,i_1,i_2,i_3)$ and $J=(j_0,j_1,j_2,j_3)$ corresponding to monomials in S, we define the reverse lexicographical (revlex) ordering on monomials by saying that $I>J$ if $\sum i_k > \sum j_k$ or $\sum i_k = \sum j_k$ and there exists l such that $i_l < j_l$ and $i_n = j_n$ for $n > l$. For each homogeneous polynomial $0 \neq f = \sum a_i x^I$, we define its *initial monomial* by $\text{in}(f) = \max\{x^I \colon a_I \neq 0\}$. Similarly, for a homogeneous ideal $I \subset S$ we define its *initial ideal* in(I) by taking the ideal generated by $\{\text{in}(f) \colon f \in I\}$. It is well known that the initial ideal in(I) is a monomial ideal having the same Hilbert function as I.

If $g \in \text{GL}(4)$ is a change of coordinates, then $g(I) \subset S$ and we may consider its initial ideal $\text{in}(g(I))$. There is an open subset $U \subset \text{GL}(4)$ such that $\text{in}(g(I))$ is constant for all $g \in U$: this constant monomial ideal is called the *generic initial ideal* of I, denoted gin(I). The generic initial ideal is known to be Borel-fixed, which means (when char $k=0$) that if $m \in \text{gin}(I)$ is a monomial and x_j divides m, then $x_i/x_j \cdot m \in \text{gin}(I)$ for each $i<j$.

Now, let $C \subset \mathbb{P}^3$ be a locally Cohen–Macaulay curve defined by the saturated homogeneous ideal $I=I_C$. In this case gin(I) is a saturated ideal which defines a subscheme of $\mathbb{P}^3$ supported on the line $\{x_0=x_1=0\}$ which is locally Cohen–Macaulay except possibly at the point $\{x_0=x_1=x_2=0\}$. We define the *lower triangle diagram* associated to C to be the function $\Delta_0(C)=\Delta_0 : \mathbb{N}\times\mathbb{N} \to \mathbb{N}\cup\{\infty\}$ given by $\Delta_0(i,j) = \min\{k \colon x_0^i x_1^j x_2^k \in \text{gin}(I)\}$ ($\Delta_0(i,j)=\infty$ if there is no such k).

Similarly, we may define an *upper triangle diagram* associated to C as follows. Let X be a complete intersection curve containing X whose generic initial ideal is (x_0^r, x_1^s) for some $r,s>0$. If D is the curve algebraically linked to C by X, then the upper triangle diagram of C is the function $\Delta_1 : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \cup \{\infty\}$ given by

$$\Delta_1(i,j) = \begin{cases} \Delta_0(D)(r-1-i, s-1-j) & \text{if } i \le r-1 \text{ and } j \le s-1, \\ \infty & \text{otherwise.} \end{cases}$$

It is proven by Liebling that this definition does not depend on the complete intersection X chosen, and further that $\Delta_0(C)(i,j) = \infty$ if and only $\Delta_1(C)(i,j) \neq \infty$ (see [8, Proposition 3.4.4]). In this way we can fit the data of these functions together in one tableau of finite numbers. We say that the pair $\Delta = (\Delta_0(C), \Delta_1(C))$ is the triangle diagram associated to the curve C.

Remark 2.1. The actual definition of the lower triangle given by Liebling is more involved (it is based on his notion of the initial module to a curve), but it is proven to be equivalent to the one given here in [8, Proposition 3.4.4]. There is also a notion of higher initial ideal due to Fløystad, who together with Green proves that knowing the higher initial ideal of a curve is equivalent to knowing the generic initial ideal of a linked curve [4, Corollary 7.11]. While the definitions of the initial module and higher initial ideal appear to be quite different, these results make it clear that they must be closely related. It would be interesting to discover the precise relationship between them.

The next two results describe some restrictions on triangles.

Proposition 2.2. *Let $C \subset \mathbb{P}^3$ be a curve. Then the triangle diagram $\Delta = (\Delta_0, \Delta_1)$ satisfies the following conditions*:

1. $\Delta_1(0,0) = 0$.
2. *There exists $N > 0$ such that $\Delta_0(i,j) = 0$ for all (i,j) satisfying $i + j > N$.*
3. $\Delta_1(i,j) = \infty \Leftrightarrow \Delta_0(i,j) \neq \infty$.

4a. *If $\Delta_0(i,j) \neq \infty$, then $\Delta_0(i,k) \neq \infty$ for $k \ge j$.*
4b. *If $\Delta_0(i,j) \neq \infty$ and $j > 0$, then $\Delta_0(i+1, j-1) \neq \infty$.*

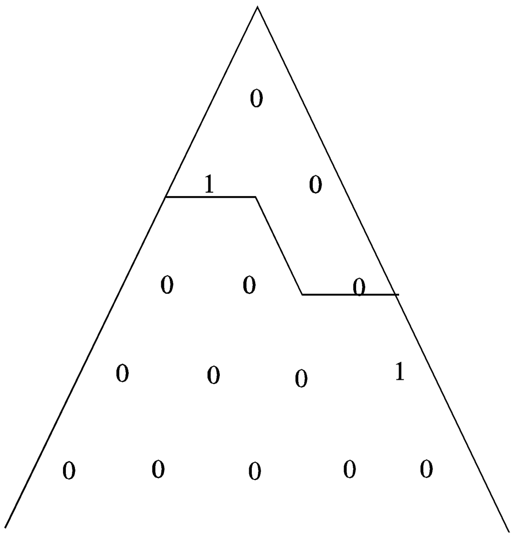
5. *Letting $\mathscr{A}_\Delta = \{(i,j)\colon 0 < \Delta_1(i,j) < \infty\}$ and $\mathscr{B}_\Delta = \{(i,j)\colon 0 < \Delta_0(i,j) < \infty\}$, it holds that*

$$\sum_{(i,j)\in\mathscr{A}_\Delta} \Delta_0(i,j) = \sum_{(i,j)\in\mathscr{B}_\Delta} \Delta_1(i,j).$$

Proof. See [8, Definitions 2.3.18 and 3.4.1 and Proposition 3.4.4]. □

Condition 3 says that the triangle is separated into two distinct regions (where either $\Delta_1 < \infty$ or $\Delta_0 < \infty$). Condition 4 describes the nature of the dividing line between these two regions. The typical way to draw these triangles is with the origin $(0,0)$ at the top with the x-axis dropping to the lower left, the y-axis dropping to the lower right. In this case, condition 4 says that the dividing line between the regions does not increase as it goes from left to right. We say that a pair of functions (Δ_0, Δ_1) is a *weak triangle diagram* (or simply a weak triangle) if it satisfies the conclusion of the above proposition.

Example 2.3. Consider the curve Y with total ideal $(x^2, xy, y^4, xw^3 - y^3z)$. Y is an extremal curve of degree 4 and arithmetic genus 0 (see [10, Proposition 0.5]). Here the generic initial ideal is (x^2, xy, y^4, y^3z) and the triangle diagram is given below. There is a line dividing the upper and lower triangle regions, and the zeros continue at the bottom of the diagram.



Proposition 2.4. *Let $C \subset \mathbb{P}^3$ be a curve. Then the weak triangle diagram $(\Delta_0(C), \Delta_1(C))$ satisfies the following conditions*:
1. $\Delta_0(i, j+1) < \Delta_0(i,j)$ *unless both are* 0 *or* ∞.
2. $\Delta_0(i+1, j-1) \le \Delta_0(i,j)$ *unless both are* 0 *or* ∞.
3. $\Delta_1(i,j) < \Delta_1(i+1,j)$ *unless both are* 0 *or* ∞.
4. $\Delta_1(i,j) < \Delta_1(i,j+1)$ *unless both are* 0 *or* ∞.
5. *For each $n \in \mathbb{N}$ we have the inequality*

$$\sum_{\substack{i+j \le n \\ (i,j) \in \mathscr{A}_\Delta}} \Delta_1(i,j) \ge \sum_{\substack{i+j \le n \\ (i,j) \in \mathscr{B}_\Delta}} \Delta_0(i,j).$$

Proof. The first four statements follow from Borel fixedness of generic initial ideals, and can be found in [8, 5.1.1 and 5.1.4]. The fifth statement is [8, Proposition 5.1.8]. □

Proposition 2.5. *Let $C \subset \mathbb{P}^3$ be a curve having triangle diagram $\Delta = (\Delta_0, \Delta_1)$. Define the functions $A, B : \mathbb{Z} \to \mathbb{N}$ by*

$$A(n) = \#\{(i,j) \colon i + j - \Delta_1(i,j) = n\},$$
$$B(n) = \#\{(i,j) \colon i + j + \Delta_0(i,j) = n\}.$$

Then the dimensions of the cohomology groups of the ideal sheaf $\mathscr{I}_C$ *are*

0. $h^0(\mathscr{I}_C(n)) = \sum_{k=0}^{n} (n-k+1)B(k)$.
1. $$h^1(\mathscr{I}_C(n)) = \sum_{(i,j)\in\mathscr{A}_\Delta} \min(\Delta_1(i,j), \max(n+1-i-j+\Delta_1(i,j),0)) - \sum_{(i,j)\in\mathscr{B}_\Delta} \min(\Delta_0(i,j), \max(n+1-i-j,0)).$$
2. $h^2(\mathscr{I}_C(n)) = \sum_{k=n+2}^{\infty} (k-n-1)A(k)$.
3. $h^3(\mathscr{I}_C(n)) = \binom{-n-1}{3}$.

Moreover, the function A *is the spectrum of* C.

Proof. See [8, Proposition 3.5.1]. $\square$

Definition 2.6. The *cohomology* $h^i(\Delta,n)$ of a weak triangle diagram $\Delta=(\Delta_0,\Delta_1)$ is given by the formulas for $h^i(\mathscr{I}_C(n))$ in Proposition 2.5 and the condition that $h^i(\Delta,n)=0$ for $i<0$ and $i>3$.

Remark 2.7. From the cohomology of a triangle, we can of course define other typical curve invariants associated to a triangle.

(a) If Δ is the triangle of a curve $C\subset\mathbb{P}^3$, then by definition $h^i(\Delta,n)=h^i(\mathscr{I}_C(n))$ for all integers i and n.

(b) In view of the previous definition, we also have a notion of the *spectrum* of Δ. In particular, we have formulas for the degree, genus, and minimal surfaces degree:
$d(\Delta) = \#\{(i,j)\colon \Delta_1(i,j)\neq\infty\}$,
$g(\Delta) = \sum_{i=0}^{i=3} (-1)^i h^i(\Delta,0)$,
$s(\Delta) = \min\{l\colon h^0(\Delta,l)\neq 0\}$.

(c) We can also use the notion of Euler characteristic by setting $\chi(\Delta,n) = \sum_i (-1)^i h^i(\Delta,n)$. If d,g are the degree and genus of the triangle Δ, then an induction proof on $\sum_{(i,j)\colon\ \Delta_0(i,j)\neq\infty} \Delta_0(i,j)$ shows that

$$\chi(\Delta,n) = \binom{n+3}{3} - dn - 1 + g.$$

Since the induction step is quite easy, we give the induction base case here. When the sum is zero, the functions Δ_0 and Δ_1 only take the values 0 and ∞. In this case, the monomials $\{x^iy^j\colon \Delta_0(i,j)\neq\infty\}$ generate the total ideal of an arithmetically Cohen–Macaulay curve C having triangle diagram Δ, when the result follows from Proposition 2.5.

It is also possible to read of the numerical character of the general hyperplane sections from the triangle diagram.

Proposition 2.8. *Let $C \subset \mathbb{P}^3$ be a curve having triangle diagram $\Delta = (\Delta_0, \Delta_1)$. Let $\sigma = \min\{i\colon \Delta_1(i,0) = \infty\}$ and for each $0 \le i < \sigma$ let $\lambda_i = \min\{j\colon \Delta_1(i,j) = \infty\}$. Then the integers $n_i = \lambda_i + i$ give the numerical character $\{n_0, n_1, \dots, n_{\sigma-1}\}$ of the general hyperplane section $C \cap H$.*

Proof. See [8, Remark 5.3.6]. □

Corollary 2.9. *Let $C \subset \mathbb{P}^3$ be a curve with invariants d and s. Then $\Delta_1(0, d-s+1) = \infty$.*

Proof. If $\Delta_1(0, d-s+1) \neq \infty$, then from Proposition 2.8 we see that the numerical character $\{n_i\}$ of the general hyperplane section $C \cap H$ satisfies $n_0 > d-s+1$. However, the numerical character determines all the cohomology of $\mathscr{I}_{C\cap H,H}$ and in particular we find that $h^1(\mathscr{I}_{C\cap H,H}(d-s)) \neq 0$ for general H. Following the proof of Corollary 1.6, we see that in this case $s_0(C) < s$, a contradiction. □

Proposition 2.10. *Let $C \subset \mathbb{P}^3$ be a curve with invariants d, g, s and assume that $d \ge 2s$. Then*

$$h^1(\Delta, l) \le \binom{d-s}{2} - \binom{s-1}{3} - g$$

for all $s \le l \le d-s$.

Proof. We will in fact prove this for all weak triangles Δ with invariants d, g, s as above satisfying the additional condition that $\Delta_1(0, d-s+1) = \infty$, which we may assume from Corollary 2.9. We achieve this by altering the triangle while not decreasing h^1 on $[s, d-s]$, and then calculating h^1 of the resulting triangle. We induct on $h = \min\{l\colon \Delta_1(0, d-s-l+1) = \infty\}$. Since $\Delta_1(0, d-s+1) = \infty$, we have that $h \ge 0$.

The induction base $h = 0$ will require two modifications. For the first modification, we define Δ^1 by $\Delta^1_0 = \Delta_0$, $\Delta^1_1(0,1) = \Delta_1(0,1) + \sum_{l=s}^{d-s} \Delta_1(0,l)$ and $\Delta^1_1(0,l) = 0$ for $s \le l \le d-s$. Δ^1 is a weak triangle and from the formula of Proposition 2.5(1), we see that $h^1(\Delta, l) \le h^1(\Delta^1, l)$ for $s \le l \le d-s$.

For the second modification, note that $s = s(\Delta)$ implies that if $i+j < s$ and $\Delta_0(i,j) \neq \infty$, then $\Delta_0(i,j) \ge s-i-j$ (if this fails, then the function B of 2.5 is nonzero for some value less than s). We now define Δ^2 by minimizing these values. Specifically, Δ^2 is defined by $\Delta^2_1 = \Delta^1_1$, $\Delta^2_0(i,j) = s-i-j$ for (i,j) satisfying $i+j < s$ and $\Delta^1_0(i,j) < \infty$ and $\Delta^2_0(i,j) = 0$ for (i,j) satisfying $s \le i+j \le d-s$ and $\Delta^1_0(i,j) < \infty$. So far we have decreased $\sum \Delta_0(i,j)$ by

$$M = \sum_{\substack{i+j<s \\ \Delta^1_0(i,j)<\infty}} \Delta^1_0(i,j) - s + i + j + \sum_{\substack{s \le i+j \le d-s \\ \Delta^1_0(i,j)<\infty}} \Delta^1_0(i,j).$$

Thus, to maintain Property 5 of weak triangle, we also set $\Delta^2_0(0, d-s+1) = \Delta^1_0(0, d-s+s) + M$. Once again Δ^2 is a weak triangle satisfying the hypothesis of the proposition, and $h^1(\Delta^1, l) \le (\Delta^2, l)$ for $s \le l \le d-s$.

Finally, we compute h^1 of the new triangle Δ^2. Observe that $h^1(\Delta^2, l)$ is constant for $s-1 \le l \le d-s$ because both sums in the formula of Proposition 2.5(1) are constant in this range. Thus, it suffices to compute $h^1(\Delta^2, s-1)$. Given the condition on the shape of a weak triangle (condition 4 of Proposition 2.2), the fact that $\Delta_1^2(0, d-s) < \infty$ forces $\Delta_1^2(i,j) = \infty$ for all (i,j) satisfying $i+j \ge s$ and $i > 0$ (here the condition $d \ge 2s$ is needed). By construction, we have $\Delta_1^2(0,l) = 0$ for $s \le l \le d-s$. It follows that if A is the function from Proposition 2.5, then $A = 1$ on $[s, d-s]$ and $A = 0$ on $(d-s, \infty)$. From this we find that $h^2(\Delta^2, s-1) = \binom{d-2s+1}{2}$. Now, we can use Remark 2.7(c) and the fact that $h^0(\Delta^2, s-1) = h^3(\Delta^2, s-1) = 0$ to see that

$$h^1(\Delta^2, s-1) = \binom{d-s}{2} - \binom{s-1}{3} - g.$$

The induction step $h > 0$ is relatively easy, as only one alteration to the triangle is required to apply the induction hypothesis. The condition $d \ge 2s$ implies that there exists (i,j) such that $i > 0$ and $\Delta_1(i,j) \ne \infty$. Let $I = \max\{i\colon \Delta_1(i,0) \ne \infty\}$ and $J = \max\{j\colon \Delta_1(I,j) \ne \infty\}$. We will alter the shape by moving the cell at (I,J) to $(0, d-s-h+1)$. For ease of notation, we set $K = d-s-h+1$ (so we move from (I,J) to $(0,K)$).

Define Δ' as follows. Away from the pairs $(I,J), (0,K)$ and $(0,K+1)$, we set $\Delta' = \Delta$. Let $\Delta_1'(0,K) = \Delta_1(I,J) + K - I - J$, $\Delta_1'(I,J) = \infty$ and $\Delta_1'(0,K+1) = \infty$. Note that the function A of Proposition 2.5 is the same for Δ and Δ', hence the degree and genus are also unchanged. Next set $\Delta_0'(I,J) = \max\{s-I-J, 0\}$ (this assures that the value of s is the same for both triangles) and $\Delta_0'(0,K) = \infty$. So far the sum $\sum \Delta_1(i,j)$ has increased by $K-I-J$ while $\sum \Delta_0(i,j)$ has increased by $\max\{s-I-J, 0\} - \max\{s-K, 0\} - \Delta_0(0,K)$. To maintain Property 5 we set $\Delta'(0,K+1) = \Delta_0(0,K+1) + \Delta_0(0,K) - \max\{s-I-J,0\} + \max\{s-K,0\} + K - I - J$ (this number is nonnegative, and strictly positive when $I+J < s < K$). Thus, Δ' is a weak triangle with the same invariants whose value of h is one less. Moreover, $h^1(\Delta', l) \ge h^1(\Delta, l)$ for all l (they are equal when $I+J < K \le s$). Applying the induction hypothesis to Δ' gives the bound. $\square$

Corollary 2.11. *Let $C \subset \mathbb{P}^3$ be a curve with invariants d, g, s and assume that $d \ge 2s$. Then $g \le \binom{d-s}{2} - \binom{s-1}{3}$.*

This bound on the genus was originally proved by Beorchia [2] assuming char $k = 0$ and later by Schlesinger [15] in arbitrary characteristic.

Corollary 2.12. *Let $C \subset \mathbb{P}^3$ be a curve of degree d, genus g and minimal surface degree s. If $d \ge 2s$, then the Rao function satisfies the following bounds*:

$$h^1(\mathscr{I}_C(l)) \le \begin{cases} \binom{d-s}{2} - \binom{s-1}{3} - g, & s \le l \le d-s, \\ \left[\binom{d-s}{2} - \binom{s-1}{3} - g + d - s - l\right]_+, & d-s < l. \end{cases}$$

If we further assume that $e = d - s - 2$, *then we have the bounds*

$$h^1(\mathscr{I}_C(l)) \leq \begin{cases} \left[l - g + \binom{d-s}{2} \right]_+, & l \leq 0, \\ \binom{d-s-l}{2} - \binom{l+3}{3} + dl + 1 - g, & 0 \leq l < s. \end{cases}$$

Looking at the left and right ends of the Rao function, we obtain a generalization of [9, Corollaire 2.6].

Corollary 2.13. *Let* C *be as in Corollary* 2.12. *Then*

$$r_0(C) \leq \binom{d-s+1}{2} - \binom{s-1}{3} - g - 1$$

and if further $e = d - s - 2$ *then*

$$r_a(C) = \min\{l \colon h^1(\mathscr{I}_C(l)) \neq 0\} \geq g + 1 - \binom{d-s}{2}.$$

3. Examples

In this section we show that the bounds given in the previous section are sharp for $s \leq 4$ and mention a few problems left open.

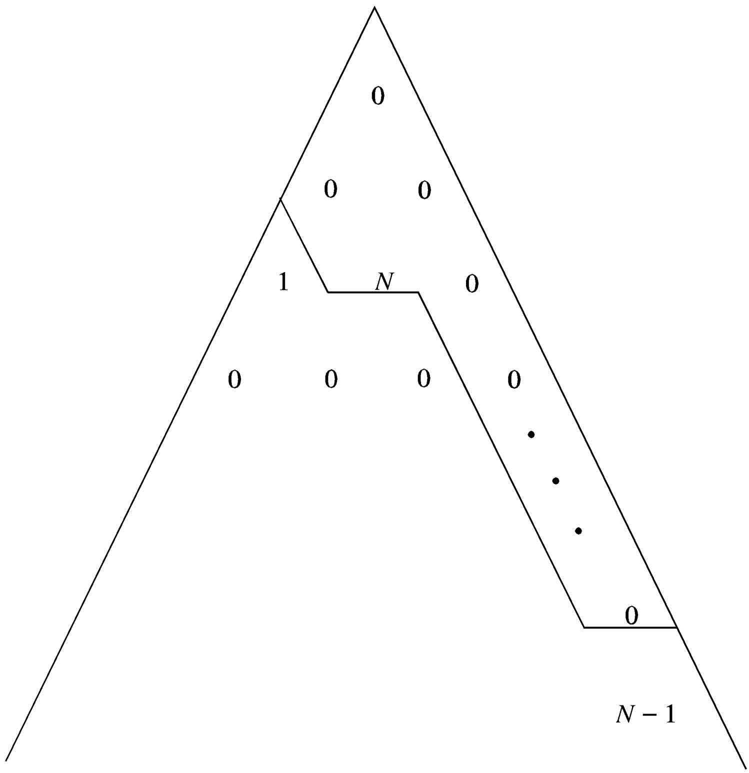
Example 3.1. For $s = 2$, the bounds of Corollaries 2.12 and 2.13 were proved by Martin-Deschamps and Perrin [9] for all $d \geq 2$ (not only $d \geq 2s = 4$) and $g \leq \binom{d-2}{2}$. In fact, the extremal curves studied in [10] show that the bound is sharp for all such pairs (d, g).

Example 3.2. For $s = 3$ we construct curves which give equality in Corollaries 2.12 and 2.13 for all $d \geq 4$ and $g \leq \binom{d-3}{2}$. Let Z be a double line of genus $g - \binom{d-3}{2} \leq 0$. If H is a plane containing the support of Z, then the general plane curve P of degree $d - 3$ will meet Z in $d - 3$ reduced points, so that the union $E = Z \cup P$ is a curve of genus exactly g. Moreover, E is an extremal curve by the criterion of Ph. Ellia [3, Section 2, Theorem 8] and this determines its Rao function (see [13, Proposition 2.1] for the Rao function). Now, we take our curve C to be the union of E and a general line L which meets P at a point Q not in the support of Z. It now follows from the exact sequence

$$0 \to \mathscr{I}_C \to \mathscr{I}_E \oplus \mathscr{I}_L \xrightarrow{\pi} \mathscr{I}_Q \to 0$$

that $h^1(\mathscr{I}_C(l))$ achieves the bound of Corollaries 2.12 and 2.13.

Remark 3.3. We can make a stronger statement in the case $s = 3$. We proved the upper bound on the Rao function on the range $[s, d-s]$ by manipulating triangle diagrams. The final maximizing triangle in the case $s = 3$ can be further modified (without decreasing the Rao function) to the triangle described as follows. Setting $N = \binom{d-3}{2} - g + 1$, we let $\Delta_0(1,1) = N$, $\Delta_0(i,j) = 0$ for $(i,j) \in \{(1,0),(0,0),(0,1),\dots,(0,d-3)\}$ and $\Delta_0(i,j) = \infty$ otherwise. We set $\Delta_1(i,j) = \infty$ when $\Delta_0(i,j) < \infty$. For the finite values, set $\Delta_1(2,0) = 1$, $\Delta_1(0,d-s+1) = N-1$ and $\Delta_1(i,j) = 0$ for the rest.



This triangle maximizes the Rao function everywhere (subject to the necessary conditions on a triangle), and yields precisely the upper bound of Corollary 2.12 for $d \geq 5$ (a similar ad hoc triangle argument shows that the bound also holds for $d = 4$). Thus for $d \geq 4$, the bounds of Corollary 2.12 hold and Examples 3.2 give sharpness.

This leaves the case $d=3$ and $s=3$. However [12], gives a classification of the curves of degree 3 along with the calculation of their Rao modules. Using this classification, we find that a curve C of degree 3, minimal surface degree 3 and genus $g \leq -3$ (for higher g no such curves exist) has Rao function bounded by

$$h^1(\mathscr{I}_C(l)) \leq \begin{cases} 0, & l \leq g+1, \\ l-g-1, & g+1 < l < 0, \\ -g, & 0 \leq l \leq 1, \\ -g-l-1, & 1 < l < 1-g, \\ 0, & l \geq 1-g. \end{cases}$$

Thus, we have a complete solution to the problem for $s = 3$.

Example 3.4. Here we show that the bounds of Corollaries 2.12 and 2.13 are sharp on the left when $s=4$. Fix $d \geq 8$ and $g \leq \binom{d-4}{2} - 1$. Setting $b = \binom{d-4}{2} - 1 - g \geq 0$, we let Z_3 be a triple structure on a line L of type $(0,b)$, which means that the Cohen–Macaulay filtration of Z_3 gives rise to exact sequences (see [1])

$$0 \to \mathcal{O}_L(0) \to \mathcal{O}_L \to \mathcal{O}_{Z_2} \to 0, \tag{3}$$

$$0 \to \mathcal{O}_L(0) \to \mathcal{O}_{Z_2} \to \mathcal{O}_{Z_3} \to 0, \tag{4}$$

where Z_2 is the underlying double structure on L (see [12, Section 2] for existence of such Z_3. Here Z_2 is the double line of genus -1, which has Rao module k in degree 0). Let H be a general plane which meets L at the point p, and let $R = Z_2 \cap H$. $R \subset H$ is a zero-dimensional scheme of length 2 supported at p and we can find a plane curve $P = P_{d-3} \subset H$ of degree $d-3$ which contains R but does not contain $Z_3 \cap H$. In this situation we see that $Z_3 \cap P = R$.

Consider the curve $C = Z_3 \cup_R P$. First, we check that $s=4$. That $s_0(Z_3) = 3$ follows from the classification in [12], so certainly we have $s \leq 4$, P being planar. Since $d \geq 8$, the cubic surfaces containing P have H as a component. If such a cubic $H \cup Q$ contains Z_3, then since H meets Z_3 properly we would have $Z_3 \subset Q$, a contradiction. Thus, there are no cubics containing both P and Z_3, showing that $s \geq 4$. A careful analysis of the exact sequence

$$0 \to \mathcal{I}_C \to \mathcal{I}_P \oplus \mathcal{I}_{Z_3} \xrightarrow{\pi} \mathcal{I}_R \to 0 \tag{5}$$

shows that $h^1(\mathcal{I}_C(l))$ achieves the upper bound of Corollary 2.12 for $l < 0$ and the bound for $r_a(C)$ of Corollary 2.13.

Example 3.5. For sharpness of the bounds on the right in the case $s = 4$, we take the union of an extremal curve E and a double line L_2 meeting at a double point. Specifically, E to be the extremal curve with total ideal

$$I(E) = (x^2, xy, hy^2, xw^{a+d-4} + hyz^a)$$

and L_2 to be the double line with total ideal $I(L_2) = (z^2, zw, w^2, x^2z - y^2w)$ we find that the bounds on the right are achieved and the bound for $r_0(C)$ for Corollary 2.13 as well.

Remark 3.6. It would be interesting to find sharp bounds for curves satisfying $s \leq d < 2s$. In view of the examples above, we can expect the worst cohomological behavior of these curves to give insight into the problem of finding a sharp bound on the Rao function when for $s > 4$.

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