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Developments in Noether-Lefschetz Theory

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Dedicated to Phillip Griffiths

Abstract. The Noether-Lefschetz theorem was stated in the 1880s, proved in the 1920s, and flourished in the 1980s, in large part due to the influence of Phillip Griffiths and his students. In this survey we examine the refinements to the original theorem and work done on the Noether-Lefschetz components.

Introduction

Let X be a smooth complex projective variety. The Picard group $\operatorname{Pic} X$ of line bundles on X modulo linear equivalence is a classical invariant. While a curve X is essentially determined by $\operatorname{Pic} X$ (or rather its jacobian[1]), this is far from true in higher dimensions. When X is a smooth quadric or cubic surface in $\mathbb{P}^3$, the Picard group is understood in terms of the geometry of lines lying on these surfaces, but in the absence of special geometry $\operatorname{Pic} X$ is difficult to compute. On the other hand, there are good results comparing the Picard group of a variety X to that of the general member Y of a complete linear system associated to an ample line bundle. Specifically, one can ask whether the restriction map $\operatorname{Pic} X \to \operatorname{Pic} Y$ is an isomorphism; this is typically false if $\dim X = 2$, true if $\dim X \geq 4$, and difficult if $\dim X = 3$. The precise result for $X = \mathbb{P}^3_{\mathbb{C}}$ is that for $d \geq 4$, the restriction map $\operatorname{Pic} X \to \operatorname{Pic} S$ is an isomorphism for all surfaces S outside of a countable union of proper subvarieties of the space of degree-d surfaces. This is Noether's theorem, a high point in algebraic geometry and Hodge theory. Here we aim to survey this important theorem and surrounding developments.

We begin with the history before 1980, starting with Noether's original 1882 statement and his likely motivations in Section 1.1. Lefschetz proved the theorem in the 1920s using topological methods – to quote Lefschetz, "It was my lot to plant the harpoon of algebraic topology into the body of the whale of algebraic geometry." We sketch his proof in Section 1.2, describing Lefschetz pencils, vanishing cycles, the monodromy action and the Picard-Lefschetz formula. The theorem lay dormant

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[1]considered as a principally polarized abelian variety, this is the Torelli theorem [**3**, VI, § 3].

until the late 1950s, when algebraic geometry got a boost from Grothendieck's unifying theory of schemes and mathematicians looked at old problems in a new light. By the late 1960s the theorem had been revisited: Moishezon used Lefschetz' ideas to extend Noether's theorem to general smooth complex projective varieties while Grothendieck, Deligne and Katz used l-adic cohomology to prove a generalization for algebraic cohomology classes on complete intersections in projective spaces valid in characteristic $p > 0$. These matters are discussed in Sections 1.3 and 1.4. In Section 1.5 we discuss the Grothendieck-Lefschetz theorem for higher dimensions, which arises from completely different considerations.

Section 2 is devoted to refinements of Noether's theorem since 1980. The subject was infused with new ideas coming from infinitesimal variations of Hodge structures, as established in the 1983 foundational paper of Griffiths and his students Carlson, Green and Harris, setting off a flurry of results from many contributors. In Section 2.1 we review Hodge structures and their infinitesimal deformations, especially for families of hypersurfaces in $\mathbb{P}^n$. With this language we state the infinitesimal Noether theorem in Section 2.2, giving a brief sketch of the proof. In Section 2.3 we discuss Green's theory of Koszul cohomology and its role in the explicit Noether theorem, which improves the classical theorem by giving a sharp lower bound on the dimensions of the Noether-Lefschetz components, that is, those components of the family of surfaces for which Noether's theorem fails. The year 1985 saw two more important developments: Griffiths and Harris offered a new proof of the classic theorem by a degeneration argument which almost entirely avoids Hodge theory; Ein extended Noether's theorem to general dependency loci of vector bundles of any rank on an arbitrary projective n-fold. These are covered in Sections 2.4 and 2.5 respectively. In 1995 Joshi proved a variant for smooth threefolds using an infinitesimal approach from unpublished notes of Mohan Kumar and Srinivas, but more importantly he observed that the result holds for general *singular* surfaces, this is the topic of Section 2.6. Section 2.7 describes the more recent work of Ravindra and Srinivas which give analogs of Noether's theorem and the Grothendieck-Lefschetz theorem for class groups of hyperplane sections of normal ambient varieties. The last section discusses our variant of Noether's theorem with base locus.

In Section 3 we discuss the irreducible components V of the Noether-Lefschetz locus, the countably infinite union of proper families of degree-d surfaces in $|\mathcal{O}_{\mathcal{P}^3}(d)|$ for which Noether's conclusion fails. The codimension $c(V) = \text{codim}(V, |\mathcal{O}_{\mathcal{P}^3}(d)|)$ satisfies $d-3 \leq c(V) \leq p_g = \binom{d-1}{3}$: the components V satisfying $c(V) = p_g$ are *general* and the rest are *special*. In Section 3.1 we present the density theorem of Ciliberto, Harris and Miranda stating that the general Noether-Lefschetz components are Zariski dense in $|\mathcal{O}_{\mathcal{P}^3}(d)|$ and sketch Green's proof of density in the Euclidean topology. Section 3.2 is devoted to the work of Green and Voisin on the components of small codimension, along with Otwinowska's more recent asymptotic results on these components. We discuss the construction of Ciliberto and Lopez which gives a range in which describes the distribution of codimensions of the components in Section 3.3. We close with Voisin's counterexample to Harris' conjecture that there should be finitely many special components.

The last section discusses applications of Noether's theorem to questions of Srinivas about class groups of complete local rings.

We assume that the reader is familiar with algebraic geometry [**38, 46**] and Hodge theory [**89, 90**]. We have tried to write in chronological order of ideas, noting later refinements before going on, and sometimes giving simpler versions of theorems for easier reading. To make comparisons of results easier, we maintain the notation of the first paragraph: X is the ambient space and Y is a smooth member of the linear system corresponding to an ample line bundle L. Most of the results are given at least a sketch of the proof or the new ideas used.

We dedicate this survey with pleasure to Phillip Griffiths for his influential work on the subject. Voisin's books on Hodge theory [**89, 90**] have been very helpful. The second author thanks Phillip Griffiths for his lectures at the 2012 CBMS conference on Hodge theory and representation theory of June 18-23.

1. The classical theorem

In this section we examine the work related to Noether's theorem before 1980. After explaining Noether's motivations [**65**], we sketch the topological methods used by Lefschetz [**60**] to prove the theorem, including Lefschetz pencils, vanishing cycles, monodromy action and the Picard-Lefschetz formula. With the same ideas, Moishezon extended the statement to algebraic homology classes in higher dimensions and characterized the smooth complex threefolds for which Noether's conclusion holds and [**64**]. Using l-adic cohomology groups, Grothendieck, Deligne and Katz interpreted these constructions to obtain a statement meaningful in arbitrary characteristic $p \geq 0$ [**43, 15**]. We close with the higher dimensional dimensional analog known as the Grothendieck-Lefschetz theorem [**42, 44**].

1.1. Noether's idea. In his 1882 treatise on space curves [**65**], Max Noether claimed that the only curves on a general surface $Y \subset \mathbb{P}^3_{\mathbb{C}}$ of degree $d > 3$ are complete intersections of Y with another surface. Looking more closely, it wasn't even a claim, but a subordinate clause with no hint of a proof [**47**, pp. 136-137]. To explain his idea in modern language, consider the projective space $|\mathcal{O}_{\mathbb{P}^3}(d)|$ of all surfaces of fixed degree d and the Hilbert flag scheme $F(n, g, d)$ parametrizing flags $C \subset Y$ in which C is a curve of degree n and genus g and Y is a surface of degree d. The flag scheme $F(n, g, d)$, is a projective algebraic set, typically reducible, and the second projection $\rho : F(n, g, d) \to |\mathcal{O}_{\mathbb{P}^3}(d)|$ is proper, hence the image of the closed subset $W \subset F(n, g, d)$ consisting of flags $C \subset Y$ with C not a complete intersection of Y with another surface is a finite union of subvarieties. Taking the union over all pairs (n, g) gives a countable union. Noether claims that each of these subvarieties is *proper*, and hence their union cannot fill out all of $|\mathcal{O}_{\mathbb{P}^3}(d)|$. Examples 1.1 and 1.2 illustrate this fact: it is likely that Noether computed many such examples to arrive at his conclusion. A rigorous proof along these lines would require such a calculation for all triples (n, g, s) and analysis of the second projection map ρ. This analysis becomes difficult as n grows.

Example 1.1. Let $V \subset |\mathcal{O}_{\mathbb{P}^3}(d)|$ be the family of smooth surfaces which contain a line. For a fixed line $L \subset \mathbb{P}^3$, the family V_L of surfaces containing L is given by the linear subspace $H^0(\mathcal{I}_L(d)) \subset H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d))$. Tensoring the exact sequence

$$0 \to \mathcal{I}_L \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_L \to 0$$

with $\mathcal{O}_{\mathbb{P}^3}(d)$ and considering global sections, we see that V_L has codimension equal to $\dim_{\mathbb{C}} H^0(L, \mathcal{O}_L(d)) = d+1$ because $H^1(\mathcal{I}_L(d)) = 0$. Since lines are parametrized by the 4-dimensional Grassmann variety $\mathbb{G}(1, 3)$ and the general surface in V does not contain two lines, V has codimension $c(V) = d+1-4 = d-3$ in $|\mathcal{O}_{\mathbb{P}^3}(d)|$ and hence is *proper* for $d \geq 4$. For $d \leq 3$, it is well known that the general surface does contain a line, so V is not proper.

EXAMPLE 1.2. For a more sophisticated example [**14**], consider curves C cut out by maximal minors of a general $(d-1) \times (d-2)$ matrix M of linear forms. Such a curve is arithmetically Cohen-Macaulay and its ideal sheaf $\mathcal{I}_C$ has a *linear* resolution

$$0 \to \mathcal{O}(1-d)^{d-2} \to \mathcal{O}(2-d)^{d-1} \to \mathcal{I}_C \to 0. \tag{1.1}$$

It follows that $H^1(C, \mathcal{N}_C) = 0$ [**47**, Exercise 12.2] and therefore C is a smooth point on its Hilbert scheme component W of dimension $4 \deg C = 4\binom{d-1}{2}$. From sequence (1.1) we read off $\dim H^0(\mathcal{I}_C(d)) = 6d-2$, so if $V \subset |\mathcal{O}_{\mathbb{P}^3}(d)|$ is the family of surfaces containing a curve from W, then $\dim W = \dim V + \dim H^0(\mathcal{I}_C(d)) - 1 = 2d^2+1$. Subtracting from $\dim |\mathcal{O}_{\mathbb{P}^3}(d)| = \binom{d+3}{3}$ we find that the family V has codimension $\binom{d-1}{3}$ in $|\mathcal{O}_{\mathbb{P}^3}(d)|$.

In the 1920s Solomon Lefschetz proved the celebrated Noether-Lefschetz theorem [**57**], which in the literature is more often called simply *Noether's theorem*:

THEOREM 1.3. **(Noether's theorem)** *For $d > 3$, every curve on the very general surface $Y \in |\mathcal{O}_{\mathbb{P}^3}(d)|$ is the complete intersection of Y with another surface:* $\operatorname{Pic} Y = \langle \mathcal{O}_Y(1) \rangle$.

The term *very general* refers to a countable intersection of Zariski open sets, or the complement of a countable union of proper Zariski closed sets. Letting $U_d \subset |\mathcal{O}_{\mathbb{P}^3}(d)|$ denote the open set corresponding to smooth surfaces, the countable union of Theorem 1.3 is the *Noether-Lefschetz locus*

$$NL(d) = \{Y \in U_d : \operatorname{Pic} Y \neq \langle \mathcal{O}_Y(1) \rangle\}.$$

REMARK 1.4. In view of Theorem 1.3, the general quartic surface $Y \subset \mathbb{P}^3$ has Picard group $\operatorname{Pic} Y = \langle \mathcal{O}_Y(1) \rangle$, so in the 1960s Mumford challenged mathematicians to produce an actual *equation* of one. This challenge was finally met by van Luijk in 2007. One such surface [**83**, Remark 3.7] has equation

$$w(x^3+y^3+x^2z+xw^2) = 3x^2y^2 - 4x^2yz + x^2z^2 + xy^2z + xyz^2 - y^2z^2.$$

It is interesting to note that this quartic contains the line $z = w = 0$ if we specialize modulo $p = 3$, but contains no line in characteristic zero.

1.2. The proof of Lefschetz. Lefschetz understood that the analogous problem in higher dimensions is much easier. He proved this by observing that the exponential sequence $0 \to \mathbb{Z} \to \mathcal{O}_Y \to \mathcal{O}_Y^* \to 0$ associated to any smooth hypersurface $Y \subset \mathbb{P}^n$ gives rise to the exact cohomology diagram

$$\begin{array}{ccccccc} H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) & \to & H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}^*) & \to & H^2(\mathbb{P}^n, \mathbb{Z}) & \to & H^2(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) \\ \downarrow & & \downarrow \alpha & & \downarrow \beta & & \downarrow \\ H^1(Y, \mathcal{O}_Y) & \to & H^1(Y, \mathcal{O}_Y^*) & \to & H^2(Y, \mathbb{Z}) & \to & H^2(Y, \mathcal{O}_Y). \end{array} \tag{1.2}$$

The cohomology groups in the four corners are zero for $n > 3$ and α is identified with the restriction map $\operatorname{Pic} \mathbb{P}^n \to \operatorname{Pic} Y$. The Lefschetz hyperplane theorem says that the maps $H^k(\mathbb{P}^n, \mathbb{Z}) \to H^k(Y, \mathbb{Z})$ are isomorphisms for $k < n-1$ and injective for $k = n-1$. Thus β is an isomorphism for $n > 3$ and therefore α as well.

For surfaces in $\mathbb{P}^3$ things are more complicated: it is easy to construct surfaces for which the map β in diagram (1.2) is not an isomorphism, such as Examples 1.1 and 1.2. Lefschetz' idea is that if $\operatorname{Pic} Y \neq \langle \mathcal{O}_Y(1) \rangle$, then there is a class $\lambda \in H^2(Y,\mathbb{Z}) \cap H^{1,1}(Y,\mathbb{C})$ representing a curve $C \subset Y$ which is not a complete intersection of Y with another surface. As Y varies, the class $\lambda \in H^2(Y,\mathbb{Z})$ is locally constant because $H^2(Y,\mathbb{Z})$ is discrete, but with deformation the class λ tends to move from $H^{1,1}(Y,\mathbb{C})$ to $H^{0,2}(Y,\mathbb{C})$ and therefore becomes non-algebraic. We sketch his proof below.

1.2.1. *A fixed surface.* Since a fixed smooth surface $Y \subset \mathbb{P}^3$ of degree d is simply connected, the cohomology group $H^2(Y,\mathbb{Z})$ is torsion free and therefore injects into $H^2(Y,\mathbb{C})$ [**17**, Cor. 2.4 and Lemma 3.1]. Thus Lefschetz viewed $\operatorname{Pic} Y$ as the intersection $H^{1,1}(Y,\mathbb{C}) \cap H^2(Y,\mathbb{Z}) \subset H^2(Y,\mathbb{C})$, as follows from the Lefschetz $(1,1)$ theorem [**89**, Theorem 7.2]. The Poincaré duality intersection pairing $\langle , \rangle$ on $H^2(Y,\mathbb{Z})$ gives the decomposition

$$H^2(Y,\mathbb{Z}) = \underbrace{\langle c_1(\mathcal{O}_Y(1)) \rangle}_{H^2_f(Y,\mathbb{Z})} \oplus \underbrace{\langle c_1(\mathcal{O}_Y(1)) \rangle^{\perp}}_{H^2_v(Y,\mathbb{Z})} \tag{1.3}$$

of $H^2(Y,\mathbb{Z})$ into the *fixed* classes $H^2_f(Y,\mathbb{Z}) := \langle c_1(\mathcal{O}_Y(1)) \rangle \subset H^2(Y,\mathbb{Z})$ coming from $\mathbb{P}^3$ and the *variable* orthogonal complement $H^2_v(Y,\mathbb{Z}) = c_1(\mathcal{O}_Y(1))^{\perp}$.

1.2.2. *Lefschetz pencils.* The choice of a line $L \cong \mathbb{P}^1 \subset |\mathcal{O}_{\mathbb{P}^3}(d)|$ gives a family of surfaces $\{Y_t\}_{t \in \mathbb{P}^1}$. For general L, the base locus $B = \cap_{t \in \mathbb{P}^1} Y_t$ is smooth and finitely many Y_t are singular, having a single rational double point of type $\mathbf{A}_1$: such a family is called a *Lefschetz pencil.* Letting $U \subset \mathbb{P}^1$ correspond to the smooth surfaces, with corresponding family $\phi : \mathcal{Y} \to U$, the fibres Y_t are diffeomorphic for $t \in U$ and we obtain a local system $R^2\phi_*\mathbb{Z}$ of free abelian groups on U with stalks $H^2(Y_t,\mathbb{Z})$, which decomposes orthogonally into two local subsystems with stalks $H^2_f(Y_t,\mathbb{Z})$ and $H^2_v(Y_t,\mathbb{Z})$ as in (1.3) above. For fixed base point $0 \in U$, the fundamental group $\pi_1(U,0)$ acts on $H^2(Y_0,\mathbb{Z})$, fixing the subsystem $H^2_f(Y_0,\mathbb{Z}) = \langle \mathcal{O}(1) \rangle$.

1.2.3. *Vanishing spheres and irreducibility of monodromy action.* To understand the action of $\pi_1(U,0)$ on $H^2_v(Y_0,\mathbb{Z})$, Lefschetz constructed geometric generators for $H^2_v(Y_0,\mathbb{Z})$. Starting with a small disk Δ_i about each point $t_i \in \mathbb{P}^1 - U$, he used the local structure of the double point in Y_{t_i} to construct a 2-sphere $S^2_t \subset Y_t$ for $t \in \Delta_i - \{t_i\}$. Using a path from t to 0, he transported the class of S^2_t to Y_0, obtaining a *vanishing sphere* $\delta_i \in H^2(Y_0,\mathbb{Z})$. For a loop $\gamma_i \in \pi_1(U,0)$ going around t_i once (and no other t_j), the action $T(\gamma_i)$ of γ_i on $H^2(Y_0,\mathbb{Z})$ is given by the *Picard-Lefschetz formula*

$$T(\gamma_i)(h) = h \pm \langle h, \delta_i \rangle \delta_i, \tag{1.4}$$

the sign depending on the orientation of γ_i.

Since T fixes $\mathcal{O}(1)$, Formula (1.4) implies the orthogonality $\langle c_1(\mathcal{O}(1)), \delta_i \rangle = 0$, hence $\delta_i \in H^2_v(Y_0,\mathbb{Z})$: in fact, the δ_i generate $H^2_v(Y_0,\mathbb{Z})$ for any choice of Lefschetz pencil ([**90**, Lemma 2.26]. The orthogonality also gives $\langle \delta_i, \delta_i \rangle < 0$ by the Hodge index theorem, so $\langle , \rangle$ is negative definite on $H^2_v(Y_0,\mathbb{Z})$. Moreover, the δ_i are in the same $\pi_1(U,0)$-orbit [**90**, Corollary 3.24], hence $\pi_1(U,0)$ acts irreducibly on $H^2(Y_0,\mathbb{Q}) = H^2(Y_0,\mathbb{Z}) \otimes \mathbb{Q}$. Indeed, if $F \subset H^2(Y_0,\mathbb{Q})$ is a $\pi_1(U,0)$-invariant subspace with $0 \neq a \in F$, then $\langle a, \delta_i \rangle \neq 0$ for some i, so $T(\gamma_i)(a) - a = \pm \langle h, \delta_i \rangle \delta_i \in F$, so $\delta_i \in F$ by (1.4) (this is why we tensored with $\mathbb{Q}$), whence all $\delta_i \in F$ so that $F = H^2_v(Y_0, \mathbb{Q})$.

1.2.4. *Properness of Noether-Lefschetz components.* Let $V \subset NL(d)$ be a component of the Noether-Lefschetz locus, meaning that V is the projection $\rho(W)$ with W a family of flags $C \subset Y$ with C not a complete intersection on Y. If V is not proper, then $L \subset V$ for any Lefschetz pencil L and $\rho^{-1}(L) \subset W$ gives a family of such flags. We can choose a 1-dimensional family $W' \subset W$ which dominates L under ρ, though it may be multi-valued so that $W' \to L$ is a degree $r \geq 1$ covering of curves. The classes $\lambda_1, \dots \lambda_r \in H^2(Y_t, \mathbb{Z})$ corresponding to curves in the flag given by $\rho^{-1}(t) \cap W'$ along with $\mathcal{O}(1)$ generate a local subsystem $S \subset H^2(Y_t, \mathbb{Q})$. Then $G = F \cap H^2_v(Y_t, \mathbb{Q})$ forms a local subsystem of $H^2_v(Y_t, \mathbb{Q})$, which must be 0 or all of $H^2_v(Y_t, \mathbb{Q})$ by irreducibility. If $G = H^2_v(Y_t, \mathbb{Q})$, then since G is generated by algebraic curve classes of Hodge type $(1,1)$ we must conclude that $H^2(Y_0, \mathbb{C}) = H^2_f(Y_0, \mathbb{C}) \oplus H^2_v(Y_0, \mathbb{C}) \subset H^{1,1}(Y_0)$, but this is impossible for $d \geq 4$ because $H^{2,0}(Y_0) \neq 0$ due to adjunction. Thus $G = 0$, meaning that the curve classes λ_i were (generically) in $\langle \mathcal{O}_{Y_t}(1) \rangle$ after all.

Remark 1.5. Similarly, Lefschetz proved [**58**] that for a smooth 3-dimensional complete intersection $X \subset \mathbb{P}^N_{\mathbb{C}}$ and very general hyperplane section $Y \in |\mathcal{O}_X(1)|$ which is not rational, $\operatorname{Pic} X \to \operatorname{Pic} Y$ is an isomorphism.

The condition $d \geq 4$ appears only at the end of the Lefschetz proof to ensure that $H^2_v(Y, \mathbb{C}) \cap H^{2,0}(Y) \neq 0$. For an arbitrary smooth threefold X, Voisin defines $H^2_{\text{van}}(Y, \mathbb{C})$ as the kernel of the map $j_* : H^2(Y, \mathbb{C}) \to H^4(X, \mathbb{C})$), proving that it is generated by the classes of vanishing spheres for any Lefschetz pencil [**90**, 2.26]. With this notation she proves [**90**, Theorem 3.33]:

Theorem 1.6. *Let L be a very ample line bundle on a smooth complex projective threefold X such that $H^{2,0}(Y) \cap H^2_v(Y, \mathbb{C}) \neq (0)$ for smooth $Y \in |L|$. Then the Noether-Lefschetz locus*

$$NL(L) = \{Y \in |L| : \operatorname{Pic} X \to \operatorname{Pic} Y \textit{ is not an isomorphism}\}$$

is a countable union of proper subvarieties.

1.3. Algebraic homology classes. After the 1924 treatise of Lefschetz [**60**] things were quiet until the 1950s, when algebraic geometry was experiencing a revival from Grothendieck's theory of schemes. In 1957 Andreotti and Frankel reproved the Lefschetz theorem on hyperplane sections for a modern audience [**1**]. Wallace updated Lefschetz' work in 1960 [**84**]. In 1967 Moishezon adapted the proof of Lefschetz to smooth complex projective varieties, obtaining a statement about algebraic homology classes [**64**, Theorem 5.4]:

Theorem 1.7. *Let $X \subset \mathbb{P}^N_{\mathbb{C}}$ be a smooth n-fold and let $Y \subset X$ be a very general hyperplane section. Then for $2k \neq n-1$ the algebraic homology classes in $H_{2k}(Y, \mathbb{Q})$ are exactly those cut out by the algebraic classes in $H_{2k+2}(X, \mathbb{Q})$; the same holds for $2k = n-1$ if $h^{n-1,0}(Y) > h^{n-1,0}(X)$.*

Remark 1.8. By duality, Theorem 1.7 says that $H^{m,m}(Y, \mathbb{Q}) \subset H^{2m}(Y, \mathbb{Q})$ is the image of $H^{m,m}(X, \mathbb{Q})$ under the restriction map if $\dim X = 2m+1$ and $Y \in |\mathcal{O}_X(1)|$ is very general. Of course this need not happen for all $Y \in |\mathcal{O}_X(1)|$; for example, if $Y \subset X = \mathbb{P}^{2m+1}$ is a hypersurface of degree $d > 2m$ containing a linear subspace M of dimension m, then the class of M is not in the image of $H^{m,m}(X, \mathbb{Q})$ by reason of degree. (*Cf.* [**68**, p.308].)

Moishezon makes a more detailed analysis for threefolds [**64**, Theorem 7.5]:

THEOREM 1.9. *Let $X \subset \mathbb{P}^N_{\mathbb{C}}$ be a smooth threefold and let $Y \subset X$ be a very general hyperplane section. Then the restriction* $\operatorname{Pic} X \to \operatorname{Pic} Y$ *is an isomorphism if and only if*

(a) *There is an equality of Betti numbers $b_2(Y) = b_2(X)$ or*
(b) *$h^{2,0}(X) < h^{2,0}(Y)$.*

This remarkable theorem *characterizes* the threefolds X and the very ample line bundles $L = \mathcal{O}_X(1)$ for which the conclusion of the Noether's theorem holds. When $X = \mathbb{P}^3$, condition (a) picks up the "missing" case where the Noether's conclusion holds, namely when $L = \mathcal{O}(1)$ and $Y \subset \mathbb{P}^3$ is a plane.

REMARK 1.10. Conditions (a) and (b) translate to algebro-geometric language as follows. The inclusion $Y \hookrightarrow X$ induces the restriction $H^2(X, \mathbb{C}) \to H^2(Y, \mathbb{C})$, which decomposes via the Hodge isomorphism into the three maps

$$\begin{aligned} H^2(X, \mathcal{O}_X) &\to H^2(Y, \mathcal{O}_Y) \\ H^1(X, \Omega_X) &\to H^1(Y, \Omega_Y) \\ H^0(X, \Omega^2_X) &\to H^0(Y, \Omega^2_Y) \end{aligned} \tag{1.5}$$

Each is injective by Kodaira vanishing. For example, the third map can be written as the composition $H^0(X, \Omega^2_X) \to H^0(Y, \Omega^2_X|_Y) \to H^0(Y, \Omega^2_Y)$ where the kernel of the first map is $H^0(X, \Omega^2_X(-Y)) = 0$ and the kernel of the second map is $H^0(Y, \Omega_Y(-Y)) = 0$. Consequently $b_2(X) \le b_2(Y)$ with equality exactly when all three are isomorphisms. Conditions (a) and (b) of Theorem 1.9 become

(a′) the three maps in (1.5) are isomorphisms or
(b′) the first map in (1.5) is not surjective.

In particular, Moishezon notes that condition (b′) holds for large tensor powers L^k due to the exact cohomology fragment

$$H^2(X, \mathcal{O}_X) \to H^2(Y, \mathcal{O}_Y) \to H^3(X, L^{-k}) \to H^3(X, \mathcal{O}_X)$$

because $h^3(X, L^{-k}) = h^0(X, K_X \otimes L^k) \to \infty$ as $k \to \infty$.

REMARK 1.11. Restricting decomposition (1.3) for $H^2(Y, \mathbb{Z})$ to $H^2(Y, \mathcal{O}_Y)$, we can interpret $H^2_v(Y, \mathbb{C}) \cap H^{2,0}(Y)$ as the cokernel of the first map in (1.5), so the hypothesis of Theorem 1.6 is equivalent to condition (b) of Theorem 1.9.

1.4. Interpretation in finite characteristic. The 1966-69 Seminaire de geometrie algebrique du Bois-Marie (SGA 7) of Deligne, Grothendieck and Katz was devoted to the study of monodromy group in characteristic $p > 0$. In this setting the usual definitions of singular homology and cohomology don't make sense, but Grothendieck devised analogs by using l-adic cohomology and category theory [**43**]. The seminar of Deligne and Katz extended Lefschetz pencils, vanishing cycles and the Picard-Lefschetz formula to this setting [**15**, Exposés XIII, XV and XVIII]. With the path prepared, they obtained the following generalization of Noether's theorem [**15**, Exposé XIX, Theorem 1.3]:

THEOREM 1.12. *Let k be a field of arbitrary characteristic and let $V \subset \mathbb{P}^{2n+d}_k$ be a $2n$-dimensional generic complete intersection of hypersurfaces having degrees $\underline{a} = (a_1, a_2, \ldots, a_d)$. Further assume that either*

(a) $\operatorname{Char} k \neq 2$ *and $(2n, \underline{a})$ is not equal to $(2n, 2), (2n, 2, 2)$ or $(2, 3)$ or*

(b) *The Hodge number* $h^{2n,0}(V) \neq 0$, i.e., $\sum(a_i - 1) > 2n$.

Then every algebraic cohomology class in $H^{2n}(V, \mathbb{Q}_l(n))$ *is a rational multiple of the class* η^n, *where* $\eta = \mathcal{O}_V(1)$.

Remark 1.13. The word *generic* takes on a different meaning here, meaning that over a field extension of the ground field k it is projectively isomorphic to a complete intersection defined by equations in which the coefficients are algebraically independent over k.

Remark 1.14. Most of the proof of Deligne and Katz works for an arbitrary smooth projective variety V, but there is one sticking point. It is not known whether the analog of the hard Lefschetz theorem holds in characteristic $p > 0$. They conjecture this for all smooth V and give several known cases which include complete intersections [**15**, 5.2.2.1]. This explains the restriction in the theorem. The next corollary specializes to surfaces (compare with Remark 1.5).

Corollary 1.15. *Let* $S \subset \mathbb{P}^n$ *be a generic complete intersection surface. Then* S *is smooth and* $\operatorname{Pic} S = \langle \mathcal{O}_S(1) \rangle$ *with the following exceptions:*

(1) *Quadric surfaces in* $\mathbb{P}^3$.
(2) *Complete intersections of two quadrics in* $\mathbb{P}^4$.
(3) *Cubic surfaces in* $\mathbb{P}^3$.

Theorem 1.12 says that algebraic cohomology classes of S are rational multiples of $\mathcal{O}(1)$, but in fact they are *integral* multiples by [**15**, Theorem 1.8, Exposé XI].

1.5. Higher Dimensions. Grothendieck used a different approach for general ambient varieties in the late 1960s. His result follows [**42, 44**]:

Theorem 1.16. **(Grothendieck-Lefschetz Theorem)** *Let* X *be a smooth projective variety of dimension* $n \geq 4$. *Then for* any *effective ample divisor* $Y \subset X$, *the restriction map* $\operatorname{Pic} X \to \operatorname{Pic} Y$ *is an isomorphism.*

Notice that Y need not even be reduced here! For example, *every* closed subscheme $Y \subset \mathbb{P}^4$ defined by a homogeneous polynomial has Picard group $\operatorname{Pic} Y$ generated by $\mathcal{O}_Y(1)$. It is stated for Y smooth [**44**, IV, Corollary 3.3], but Lazarsfeld notes that the argument goes through for arbitrary Y [**56**, Remark 3.1.26]. This proof contains a beautiful idea due to Grothendieck [**42**, Exposé X]. He considers an open neighborhood U of Y in the formal completion $\hat{X}$ of X along Y, showing that the sequence of induced maps

$$\operatorname{Pic} X \to \operatorname{Pic} U \to \operatorname{Pic} \hat{X} \to \operatorname{Pic} Y$$

are all isomorphisms. The most difficult part is the isomorphism $\operatorname{Pic} \hat{X} \cong \operatorname{Pic} U$, for which Grothendieck defines *Lefschetz conditions* that are satisfied the pair (X, Y). The last isomorphism is obtained by considering the infinitesimal neighborhoods $Y_n \subset X$ defined by ideals $\mathcal{I}_Y^n$. The Kodaira vanishing theorem implies that $H^i(Y, \mathcal{I}_Y^n/\mathcal{I}_Y^{n+1}) = 0$ for $i = 1, 2$ and therefore the exact sequences $0 \to \mathcal{I}_Y^n/\mathcal{I}_Y^{n+1} \to \mathcal{O}_{Y_n}^* \to \mathcal{O}_{Y_{n+1}}^* \to 0$ give isomorphisms $\operatorname{Pic} Y_n \cong \operatorname{Pic} Y_{n+1}$ for $n > 0$ and hence $\operatorname{Pic} \hat{X} \cong \varprojlim \operatorname{Pic} Y_n \cong \operatorname{Pic} Y$.

2. New ideas: the infinitesimal approach

By the mid 1970s it seemed as if the ideas of Lefschetz had been pushed as far as they could go. Indeed, there was relative silence for almost a decade until Carlson, Green, Griffiths and Harris introduced infinitesimal methods to the problem [**9**]. This led to the infinitesimal Noether theorem, which sparked new interest in the subject and instigated developments over the next decade. There soon followed Green's explicit Noether-Lefschetz theorem [**25, 26**] (see also [**85**]) and Ein's extension to dependency loci of sections of vector bundles [**19**]. Griffiths and Harris developed an algebraic degeneration argument in 1985 [**40**]. Later, two papers based on notes of Mohan Kumar and Srinivas [**54**] led to a new variant for smooth threefolds of Joshi in 1995 [**50**] and a generalization to normal ambient spaces of Ravindra and Srinivas in 2009 [**74**]. Our extension of the theorem for hypersurfaces in $\mathbb{P}^n$ with base locus came out in 2011 [**4**].

2.1. Variations of Hodge structures and projective hypersurfaces. In 1983 Carlson, Green, Griffiths and Harris established the theory of infinitesimal variations of Hodge structures in their foundational paper [**9**]. It remains an active area of research today [**8, 30, 31, 12, 75**]. In this section we review Hodge structures, their deformations, and give Griffiths' interpretation of the differential to the period map for the family of smooth projective hypersurfaces of fixed degree.

2.1.1. *Hodge structures.* Hodge theory assigns to any compact Kähler manifold Y a decomposition $H^k(Y,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(Y)$, where each complex subspace $H^{p,q}(Y) \subset H^k(Y,\mathbb{C})$ consists of classes represented by closed (p,q)-forms and $H^{p,q}(Y)$ is isomorphic to $H^{q,p}(Y)$ via complex conjugation [**89**, §6.1], while the universal coefficients theorem from algebraic topology provides and isomorphism $H^k(Y,\mathbb{C}) \cong H^k(Y,\mathbb{Z}) \otimes \mathbb{C}$. Abstracting this data leads to a *Hodge structure of weight* k, a pair $(H_{\mathbb{Z}}, H^{p,q})$ consisting of a free abelian group $H_{\mathbb{Z}}$ ($H^k(Y,\mathbb{Z})/(\text{torsion})$ in the above setting) and a decomposition $H_{\mathbb{Z}} \otimes \mathbb{C} = H = \oplus_{p+q=k} H^{p,q}$ satisfying $H^{p,q} = \overline{H^{q,p}}$. The *Hodge filtration*

$$F^p = \bigoplus_{r \geq p} H^{r,k-r} = H^{p,k-p} \oplus H^{p+1,k-p-1} \oplus \cdots \oplus H^{k,0} \subset H$$

satisfies

1. $(0) = F^{k+1} \subset F^k \subset \cdots \subset F^0 = H$.
2. $F^p \oplus \overline{F^{k-p+1}} = H$.

These two properties suffice to recover $H^{p,q} = F^p \cap \overline{F^q}$, so one can define a Hodge structure by the data $\{H_{\mathbb{Z}}, H^{p,q}\}$ or $\{H_{\mathbb{Z}}, F^p\}$.

2.1.2. *Polarized Hodge structures.* Now suppose our Kähler manifold Y with the natural Hodge structure $(H_{\mathbb{Z}}, H^{p,q})$ carries an ample line bundle L polarizing Y by the integral class $\omega = c_1(L)$. This gives rise to the bilinear form

$$Q : H_{\mathbb{Z}} \times H_{\mathbb{Z}} \to \mathbb{Z}$$

via $Q(\alpha,\beta) = \int_X \alpha \wedge \beta \wedge \omega^{n-k}$ with $n = \dim Y$. The form Q satisfies

1. $Q(\alpha,\beta) = (-1)^k Q(\beta,\alpha)$.
2. $Q(\alpha,\beta) = 0$ for $\alpha \in H^{p,q}, \beta \in H^{r,s}, (p,q) \neq (s,r)$.
3. $i^{p-q} Q(\alpha,\overline{\alpha}) > 0$ if $0 \neq \alpha \in H^{p,q}$.

Thus a *polarized Hodge structure of weight* k is a triple $\{H_{\mathbb{Z}}, H^{p,q}, Q\}$ in which $\{H_{\mathbb{Z}}, H^{p,q}\}$ is a Hodge structure and Q is a bilinear map as above.

Example 2.1. Another important example of a (polarized) Hodge structure arising from geometry comes from the Lefschetz decomposition. For Y of dimension n and $\omega = c_1(L)$ as above (or any integral Kähler form), the *Lefschetz operator* $L : H^k(Y,\mathbb{Z}) \to H^{k+2}(Y,\mathbb{Z})$ acts via cup product with ω. The *hard Lefschetz theorem* says that the maps $L^{n-k} : H^k(Y,\mathbb{Z}) \to H^{2n-k}(Y,\mathbb{Z})$ are isomorphisms. The kernel of one higher power of L defines the primitive cohomology, that is

$$H^k_{\text{prim}}(Y,\mathbb{Z}) = \ker(L^{n-k+1} : H^k(Y,\mathbb{Z}) \to H^{2n-k+2}(Y,\mathbb{Z})).$$

The primitive cohomology groups form building blocks for all cohomology groups in the sense that each $\alpha \in H^k(Y,\mathbb{Z})$ can be uniquely written in the form

$$\alpha = \sum_{r\geq 0} L^r \alpha_r \text{ with } \alpha_r \in H^{k-2r}(Y,\mathbb{Z})_{\text{prim}}.$$

This is known as the *Lefschetz decomposition.* Now setting

$$H_{\mathbb{Z}} = H^k(Y,\mathbb{Z}) \cap H^k_{\text{prim}}(Y,\mathbb{Q})$$

$$H^{p,q} = H^{p,q}(Y,\mathbb{C}) \cap H^k_{\text{prim}}(Y,\mathbb{C})$$

$$Q(\varphi,\psi) = (-1)^{\frac{(k(k-1)}{2}} \int_Y \varphi \wedge \psi \wedge \omega^{n-2k}$$

gives a polarized Hodge structure of weight k.

2.1.3. *Variations of Hodge structure.* Hodge structures have been useful in some classification results [**22, 51**], but most fixed abstract Hodge structures do *not* come from geometry, and it is more productive to study how they vary in a family. Let $f : \mathcal{Y} \to B$ be a projective morphism with smooth fibers Y_b of dimension n. In the differentiable category this fibration is trivial over simply connected open sets $U \subset B$ by Ehresmann's theorem [**18**], so we can identify $H^k(Y_s,\mathbb{Z})$ with $H^k(Y_0,\mathbb{Z})$ for fixed $0 \in U$. Gluing these together gives the *Hodge bundle* $\mathcal{H} = R^k f_*\mathbb{Z} \otimes_{\mathbb{Z}} \mathcal{O}_B$, a holomorphic vector bundle on B with a filtration

$$(0) = \mathcal{F}^{k+1} \subset \mathcal{F}^k \subset \cdots \subset \mathcal{F}^0 = \mathcal{H}$$

by holomorphic subbundles whose fiber over $b \in B$ is the corresponding Hodge filtration. Differentiation on local trivializations for $\mathcal{H}$ gives rise to the *Gauss-Manin connection* $\nabla : \mathcal{H} \to \Omega_B \otimes \mathcal{H}$ which satisfies *Griffiths transversality* [**33, 34, 35**], the condition that $\nabla(\mathcal{F}^p) \subset \Omega_B \otimes \mathcal{F}^{p-1}$. A *variation of Hodge structures* is the data $\{\mathcal{H}, \mathcal{F}^\bullet, \nabla\}$ on B, or $\{\mathcal{H}, \mathcal{F}^\bullet, Q, \nabla\}$ if there is a polarization involved.

2.1.4. *Infinitesimal variation of Hodge structures.* For a given family $\mathcal{Y} \to B$ as above, consider the (holomorphic) tangent space $T = T_{B,0}$ at $0 \in B$. Griffiths transversality induces $\mathcal{O}_B$-linear maps

$$\overline{\nabla}_0 : \mathcal{F}^p/\mathcal{F}^{p+1} \to \Omega_B \otimes \mathcal{F}^{p-1}/\mathcal{F}^p$$

for each p. The fibre at $s = 0$ gives maps $\overline{\nabla}_0 : H^{p,q} \to \text{Hom}(T, H^{p-1,q+1})$, which can be reassembled to obtain the *differential of the period map*

$$\delta : T \to \oplus_p \text{Hom}(H^{p,q}, H^{p-1,q+1}) \tag{2.1}$$

given by $\delta(v)(\xi) = \overline{\nabla}_0(v)(\xi)$. Alternatively, this map can be obtained by forming the classifying space D for polarized Hodge structures of fixed dimensions $h^{p,q} = \dim H^{p,q}$, which has the structure of a homogeneous manifold [**9**, 1.a.4]. The variation of Hodge structures associated to $\mathcal{Y} \to B$ defines a local period map

$B \to D$ and computing the differential at 0 yields the map δ [**9**, 1.a.11, 1.a.12 and 1.a.13], which satisfies

(1) $\delta(\xi_1)\delta(\xi_2) = \delta(\xi_2)\delta(\xi_1)$ for $\xi_1, \xi_2 \in T$.
(2) $Q(\delta(\xi)\varphi, \psi) + Q(\varphi, \delta(\xi)\psi) = 0$ for $\xi \in T, \varphi \in F^p, \psi \in F^{k-p+1}$.

An *infinitesimal variation of Hodge structure* is the data $V = \{H_{\mathbb{Z}}, H^{p,q}, Q, T, \delta\}$ satisfying the above properties.

REMARK 2.2. When the infinitesimal variation of Hodge structure arises from a family $\mathcal{Y} \to B$ as above, Griffiths showed that δ is given by the cup product $\delta(v)(\xi) = \kappa(v) \cup \xi$, where $\kappa : T \to H^1(Y_0, \mathcal{T}_{Y_0})$ is the Kodaira-Spencer map, where $\mathcal{T}_Y$ denotes the tangent bundle on Y [**34**, Proposition 1.20].

2.1.5. *Hypersurfaces in projective space.* Let $B \subset |\mathcal{O}_{\mathbb{P}^N}(d)|$ be the open subset corresponding to smooth hypersurfaces, where $N = 2m + 1$ is odd. If $\mathcal{Y} \to B$ is the universal family with fibers Y_b of dimension $2m$, one can consider a generalized Noether-Lefschetz locus

$$NL(d) = \{Y \in B | H^{2m}(Y, \mathbb{Q}) \cap H^{m,m}(Y) \neq \mathbb{Q}\}. \tag{2.2}$$

Note that $Y_0 \in NL(d)$ if and only if there is a class $\lambda \in H^{2m}(Y_0, \mathbb{Q}) \cap H^{m,m}(Y_0)$ which is not a multiple of $c_1(\mathcal{O}(1))^n$, which is equivalent to the existence of a primitive class $0 \neq \lambda \in H^{m,m}_{\text{prim}}(Y_0, \mathbb{Q})$. If $0 \in U$ is a simply connected open neighborhood, then λ defines a local section of the Hodge bundle and one can consider the local Hodge locus $U_\lambda = \{b \in U : \lambda \in \mathcal{F}^{m+1}_b\}$. Cattani, Deligne and Kaplan have shown that U_λ are algebraic [**11**] and consequently $NL(d) = \cup_\lambda U_\lambda$ is a countable union of algebraic sets.

The differential to the period map δ at $Y \in B$ has an algebraic interpretation due to Carlson and Griffiths [**7, 36**]. If Y is given by equation $F = 0$, the Jacobi ideal is $J = (\partial F/\partial x_0, \partial F/\partial x_1, \ldots, \partial F/\partial x_{n+1})$ and the Jacobi ring is $R = S/J$, where S is the homogeneous coordinate ring for $\mathbb{P}^{n+1}$. With this notation, there are isomorphisms

$$T_{B,Y} \cong R_d, \qquad H^{n-q,q}_{\text{prim}} \cong R_{(q+1)d-n-2} \text{ for all } q \geq 0 \tag{2.3}$$

and the maps $T_{B,Y} \otimes H^{n-q,q} \to H^{n-q-1,q+1}$ arising from the differential δ are identified with the multiplication maps $R_d \otimes R_{(q+1)d-n-2} \to R_{(q+2)d-n-2}$. Moreover, if $n = 2m$ and $0 \neq \lambda \in H^{m,m}_{\text{prim}}(Y, \mathbb{Q})$ as above, then $T_{U_\lambda, Y} \subset T_{B,Y}$ is the left annihilator of λ.

2.2. The infinitesimal Noether theorem. This result describes the Hodge classes of the middle cohomology group of a sufficiently ample hypersurface whose Hodge type is fixed under first order deformations. In accordance with our conventions, let L be an ample line bundle on a smooth complex variety X of dimension n with smooth member $Y \in |L|$. Let $j : Y \hookrightarrow X$ be the inclusion and let $\omega = c_1(L) \in H^2(X, \mathbb{Z})$ be the first Chern class. We must make some definitions before we can state the theorem: we first extend the decomposition of the middle cohomology of Y used by Lefschetz in his original proof (see section 1.2).

2.2.1. *Fixed cohomology and variable cohomology.* The Poincáre-Lefschetz dual to the homology sequence for the pair (X, Y) with complex coefficients gives rise to the commutative diagram

$$\begin{tikzcd}
H^n(X-Y) \arrow[r, "R"] & H^{n-1}(Y) \arrow[r, "j_*"] & H^{n+1}(X) \\
& H^{n-1}(X) \arrow[u, "r"] \arrow[ur, "\omega"'] &
\end{tikzcd}$$

in which the restriction r is an injection by the Lefschetz hyperplane theorem and ω is an isomorphism by the hard Lefschetz theorem. Therefore $r \circ \omega^{-1}$ is a splitting for j_* and we obtain a decomposition

$$H^{n-1}(Y) = \underbrace{rH^{n-1}(X)}_{H^{n-1}_f(Y)} \oplus \underbrace{RH^n(X-Y)}_{H^{n-1}_v(Y)}. \tag{2.4}$$

Here $H^{n-1}_f(Y) = rH^{n-1}(X)$ is the *fixed* part of $H^{n-1}(Y)$ coming from the ambient space X and $H^{n-1}_v(Y) = rH^{n-1}(X) = \ker j_*$ is the *variable* part. This coincides with Voisin's vanishing cohomology $H^{n-1}_{\text{van}}(Y)$, which is generated by classes of vanishing spheres for any Lefschetz pencil [**90**, 2.26]. The decomposition (2.4) respects Hodge structure and the summands are orthogonal with respect to the bilinear form Q in the weight $n-1$ Hodge structure [**90**, 2.27].

2.2.2. *Infinitesimally fixed cohomology.* We describe the classes whose Hodge type does not change under first order deformation in the family $|L|$. Let $S \subset |L|$ be the open subset corresponding to smooth hypersurfaces with universal family

$$\begin{array}{ccc}
\mathcal{Y} & \subset & X \times S \\
& & \downarrow \\
& & S.
\end{array}$$

Then Y is defined by the vanishing of a section $s \in H^0(L)$. The tangent space to Y in the Hilbert scheme for X is isomorphic to $H^0(Y, N)$, where N is the normal bundle and the tangent space $T_Y(|L|)$ to Y in the family $|L|$ is the image of $H^0(X, L) \to H^0(Y, N)$ arising from the exact sequence

$$0 \to \mathcal{O}_X \xrightarrow{s} L \to N \to 0.$$

In this case we are interested in tangent directions corresponding to movement of Y in the family L, i.e. we don't want to consider those induced from automorphisms of Y. To achieve this, consider the diagram

$$\begin{tikzcd}
& & & 0 \arrow[d] & \\
& & & \mathcal{T}_Y \arrow[d] & \\
0 \arrow[r] & \mathcal{T}_X(-Y) \arrow[r] & \mathcal{T}_X \arrow[r] \arrow[dr, "\sigma"'] & \mathcal{T}_X \otimes \mathcal{O}_Y \arrow[r] \arrow[d] & 0 \\
& & & N \arrow[d] & \\
& & & 0 &
\end{tikzcd}$$

and define T to be the tangent space $T_Y(|L|) \subset H^0(Y, N)$ modulo the image of σ. If L is sufficiently ample that $H^0(X, \mathcal{T}_X) \to H^0(Y, \mathcal{T}_X \otimes \mathcal{O}_Y)$ is an isomorphism, then we may view $T \subset H^1(Y, \mathcal{T}_Y)$ as a subset and taking cup product with the Kodaira-Spencer class defines the infinitesimal variation of Hodge structure as in Remark 2.2 (see also [**89**, Ch. 10]). Thus we obtain the infinitesimal variation of Hodge structures $V = \{H_{\mathbb{Z}}, H^{p,q}, Q, T, \delta\}$ where $H_{\mathbb{Z}} = H^{n-1}(Y, \mathbb{Z})/(\text{torsion})$,

$H^{p,q} = H^{p,q}(Y)$, T is the tangent space and $\delta : T \to \oplus_p \mathrm{Hom}(H^{p,q}, H^{p-1,q+1})$ is the differential to the period map. With this notation, the classes whose Hodge types do not change infinitesimally under V are precisely

$$H^{p,q}_{\mathrm{i.f.}}(Y) = \{\Psi \in H^{p,q} : \delta(\xi)\Psi = 0 \text{ for all } \xi \in T\}.$$

Carlson, Green, Griffiths and Harris prove the following result [**9**, 3.a.16]:

THEOREM 2.3. **(Infinitesimal Noether Theorem)** *If X is a smooth n-dimensional projective variety of and L is a sufficiently ample line bundle on X, then for any smooth $Y \in |L|$ the infinitesimally fixed part of the middle cohomology groups of Y is precisely the fixed cohomology coming from the ambient space X. In other words if $p + q = n - 1 = \dim Y$, then*

$$H^{p,q}_{\mathrm{i.f.}}(Y) = H^{p,q}_f(Y).$$

The proof follows from two ingredients:

2.2.3. *Hodge filtration and order of the pole.* Since $X - Y$ is affine, $H^k(X - Y)$ is isomorphic to the algebraic de Rham cohomology $H^k_{ADR}(X - Y)$ associated to the complex of differentials. The first ingredient is a theorem of Griffiths saying that under this isomorphism, the Hodge filtration on $H^k(X - Y)$ is given by the order of the pole along Y [**36**] (see also [**90**, § 6.1.2]). The precise statement is this:

THEOREM 2.4. *Assume that $H^m(\Omega^p_X(qY)) = 0$ for $p \geq 0, m > 0$ and $q > 0$. Then the image of the natural map $H^0(X, K_X(pY)) \to H^k(X - Y)$ is the Hodge filtrant $F^{k-p+1}H^k(X - Y)$ for $1 \leq p \leq k$.*

In the context of Theorem 2.3, the important consequence of Theorem 2.4 is the exact sequence

$$(2.5)\quad 0 \to H^{n-q,q}_{\mathrm{prim}}(X) \to \frac{H^0(K_X((q+1)Y)}{dH^0(\Omega^{n-1}_X(qY)) + sH^0(\Omega^n_X(qY))} \to H^{n-1-q,q}_v(Y) \to 0$$

where d and s are the natural maps and $n = \dim X$ [**9**, 3.a.8].

2.2.4. *Surjectivity of multiplication maps.* The second ingredient is a condition that comes up in several of the variants of Noether's theorem. It is the fact that if L is sufficiently ample, then the multiplication maps

$$(2.6)\qquad H^0(X, L) \otimes H^0(X, K_X \otimes L^q) \to H^0(X, K_X \otimes L^{q+1})$$

are surjective for all $q \geq 0$. Using the surjection $H^0(X, L) \to T$ from section 2.2.2 and the surjections $H^0(X, K_X \otimes L^{q+1}) \to H^{n-1-q,q}_v(Y)$ induced from (2.5), one infers from (2.6) that there are surjections

$$(2.7)\qquad T \otimes H^{p,q}_v(Y) \to H^{p+1,q-1}_v(Y) \to 0$$

if $p + q = n - 1$ (see [**90**, Theorem 6.13] for example). This leads to the proof of Theorem 2.3. For $p+q = n-1$, it's clear that the fixed cohomology is infinitesimally fixed, so $H^{p,q}_f(Y) \subset H^{p,q}_{\mathrm{i.f.}}(Y)$ and it suffices to prove the reverse inclusion, so let $\psi \in H^{p,q}_{\mathrm{i.f.}}(Y)$. Writing $\psi = \psi_f + \psi_v$ from decomposition (2.4), we need to show that $\psi_v = 0$. By definition $d(\xi)\psi = 0$ for all $\xi \in T$ and $d(\xi)\psi_f = 0$ because $\psi_f \in H_f(Y)$, so $d(\xi)\psi_v = 0$ for all $\xi \in T$ and hence

$$0 = Q(d(\xi)\psi_v, \varphi) = -Q(\psi_v, d(\xi)\varphi)$$

for all $\xi \in T$ and $\varphi \in H^{n-1}_v(Y)$. This is not possible, because the $d(\xi)\varphi$ generate $H^{p+1,q-1}_v(Y)$ and yet $Q(\psi_v, \overline{\psi_v}) \neq 0$ from general properties of Q [**9**, 1.a.2].

Remark 2.5. Looking at the proof, L must be taken sufficiently ample that

(1) The multiplication maps (2.6) are surjective for $q \geq 0$ [**9**, 3.a.18].
(2) $H^r(X, \Omega^p_X \otimes L^q) = 0$ for $r > 0, q > 0$ to use the exact sequences [**9**, 3.a.11].
(3) The map $H^0(X, \mathcal{T}_X) \to H^0(Y, \mathcal{T}_X \otimes \mathcal{O}_Y)$ is an isomorphism [**9**, 3.a.15].

In particular, these hold for $d \geq 4$ when $X = \mathbb{P}^3$ and $L = \mathcal{O}_{\mathbb{P}^3}(d)$.

By looking at tangent spaces, one sees that Noether's Theorem follows from the Infinitesimal Noether Theorem.

2.3. The explicit Noether theorem and Koszul cohomology. Noether's theorem tells us that the Noether-Lefschetz components $V \subset NL(d)$ are proper, that is, that $c(V) = \operatorname{codim}(V, |\mathcal{O}_{\mathbb{P}^3}(d)|) > 0$, but one can ask for a sharp lower bound on $c(V)$. Carlson, Green, Griffiths and Harris make the following concrete conjecture [**9**, 3.a.26]:

Conjecture 2.6. *Let $S_k \subset NL(d)$, $d \geq 4$ be the family of surfaces S containing a curve of degree k which is not a complete intersection of S with another surface. Then $c(S_k) \geq d - 3$ with equality only if $k = 1$.*

Note that equality holds when $k = 1$ by Example 1.1. That the inequality holds is the following result.

Theorem 2.7. **(Explicit Noether-Lefschetz Theorem)** *If $V \subset NL(d)$ is an irreducible component, then V has codimension at least $d-3$ in $|H^0(\mathbb{P}^3, \mathcal{O}(d))|$.*

Theorem 2.7 was proved in 1984 by Mark Green using his theory of Koszul cohomology [**24, 25**]. For a vector space V and a graded module M over the symmetric algebra S^*V, the *Koszul cohomology group* $\mathcal{K}_{p,q}(M, V)$ is the cohomology in the middle of the fragment

$$\wedge^{p+1} V \otimes M_{q-1} \to \wedge^p V \otimes M_q \to \wedge^{p-1} V \otimes M_{q+1} \tag{2.8}$$

arising from the Koszul complex. These groups arise naturally in algebraic geometry, for if $\mathcal{F}$ is a coherent sheaf on a variety X and L is a line bundle, then $M = \oplus_{d \geq 0} H^0(X, M \otimes L^d)$ is a graded S^*V-module for any subspace $V \subset H^0(X, L)$. Many classical results about generators and relations of the ideal of a projective variety can be interpreted in terms of vanishings of Koszul cohomology groups; a survey of these results can be found in [**28, 29, 20, 32**].

To state the relevant vanishing theorem, let $W \subset H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(d))$ be any base-point free linear system. Multiplication endows $M_k = \oplus_{t \in \mathbb{Z}} H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(k + td))$ with a graded S^*W-module structure for each $k \in \mathbb{Z}$.

Theorem 2.8. $\mathcal{K}_{p,0}(M_k, W) = 0$ *if* $k \geq p + d + \operatorname{codim}(W, H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(d)))$.

Green first proved this with a spectral sequence argument [**25**, Theorem 2.16] and later gave a slicker proof using a filtration of $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(d))/W$ with one dimensional quotients and regularity properties [**26**, Theorem 1]. As a special case, the multiplication map

$$W \otimes H^0(\mathbb{P}^3, \mathcal{O}(d-4)) \to H^0(\mathbb{P}^3, \mathcal{O}(2d-4)) \tag{2.9}$$

is surjective if $\operatorname{codim}(W, H^0(\mathbb{P}^3, \mathcal{O}(d))) \leq d - 4$ (take $p = 0, k = 2d - 4$).

To sketch Green's argument of Theorem 2.7, let $V \subset NL(d)$ be an irreducible component, let $Y \in V$ be a smooth point of V with tangent spaces $T_{V,Y} \subset T_{U_d,Y}$. The differential to the period map induces $\delta : T_{U_d,Y} \to \operatorname{Hom}(H^{1,1}(Y), H^{0,2}(Y))$.

Since $S \in NL(d)$, there is a nonzero primitive Hodge class $\lambda \in H^{1,1}_{\text{prim}}(Y) \cap H^2(Y, \mathbb{Z})$ which remains of type $(1,1)$ in directions $v \in T_{V,Y}$: *i.e.*, $\delta(v)(\lambda) = 0$ for each $v \in T_{V,Y}$, but

$$\delta(v)(\lambda) = 0 \iff \forall \omega \in H^{2,0}, Q(\delta(v)(\lambda), \omega) = 0 \iff \forall \omega \in H^{2,0}, Q(\lambda, \delta(v)(\omega)) = 0$$

so the image of the map $\phi : T_{V,Y} \otimes H^{2,0} \to H^{1,1}$ given by $\phi(v \otimes \omega) = \delta(v)(\omega)$ is contained in $\lambda^{\perp}$ and therefore is not surjective. In view of Section 2.1.5, it follows that the multiplication map $R_d \otimes R_{d-4} \to R_{2d-4}$ on graded pieces of the Jacobi ring is not surjective and therefore neither is the map $W \otimes S_{d-4} \to S_{2d-4}$, where $W \subset S_d$ is the preimage of $T_{V,S}$ under the quotient map $S_d \to R_d$. Now W is base-point free because it contains J (which has no zeros because S is smooth), so $\operatorname{codim}(W, S_d) \geq d - 3$ by the special case of Theorem 2.8 noted above.

Remark 2.9. We note some variants of Theorem 2.7.

(a) In 1989 Voisin gave another proof of Theorem 2.7 for $d \geq 5$ which was also based on analyzing the Jacobi ring [**85**].
(b) Green's student Sung-Ock Kim extended the explicit Noether theorem to complete intersection surfaces in $\mathbb{P}^n$ of fixed type in 1991 [**53**].
(c) Lopez and Maclean gave an explicit Noether theorem for smooth threefolds X in 2007. For L a line bundle on X, the Noether-Lefschetz locus $NL(L)$ is the set of smooth $Y \in |L|$ for which the restriction map $\operatorname{Pic} X \to \operatorname{Pic} Y$ is not surjective. In terms of Castelnuovo-Mumford regularity properties of L they give a lower bound on the codimension of the components of the Noether-Lefschetz locus [**61**].

2.4. The degeneration proof of Griffiths and Harris. While the Hodge-theoretic machinery is powerful, the degeneration argument of Griffiths and Harris' [**40**] by contrast uses almost none of this. Their idea is to deform a very general smooth surface S to a general reducible surface containing a plane, where it turns out they can compute the Picard group.

To sketch their method, start with a smooth surface T of degree $d-1$ and a plane P generic with respect to T; then choose a surface U of degree d generic with respect to both of these surfaces. Let X be the pencil of degree-d surfaces containing $T \cup P$ and U parametrized by t, where $t = 0$ corresponds to the reducible surface $T \cup P$. If the respective equations of U, T and P are G, F and L, then X has equation

$$tG - LF = 0$$

and it is clear that X has singularities at the $d(d-1)$ points p_i where all three surfaces meet in the central fiber $t = 0$. These singularities are isolated of the analytic isomorphism type of the vertex of a cone over a smooth quadric surface, and each one can be resolved by blowing up p_i; in fact, one can blow down one of the families of rulings on each exceptional surface to give a smooth family. Choose in each case to blow down the ruling containing the intersection of the quadric with the strict transform $\tilde{T}$ of T, and call the resulting smooth family $\tilde{X}$.

The central fibre $\tilde{X}_0$ is reducible with components $\tilde{T} \cong T$ and $\tilde{P}$, the latter of which is isomorphic to P with the p_i blown up; these components meet in a curve $\tilde{C} \cong C$. The Picard group $\operatorname{Pic} \tilde{X}_0$ can be understood as the fibered product $\operatorname{Pic} \tilde{P} \times_{\operatorname{Pic} \tilde{C}} \operatorname{Pic} \tilde{T}$ via the restriction maps $r_1 : \operatorname{Pic} \tilde{T} \to \operatorname{Pic} C, r_2 : \operatorname{Pic} \tilde{P} \to \operatorname{Pic} C$. The first restriction r_1 is injective with finitely generated image $\cong \operatorname{Pic} T$. Griffiths and Harris use a monodromy argument (the only hint of Hodge theory in their proof) to show that $\ker r_2$ is the subgroup generated by $\mathcal{O}_{\tilde{P}}(dH - E_1 - \cdots - E_{d(d-1)})$, where H is the pullback of the hyperplane class on P and E_i are the exceptional divisors of the blown up p_i, and that $\mathrm{Im} r_1 \cap \mathrm{Im} r_2 = \langle \mathcal{O}_{\tilde{C}}(1) \rangle$. This proves that

$$\operatorname{Pic} X_0 \cong \operatorname{Pic} \tilde{P} \times_{\operatorname{Pic} \tilde{C}} \operatorname{Pic} \tilde{T} \cong \mathbb{Z} \times \mathbb{Z}$$

with the first factor generated by $\mathcal{O}(1)$ and the second generated by a line bundle M that is $\mathcal{O}_{\tilde{P}}(dH - E_1 - \cdots - E_{d(d-1)}) \cong \mathcal{O}_{\tilde{P}}(\tilde{C})$ on $\tilde{P}$ and trivial on $\tilde{T}$. Moreover, M has a section Y such that $Y_t = P \cap X_t$ for $t \neq 0$.

Now suppose that there is a family $W_t \subset X_t$ of curves coming from an effective divisor W on X. After removing any components of $\tilde{P}$ or $\tilde{T}$ from W, the resulting curve on $\tilde{X}_0$ differs from a complete intersection $\mathcal{O}_{\tilde{X}}(n)$ by some (positive or negative) multiple of Y, which is a complete intersection away from the central fibre and therefore on U, as U was generically chosen. The more difficult (and interesting) case occurs when the family of curves is not rational over the line parametrizing the family X_t; the argument then involves delicate calculations on blowups of the surface $\tilde{X}_0$ (see [**40**, appendix]).

2.5. Dependency loci of sections of vector bundles. In 1985 Lawrence Ein extended the Noether-Lefschetz theorem to dependency loci of generic subspaces of a vector bundle E of rank r on a projective n-fold X. A t-dimensional subspace $T \subset H^0(X, E)$ defines the map $\sigma_T : T \otimes \mathcal{O}_X \to E$ along with dependency loci $D_k(\sigma_T) = \{x : \operatorname{rank} \sigma_T(x) \leq k\}$. If E is generated by global sections and $2(r + 3 - n) > \dim X$, then $D_{t-2}(\sigma_T)$ is empty and $Y_T := D_{t-1}(\sigma_T)$ is a smooth subvariety of dimension $n + t - r - 1$ so that one may ask about $\operatorname{Pic} Y_T$. To state his theorem, Ein starts with a vector bundle F of rank r and considers integers m_1, m_2, m_3, m_4 large enough that

A. The line bundle $A = \Lambda^r E \otimes K_X \otimes \mathcal{O}_X(m_1)$ is very ample.
B. $H^i(F(m-i)) = H^i(F(m) \otimes A^{-i}) = 0$ for $i > n$ and $m \geq m_2$.
C. $H^k(F \otimes \Lambda^{r-k} F \otimes \mathcal{O}_X((r-k)m) \otimes K_X) = 0$ for $0 < k < r$ and $m \geq m_3$.
D. For $0 < k \leq r$ and $m \geq m_4$, $H^{n+t-k-1}(\Lambda^k F \otimes \Theta_X \otimes \mathcal{O}_X(km) \otimes K_X) = 0$ and $H^{n+t-k-2}(\Lambda^k F \otimes \Theta_X \otimes \mathcal{O}_X(km) \otimes K_X) = 0$.

(see [**19**, 2.4A,2.4B,2.4C,2.4D]). For $E = F(m)$ with $m \geq \max\{m_1, m_2 - 1, m_3, m_4\}$ he proves the following [**19**, Thm. 2.2 and Thm. 2.4]:

Theorem 2.10. *Let X be a smooth projective n-fold and E a vector bundle of rank r on X. Suppose E is sufficiently ample (see above) and $T \subset H^0(E)$ is a general t-dimensional subspace with Y_T smooth of dimension ≥ 2, then*

(1) *If $t = 1$, then* $\operatorname{Pic} Y_T \cong \operatorname{Pic} X$.
(2) *If $t > 1$, then* $\operatorname{Pic} Y_T \cong \operatorname{Pic} X \oplus \mathbb{Z}$.

Remark 2.11. When $t > 1$, the "extra" copy of $\mathbb{Z}$ can be explained as follows. Since $\operatorname{rank} \sigma|_{Y_T} = t - 1$, the kernel of the restriction $\sigma_{Y_T} : T \otimes \mathcal{O}_{Y_T} \to E|_{Y_T}$ is a line bundle, which freely generates the cokernel of the map $\operatorname{Pic} X \to \operatorname{Pic} Y_T$.

Ein splits his proof into two cases. If $\dim Y_T > 2$, the proof follows fairly easily from diagram (1.2). If $\dim Y_T = 2$, the proof is much harder and it is only here that he uses the cohomological vanishings A-D above. While he makes no use of Hodge structures, he does prove an infinitesimal comparison similar to Theorem 2.3, using the fact that the obstructions to lifting a line bundle on Y infinitesimally are given by the cup product $H^1(\Omega_Y) \otimes H^1(\mathcal{T}_Y) \to H^2(\mathcal{O}_Y)$ [**67**].

REMARK 2.12. For comparison, in the case $r = t = 1$ when X is a threefold and E is a line bundle, his vanishings are used to deduce the following conditions, which suffice for his proof:

(1) The multiplication map $H^0(X, L) \otimes H^0(X, K_X \otimes L) \to H^0(X, K_X \otimes L^2)$ is surjective [**19**, 2.5.3].
(2) $H^1(\Omega_X) \cong H^1(\Omega_X|_Y)$ [**19**, 2.5.1].

Recall that a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$ is (Castelnuovo-Mumford) m-regular if $H^i(\mathbb{P}^n, \mathcal{F}(m-i)) = 0$ for all $i > 0$. This useful condition implies that $\mathcal{F}(m)$ is generated by its global sections. The most important application of Theorem 2.10 is the following [**19**, Thm 3.3]:

THEOREM 2.13. *Let E be a (-2)-regular rank r vector bundle on $\mathbb{P}^n$ with $n \geq 3$ and $r \geq n-2$. Let $T \subset H^0(E)$ be a generic subspace of dimension $t = r+3-n$. Then Y_T is a smooth surface and* $\operatorname{Pic} Y_T = \langle \mathcal{O}_{Y_T}(1) \rangle$ *(resp.* $\operatorname{Pic} Y_T \cong \mathbb{Z} \oplus \mathbb{Z}$*) if $t = 1$ (resp. $t > 1$) unless*

(1) *$E = \mathcal{O}_{\mathbb{P}^3}(2)$.*
(2) *$E = \mathcal{O}_{\mathbb{P}^3}(3)$.*
(3) *$E = \mathcal{O}_{\mathbb{P}^4}(2) \oplus \mathcal{O}_{\mathbb{P}^4}(2)$.*

This extends Corollary 1.15 because complete intersection surfaces are general sections of direct sums of line bundles.

REMARK 2.14. Spandaw has extended some of these results [**79, 80**].

2.6. Singular surfaces. In 1995 Joshi used ideas of Green [**26**] and Nori [**66**] sketched in unpublished notes of Mohan Kumar and Srinivas [**54**] to prove a new variant of Noether's theorem [**50**, Propositions 2.1, 3.1 and 4.1]:

THEOREM 2.15. *Let X be a smooth threefold with very ample line bundle L and assume*

(1) *$H^1(X, \Omega_X^2 \otimes L) = 0$.*
(2) *$H^1(X, M \otimes K_X \otimes L) = 0$, where $M = \ker(H^0(X, L) \otimes \mathcal{O}_X \to L)$.*

Then the the restriction map $\operatorname{Pic} X \to \operatorname{Pic} Y$ *is an isomorphism for the very general surface $Y \in |L|$.*

Joshi's proof runs as follows. Let $\mathcal{Y} \subset \mathcal{X} = X \times |L|$ be the universal family of surfaces and, for $s \in |L|$ corresponding to a smooth surface Y_s, let $T \subset |L|$ be the closed subscheme defined by the ideal sheaf m_s^2. Base extension by T gives universal infinitesimal deformations, *i.e.*, the family $\mathcal{Y}_T \subset \mathcal{X}_T$. The vanishing hypotheses in Theorem 2.15 imply the vanishing $H^2(X, \Omega_{\mathcal{Y}_T, \mathcal{X}_T}|_{X \times s}) = 0$, which implies an infinitesimal Noether theorem, which implies the global Noether theorem stated.

REMARK 2.16. The vanishing (2) in Theorem 2.15 imples the surjectivity of the multiplication maps (1) in Remark 2.12, but the first vanishing (1) doesn't appear to relate to any of Ein's vanishing assumptions.

The comparison in the preceding remark suggests the following question:

QUESTION 2.17. For a smooth threefold X with very ample line bundle L, are there cohomological vanishings implied by both Ein's hypotheses from Remark 2.12 and Joshi's from Theorem 2.15 that suffice for Noether's conclusion to hold for $Y \in |L|$?

Joshi's most important contribution is the observation that one can pick up the general *singular* surface with some extra vanishing hypotheses [**50**, Thm 5.1]:

THEOREM 2.18. *Let L be a very ample line bundle on a smooth threefold X. Then for $n \gg 0$, the Noether-Lefschetz locus has codimension ≥ 2 in the complete linear system $|L^n|$. In particular, the restriction $\operatorname{Pic} X \to \operatorname{Pic} Y$ is an isomorphism for the very general singular surface $Y \in |L^n|$.*

What Joshi actually proves is that for a fixed point $x \in X$, the very general surface Y containing the triple point defined by $\mathcal{I}^2_{X,x}$ satisfies the conclusion. Joshi's method is to consider the incomplete linear system defined by the triple point, interpreting the problem on the blow-up $\widetilde{X} \to X$ at x. For L sufficiently ample (obtained by replacing L with L^n), he works with a linear system which is base-point free and big, an idea we have already seen in Theorem 2.21.

2.7. Normal ambient threefolds. Ravindra and Srinivas have extended some of these results to hypersurfaces of normal ambient spaces. Their version of Noether's theorem for hypersurface sections of a normal threefold follows [**74**]:

THEOREM 2.19. *Let $f : X \to \mathbb{P}^n$ be a morphism of a normal projective threefold such that $\mathcal{O}_X(1)$ is ample and assume that $(f_* K_X)(1)$ is generated by its global sections. Then for the very general hyperplane $H \subset \mathbb{P}^n$, the surface $Y = f^{-1}(H)$ is normal and the restriction map $\operatorname{Cl} X \to \operatorname{Cl} Y$ on class groups is an isomorphism.*

To prove the theorem, they prove an analogous statement on a desingularization $\widetilde{X} \to X$ for big and base-point free line bundles. In this setting, their method has the same spirit as Grothendieck's proof of Theorem 1.16, using the formal completion of $\widetilde{X}$ along $\widetilde{Y}$. In particular, there is no use of Hodge theory. To see Theorem 1.3 as a special case, take $f : \mathbb{P}^3 \hookrightarrow \mathbb{P}^N$ to be the d-uple embedding. Working in $\mathbb{P}^N$ we interpret $(f_* K_{\mathbb{P}^3})(1)$ as $\mathcal{O}_{\mathbb{P}^3}(-4)(d) = \mathcal{O}_{\mathbb{P}^3}(d-4)$ which is generated by global sections for $d \geq 4$.

After explaining the statement and methods of their theorem, Srinivas says that it is not clear what the most general assertion is in the direction of Noether's theorem, as statement that would include Theorem 2.19 above and the classical statement. We agree, and pose the question:

QUESTION 2.20. *Is there a common generalization to the various forms of Noether's theorem presented here? What form would such a statement take?*

Ravindra and Srinivas have also proved a variant of the Grothendieck-Lefschetz theorem for normal ambient varieties [**73**]:

THEOREM 2.21. *Let X be a normal projective variety, L an ample line bundle on X and $V \subset H^0(X, L)$ be a base point free linear system. Then the general member $Y \in |V|$ is normal and the restriction map $r : \operatorname{Cl} X \to \operatorname{Cl} Y$ satisfies the following:*

(a) *If $\dim X \geq 4$, then r is an isomorphism.*
(b) *If $\dim X = 3$, then r is injective with finitely generated cokernel.*

Their proof reduces to an analogous result on a desingularization $\pi : \widetilde{X} \to X$ where the pull-backs $\widetilde{Y}$ of $Y \in |V|$ are smooth but no longer correspond to an ample line bundle. Instead, $\pi^* L$ is base-point free and *big*, meaning that tensor powers L^k define maps that are birational onto their images for $k \gg 0$. They prove a variant of Theorem 1.16 for base-point free big linear systems, using a method similar to that of Grothendieck, including the use of Lefschetz conditions.

2.8. Noether's theorem with base locus. We discuss our extension of the classic Noether-Lefschetz Theorem to linear systems $V \subset H^0(\mathcal{O}_{\mathbb{P}^3}(d))$ having a fixed base locus Z, the main result being a computation of the class group $\mathrm{Cl}\, Y$ of the general member $Y \in |V|$. Our theorem recovers both Theorems 1.3 and 3.8 but has something new to say when Z is non-reduced or has points of embedding dimension three.

A theorem of Lopez (see Theorem 3.8 below) says that the general surface S containing a smooth curve Z has Picard group freely generated by Z and $\mathcal{O}_S(1)$, a geometrically pleasing result [**60**]. What happens if we replace Z with an arbitrary curve, possible reducible, non-reduced and unmixed? Assuming Z lies on a normal surface, it's natural to ask what the $\mathrm{Cl}\, S$ looks like. Our answer is the follows [**4**, Theorem 1.1]:

THEOREM 2.22. *Let $Z \subset \mathbb{P}^3_{\mathbb{C}}$ be a closed subscheme of dimension at most one with at most finitely many points of embedding dimension three. If $\mathcal{I}_Z(d-2)$ is generated by global sections with $d \geq 4$, then the very general surface of degree d containing Z is normal with class group* $\mathrm{Cl}\, S$ *freely generated by $\mathcal{O}_S(1)$ and the supports of the curve components of Z.*

The hypotheses on Z are the weakest that ensure Z lies on normal surfaces of high degree. For simplicity we have stated the theorem assuming $\mathcal{I}_Z(d-2)$ is generated by global sections, but it is enough to assume that either

(1) Z is reduced of embedding dimension two or
(2) $H^0(\mathcal{I}_Z(d-2)) \neq 0$.

The proof of Theorem 2.22 is based on the degeneration argument of Griffiths and Harris (see Section 2.4). The construction is complicated by the fact that the general surface containing Z may be forcibly singular; we therefore construct an étale covering of the blown-up family in order to sort out the singularities on the fibers.

REMARK 2.23. We note special cases in which one can read off $\mathrm{Pic}\, S$.

(a) If Z is empty, then $\mathrm{Pic}\, S = \langle \mathcal{O}_S(1) \rangle$ and we recover the classic Noether-Lefschetz theorem.
(b) If $\dim Z = 0$ (*i.e.*, $r = 0$), then again $\mathrm{Cl}\, S = \langle \mathcal{O}_S(1) \rangle$; this reveals the geometrically intuitive fact that codimension two base loci don't affect the class group. Since $\mathcal{O}_S(1)$ is a line bundle, $\mathrm{Pic}\, S = \mathrm{Cl}\, S$. If Z has points of embedding dimension three, this strengthens Theorem 2.18.
(c) The most interesting case occurs when Z is a curve with irreducible components $Z_1, Z_2, \ldots Z_r$. Here are some samples:
   (i) If Z is a reduced local complete intersection and the Z_i intersect at points of embedding dimension 2, then $\mathrm{Pic}\, S = \langle \mathcal{O}_S(1), Z_1, \ldots, Z_r \rangle$.
   (ii) If Z is an integral local complete intersection curve, then the Picard group of S is $\mathrm{Pic}\, S = \langle \mathcal{O}_S(1), \mathcal{O}_S(Z) \rangle$.

(iii) If Z has embedding dimension 2, then $\operatorname{Pic} S = \langle \mathcal{O}_S(1), Z_1, \dots, Z_r \rangle$.

REMARK 2.24. If p is a point on a normal surface S, the natural restriction map $\operatorname{Cl} S \to \operatorname{Cl} \mathcal{O}_{S,p}$ is surjective and the natural map $\operatorname{Cl} \mathcal{O}_{S,p} \hookrightarrow \operatorname{Cl} \widetilde{\mathcal{O}}_{S,p}$ is an inclusion, so one can identify $\operatorname{Cl} \mathcal{O}_{S,p}$ as the image of the composite $\operatorname{Cl} S \to \operatorname{Cl} \mathcal{O}_{S,p} \to \operatorname{Cl} \widetilde{\mathcal{O}}_{S,p}$. For S a very general surface containing Z as in Theorem 2.22 above, it follows that the local class group $\operatorname{Cl} \mathcal{O}_{S,p}$ is generated by the supports of the curve components of Z, which makes the class group computable for such local rings. We used this to compute the class groups of local rings on such surfaces for various base loci [**6**]. Moreover, from this data one can use Jaffe's exact sequence

$$0 \to \operatorname{Pic} S \to \operatorname{Cl} S \to \oplus_{p \in \operatorname{Sing} S} \operatorname{Cl} \mathcal{O}_{S,p}$$

to compute the Picard groups of the singular surfaces. In Section 4 we will see applications of this technique to questions of Srinivas.

REMARK 2.25. In related work, Di Gennaro and Franco have recently computed the Néron-Severi groups of general high degree hypersurface sections of a smooth complete intersection Y of odd dimension containing a fixed base locus Z, provided that the general such hypersurface section $H \in |H^0(I_{Z,Y}(d))|$ [**16**]. Thus in the special case $Y = \mathbb{P}^3$ and Z reduced of embedding dimension two they recover the conclusion of Theorem 2.22, but their method says nothing about the class groups if Z is more general.

3. Components of the Noether-Lefschetz locus

While Noether's theorem says that each irreducible component $V \subset NL(d)$ is proper in the space $|\mathcal{O}_{\mathbb{P}^3}(d)|$ of surfaces of degree d for $d > 3$, one can ask deeper questions about these components. In this section we survey what is known about the nature of these components. Specifically, we discuss bounds on the codimensions of the components, the density of the general components due to Ciliberto, Harris and Miranda [**13**], the work of Green and Voisin classifying those of smallest codimension [**27, 85, 86**], including the asymptotic result of Ontinowska [**68**], and result of Ciliberto and Lopez on the distribution of codimensions [**14**]. We close with Voisin's example of Noether-Lefschetz loci having infinitely many special components [**88**].

3.1. The density theorem. Noether's theorem and its variations tell us that the components of the Noether-Lefschetz locus $NL(d) \subset |\mathcal{O}_{\mathbb{P}^3}(d)|$ are proper subvarieties. For such a component $V \subset NL(d)$, the explicit Noether-Lefschetz theorem gives a lower bound on the codimension of V, namely $d - 3 \le c(V)$. Hodge theory provides an upper bound as well [**9**, 3.a.25]: a class $\lambda \in H^2(S, \mathbb{C})$ is algebraic if and only if it has type $(1,1)$, which is equivalent to the vanishing $\int_\lambda \omega = 0$ for each holomorphic 2-form $\omega \in H^0(S, \Omega_S^2)$. These vanishings impose $p_g(S) = \dim H^0(S, \Omega_S^2)$ conditions which define $NL(d)$, and hence

$$d - 3 \le c(V) \le p_g = \binom{d-1}{3}. \tag{3.1}$$

One expects the conditions arising from the differentials $\omega \in H^0(S, K_S)$ to be independent, so the components of codimension p_g are called *general* while the other components are called *special*. The first following result tells us about the distribution of the general components.

THEOREM 3.1. **(The density theorem)** *The union of general components of $NL(d)$ is Euclidean dense (and hence Zariski dense) in $|\mathcal{O}_{\mathbb{P}^3}(d)|$.*

Ciliberto, Harris and Miranda proved that the union of general components of $NL(d)$ is *Zariski* dense by induction on d, using the dense union of general components in degree $d-1$ to construct infinitely many such components in degree d: to show density, they show that the closure contains the family $R(d)$ of (reducible) surfaces containing a plane (see the Griffiths-Harris degeneration proof, § 2.4) and that these components approach $R(d)$ in directions which are dense within the normal bundle to $R(d)$ [**13**]. The stronger result follows from an argument of Mark Green showing that the existence of *just one* general component already implies that $NL(d)$ is dense in the *Euclidean* topology.

We sketch an argument based on Green's, using notation from § 2.1.5. If $NL(d)$ has a general component, then there is a surface Y_0 and a Hodge class $\lambda \in H^2(Y_0, \mathbb{Q})$ for which the Hodge locus U_λ has codimension p_g. To show Euclidean density, consider an open ϵ-ball $B_b \subset B$ about an arbitrary point $b \in B$ and construct a contractible tubular neighborhood U of a path from b to 0 which contains B_b. The *real* vector bundle $\mathcal{H}^2_{\mathbb{R}}$ with fibers $H^2(Y, \mathbb{R})$ is trivial over U, so we can form the diagram

$$\begin{array}{ccccc} \mathcal{H}^{1,1}_{\mathbb{R}}|_U & \stackrel{\sigma}{\hookrightarrow} & \mathcal{H}^2_{\mathbb{R}} \cong U \times H^2(Y_0, \mathbb{R}) & \stackrel{\pi_2}{\to} & H^2(Y_0, \mathbb{R}) \\ \downarrow & & \downarrow \pi_1 & & \\ U & = & U & & \end{array}$$

where $\mathcal{H}^{1,1}_{\mathbb{R}}$ is the real subbundle with fibers $H^{1,1}(Y, \mathbb{R})$. The composite map $G = \pi_2 \circ \sigma$ is injective on fibers and $G^{-1}(H^2(Y_0, \mathbb{Q}))$ is the set of Hodge classes. If $N \subset H^2(Y_0, \mathbb{Q})$ is the complement of the subspace generated by $c_1(\mathcal{O}(1))$, then $\pi_1(G^{-1}(N)) = NL(d) \cap U$. Note that $N \subset H^2(Y_0, \mathbb{R})$ is dense because $\dim H^2(Y_0, \mathbb{R}) > 1$ for $d > 4$. One can use the complex bundles $\mathcal{H}^{2,0}_{\mathcal{C}} \oplus \mathcal{H}^{1,1}_{\mathcal{C}} \subset \mathcal{H}^2_{\mathcal{C}}$ to show that the image of G contains an open neighborhood of $\lambda \in H^2(Y_0, \mathbb{R})$, hence $S = \{x \in \mathcal{H}^{1,1}_{\mathbb{R}}|_U : dG_x \text{ has maximal rank}\}$ is non-empty [**13**, Basic Claim, p. 679]. As the complement of a closed analytic set, S is therefore dense in $\mathcal{H}^{1,1}_{\mathbb{R}}|_U$ and $G|_S : S \to H^2(Y_0, \mathbb{R})$ has the local structure of a real projection [**72**]. It follows that $G^{-1}(N)$ is dense in $\mathcal{H}^{1,1}_{\mathbb{R}}$, so $\pi_1(G^{-1}(N))$ is dense in U and in particular intersects B_b.

REMARK 3.2. Sung-Ock Kim extended Green's argument to the space of complete intersection surfaces in higher dimensional projective spaces [**53**], and Voisin proved a variant valid for variations of Hodge structures [**90**, Prop. 5.20].

3.2. Components of small codimension. Given that the general components of $NL(d)$ are dense, it is natural to investigate what the components of small codimension might look like. Are they also dense? What are the possible codimensions? How are they distributed? Ciliberto, Harris and Miranda asked the following question [**13**].

QUESTION 3.3. *Do the Noether-Lefschetz components $V \subset NL(d)$ having small codimension consist of surfaces containing curves of small degree?*

To make this question precise one would have to specify how small is "small," but the general idea is clear. The answer is yes for the smallest codimensions:

THEOREM 3.4. *For $d > 4$, each Noether-Lefschetz component $V \subset NL(d)$ has codimension $c(V) > 2d - 7$ with two exceptions:*

(a) *The family V of surfaces containing a line, for which $c(V) = d - 3$.*
(b) *The family V of surfaces containing a conic, for which $c(V) = 2d - 7$.*

Example 3.5. This theorem tells the whole story when $d = 5$: the Noether-Lefschetz locus $NL(5)$ has one component of codimension $d - 3 = 2$ (the surfaces containing a line), one component of codimension $2d-7 = 3$ (the surfaces containing a conic) and infinitely many components of codimension $p_g = 4$, which are dense in $|\mathcal{O}_{\mathbb{P}^3}(5)|$ by Theorem 3.1.

Theorem 3.4 (a) was proved by Voisin in 1988 [**85**] and independently by Green [**27**] by a surprising application of Gotzmann's persistence theorem [**23**] and Macaulay's growth bound [**62**]. Voisin obtained the complete theorem stated in 1989 [**86**]. Building on the technique of Green and Voisin, Otwinowska extended Theorem 3.4 in an asymptotic sense to components of codimension roughly bd for any integer $b > 0$ for the generalized Noether-Lefschetz locus (2.2) from in Section 2.1.5. Her interesting result gives evidence toward the Hodge conjecture for $m > 1$ and asymptotically confirms Conjecture 3.3 for $m = 1$ [**68**, Corollary 3]:

Theorem 3.6. *Fix integers $b > 0$, $m > 0$ and set $N = 2m + 1$. Then for $d \gg 0$, any component V of the Noether-Lefschetz locus $NL(d) \subset |\mathcal{O}_{\mathbb{P}^N}(d)|$ with $c(V) \leq b\frac{d^m}{m!}$ consists of hypersurfaces containing an m-dimensional subvarieties of degree $\leq b$.*

Most of Otwinowska's proof is devoted to a purely algebraic result to the effect that for d sufficiently large with respect to fixed $t < r$, if $I \subset S = \mathbb{C}[x_0, \dots, x_r]$ is a homogeneous ideal containing $r+1$ polynomials of degree $d-1$ which form a regular sequence and S/I is a finite length Gorenstein ring of socle degree $(t + 1)(d - 2)$ with $\dim(S/I)_d \leq b\frac{d^t}{t!}$, then I contains an ideal I_V defining a subscheme V of pure dimension t and degree at most b. Taking inspiration from work of Voisin [**86**], Otwinowska applies this result to ideals E_r which describe deformations of order $r + 1$ to U_λ in a neighborhood of a hypersurface F to obtain the result, though there is some extra work to show that $V_{\text{red}} \subset F$.

3.3. Distribution of codimensions. The results above give an understanding of what happens for Noether-Lefschetz components of the largest and smallest codimensions according to (3.1), but what happens in between? The answer is that there are some gaps in the codimension $c(V)$ at the bottom, but Ciliberto and Lopez showed that there are no gaps beyond codimension roughly $9/2d^{1.5}$ [**14**]:

Theorem 3.7. *For each degree $d \geq 8$ and each integer c with*

$$\min\{3/4d^2 - 17/4d + 19/3, 9/2d^{3/2}\} \leq c \leq p_g = \binom{d-1}{3}$$

there exists a component $V \subset NL(d)$ of codimension $c(V) = c$.

Their construction is lengthy, as they must construct families of each codimension. As a main tool towards building these families, they prove in general that if W is a component of the Hilbert scheme for curves in $\mathbb{P}^3$ whose general member C is smooth such that

(1) $\mathcal{I}_C$ is $(d - 1)$-regular.
(2) $H^1(\mathcal{I}_C(d - 4)) = 0$,

then the family V of degree d surfaces containing a curve $C \in W$ is a component of $NL(d)$ having codimension $c(V) = h^0(\mathcal{O}_C(d-4)) - \dim W + 4 \deg C$. With this established, they use well known families of curves with general moduli and on smooth cubic surfaces satisfying (1) and (2) to construct components and keep careful track of the codimensions they obtain. The general surface S in V has a predictable Picard group:

THEOREM 3.8. *Let $d \geq 4$ be an integer and $C \subset \mathbb{P}^3$ be a smooth connected curve such that $\mathcal{I}_C(d-1)$ is generated by global sections. Then the very general surface S containing C of degree d has Picard group $\operatorname{Pic} S \cong \mathbb{Z}^2$ generated by $\mathcal{O}_S(1)$ and C.*

Lopez proved a stronger theorem [**60**, Theorem II.3.1], but it is this special case [**60**, Corollary II.3.8] that has found many applications [**14, 10, 21**]. While Lopez used the degeneration method of Griffiths and Harris to prove his result (see §1.4 above), we give a short argument using Theorem 2.19. The linear system $H^0(\mathcal{I}_C(d)) \subset H^0(\mathcal{O}(d))$ gives a birational map $\mathbb{P}^3 \to \mathbb{P}H^0(\mathcal{I}_C(d))$ whose indeterminacy locus is C. Blowing up C yields the diagram

$$\begin{array}{ccccc} E & \subset & \widetilde{\mathbb{P}}^3 & \stackrel{\sigma}{\hookrightarrow} & \mathbb{P}H^0(\mathcal{I}_C(d)) \\ \downarrow & & \downarrow \pi & & \\ C & \subset & \mathbb{P}^3 & & \end{array}$$

where $\pi : \widetilde{\mathbb{P}}^3 \to \mathbb{P}^3$ is the blow-up and the exceptional divisor E is a $\mathbb{P}^1$-bundle over C via π. By the construction in [**71**, Proposition 4.1], σ is a closed immersion defined by the very ample line bundle $L = \pi^*\mathcal{O}(d) - E$. The canonical class on $\widetilde{\mathbb{P}}^3$ is given by $K_{\widetilde{\mathbb{P}}^3} = \pi^* K_{\mathbb{P}^3} + E$ and therefore $(\sigma_* K_{\widetilde{\mathbb{P}}^3})(1)$ may be thought of as $K_{\widetilde{\mathbb{P}}^3} \otimes L = \pi^*\mathcal{O}(d-4)$ on $\widetilde{\mathbb{P}}^3$ which is generated by global sections since $d \geq 4$. Applying Theorem 2.19 above, the map $\operatorname{Pic} \widetilde{P}^3 \to \operatorname{Pic} Y$ is an isomorphism for very general $Y \in |L|$. Since $\operatorname{Pic} \widetilde{\mathbb{P}}^3$ is freely generated by $\pi^*\mathcal{O}(1)$ and E, the same is true of $\operatorname{Pic} Y$. Finally the map π induces an isomorphism $Y \to S = \pi(Y)$ in which the class of E on Y becomes the class of C on S, therefore $\operatorname{Pic} S$ freely generated by $\mathcal{O}(1)$ and C.

Regarding Theorem 3.8, it is difficult to imagine a component $V \subset NL(d)$ whose generic surface has Picard number greater than two. Indeed, Ciliberto, Harris and Miranda ask the following natural question [**13**]:

QUESTION 3.9. *If S is a general surface in a Noether-Lefschetz component $V \subset NL(d)$, is $\operatorname{Pic} S \cong \mathbb{Z}^2$?*

3.4. Voisin's example. The results of Green and Voisin in Section 3.2 were at least partially motivated by Harris' conjecture that $NL(d)$ should have only finitely many special components for $d \geq 4$ [**27**, p. 301]. This is certainly true for $d = 4$ (because all components are general) and for $d = 5$ by Example 3.5; there's no doubt that Noether knew of many more when he stated Noether's theorem in the first place. One approach to proving it arises from infinitesimal variations of Hodge structures. Griffiths and Harris observe that if S is a general member of a special component, then $H^0(S, K_S(-\Gamma)) \subset H^{2,0}(-\gamma)$ [**39**, 4.a.4]. Green asked whether every special component V has the following special property: that for general $S \in V$ there is a canonical form $\omega \in H^0(S, K_S)$ which is the supporting divisor of the class $\lambda \in H^{1,1} \cap H_{\mathbb{Z}}$. A positive answer would yield a proof of the conjecture. For $d = 6, 7$, Voisin proved even a stronger statement [**87**, 0.3], but she notes that the two supporting lemmas are false for $d > 7$. After the work towards proving Harris' conjecture, it was finally Voisin who produced a counterexample in 1991 [**88**]:

Example 3.10. Fix $s > 0$ and let U denote the family of surfaces Σ of degree $d = 4s$ given by equations of the form

$$0 = P(F_0, F_1, F_2, F_3)$$

where F_i are polynomials of degree s in x_i having no common zeros and P is a quartic polynomial. Geometrically Σ is the pull-back of the quartic surface S with equation $0 = P(y_0, y_1, y_2, y_3)$ under the map $\psi : \mathbb{P}^3 \to \mathbb{P}^3$ given by $\psi(x_0, x_1, x_2, x_3) = (F_0, F_1, F_2, F_3)$ and hence the general member of U is smooth by Bertini's theorem. Voisin uses an infinitesimal argument to show that the general member of U is not contained in $NL(d)$. On the other hand, the finite map yields injections $\operatorname{Pic} S \to \operatorname{Pic} \Sigma$, so the infinitely many dense codimension 1 components of $NL(4)$ pull back to infinitely many dense codimension 1 families contained in $U \cap NL(d)$. Each such family is contained in a *special* component of $NL(d)$ for s large by reason of dimension: Voisin shows that $\dim U = 4(h^0(\mathcal{O}(s)) - 4) + 34$, which is strictly larger than the dimension $1+2d^2$ of a general component of $NL(d)$. Since they are dense in U, they cannot be contained in a finite union of special components from $NL(d)$. Noting that each of these families is contained in $\overline{U}$, Voisin poses the following question [**88**, 0.5]:

Question 3.11. *Is the union of special components of the Noether-Lefschetz locus $NL(d)$ Zariski dense in $|\mathcal{O}_{\mathbb{P}^3}(d)|$?*

4. Class groups and Noether's theorem

Given a normal Noetherian local domain A with completion $\hat{A}$, it is well known that the induced map $\operatorname{Cl} A \to \operatorname{Cl} \hat{A}$ on class groups is injective [**76**, Proposition 1]. In [**82**, Question 3.1] Srinivas poses the following general question:

Question 4.1. *Given $\hat{A}$, what are the possible images of this injection?*

He follows this up with a more specific question [**82**, Question 3.7]:

Question 4.2. *Given $\hat{A}$, does there exists A such that* $\operatorname{Cl} A$ *is generated by its canonical module ω_A?*

Since freeness of ω_A is equivalent to A being Gorenstein, and the Gorenstein property is stable under completion, Question 4.2 reduces in the case that $\hat{A}$ is Gorenstein to whether $\operatorname{Cl} A$ can be trivial, *i.e.*, whether A can taken to be a UFD.

Heitmann actually characterized complete local rings which are completions of a UFD, proving that a complete local ring A is the completion of a UFD if and only if A is a field, A is a DVR, or A has depth ≥ 2 and no integer is a zero-divisor of A [**48**]. Heitmann's construction, however, is set-theoretic, and the UFD he produces is in general not an excellent ring. Therefore, Srinivas poses the question in the context of so-called "geometric" rings, that is, those that are essentially of finite type over $\mathbb{C}$. In the sequel, all rings under consideration are geometric rings or completions of such, and we take Question 4.2 to be posed for geometric rings.

In studying the K-theory of the local rings defining rational double point surface singularities, Srinivas in [**81**, § 2] found that such a ring is always the completion of a (geometric) UFD. This result is greatly generalized by Parameswaran and Srinivas, who prove the following [**69**, Theorems 1 and 2]:

THEOREM 4.3. *The answer to Question 4.2 is affirmative for isolated complete intersection singularities of dimensions* 2 *and* 3 *(and therefore of all dimensions).*

REMARKS 4.4.

(1) Since a complete intersection is Gorenstein, Theorem 4.3 shows that the local ring of such a singularity has a UFD in its analytic isomorphism class.

(2) The reason that dimensions 2 and 3 are the critical ones for this result is that a theorem of Grothendieck [**42**, Exp. XI] states that a complete local ring of dimension ≥ 4 that is a complete intersection and regular in codimension 3 – true when the singularity is isolated – is already a UFD, and therefore so is any local ring that completes to it.

The method of Parameswaran and Srinivas is roughly as follows: Given the local ring A of a complete intersection isolated singularity at the origin in $\mathbb{C}^n$, they show that a sufficiently general perturbation of the polynomials defining the singularity by polynomials in a sufficiently high power of the maximal ideal of A defines a surface S with a singularity at the origin that is analytically isomorphic to A. They then construct a Lefschetz-type pencil of such surfaces and show via a monodromy argument that the general surface has the property that $\mathrm{Cl}\,\mathbb{P}^n_{\mathbb{C}} \to \mathrm{Cl}\,S$ is an isomorphism, which means that $\mathrm{Cl}\,S$ is generated by $\mathcal{O}_S(1)$, so that the class group of the local ring of S at the origin must be trivial and thus the local ring a UFD.

Using methods motivated by the classical Lefschetz proof but also involving sophisticated applications of singularity theory and adjunction theory, Parameswaran and van Straten [**70**, Thm. 1.1] give a solution for any normal surface singularity:

THEOREM 4.5. *The answer to Question 4.2 is affirmative for all normal surface singularities.*

Theorem 2.22 can be applied to Questions 4.1 and 4.2 for hypersurface singularities: In [**5**, Thm. 1.2] we prove the following:

THEOREM 4.6. *Let $A = \mathbb{C}[[x_1, \dots x_n]]/f$, where f is a polynomial defining a variety V which is normal at the origin p. Then there exists an algebraic hypersurface $X \subset \mathbb{P}^n_{\mathbb{C}}$ and a point $p \in X$ such that $R = \mathcal{O}_{X,p}$ is a UFD and $\widehat{R} \cong A$.*

Thus Question 4.2 has an affirmative answer for all normal hypersurface singularities, isolated or not.

The method of proof is as follows: By normality and the Jacobian criterion, the ideal $(f, f_{x_1}, f_{x_2}, ... f_{x_n})$ defines a subscheme Y whose components containing the origin all have codimension at least 3 in $\mathbb{P}^n_{\mathbb{C}}$. Taking Z to be a suitable thickening of these components (defined in fact by $(f, f^3_{x_1}, f^3_{x_2}, ... f^3_{x_n})$), and using power-series arguments similar to those used in the proof of Theorem 4.3 above, show that the very general hypersurface X of high degree containing Z has a singularity at the origin analytically isomorphic to that of V. On the other hand, since Z has no components in codimension 2, Theorem 2.22 implies that $\mathrm{Cl}\,X = \langle \mathcal{O}_X(1) \rangle$, so that the class group of $\mathcal{O}_{X,p}$ is trivial; therefore $\mathcal{O}_{X,p}$ is a UFD in the desired analytic isomorphism class.

For many relatively well behaved base loci in $\mathbb{P}^3_{\mathbb{C}}$, such as multiplicity structures on a smooth curve lying on a smooth surface or unions of such, we have shown that the singularities of the very general surface containing such a base locus has only *rational double point* singularities [**6**]. We exploit this fact to prove the following [**5**, Thm. 1.3], which generalizes the above-mentioned result [**81**, § 2] of Srinivas:

THEOREM 4.7. *Fix* $T \in \{\mathbf{A}_n, \mathbf{D}_n, \mathbf{E}_6, \mathbf{E}_7, \mathbf{E}_8\}$ *and a subgroup* H *of the class group of the completed local ring for a singularity of type* T*. Then there exists an algebraic surface* $S \subset \mathbb{P}^3{}_{\mathbb{C}}$ *and a rational double point* $p \in S$ *of type* T *such that* $\mathrm{Cl}\,\mathcal{O}_{S,p} \cong H$*.*

Vis a vis Question 4.1, then, in the case of rational double point surface singularities, *every* subgroup arises as the image of the natural injection $\mathrm{Cl}\,A \to \mathrm{Cl}\,\hat{A}$.

The method of proof here is to construct a 1-dimensional base locus Z supported on an irreducible curve C such that the very general surface S of sufficiently high degree containing Z has the appropriate singularity. By Theorem 2.22, $\mathrm{Cl}\,S$ is generated by C, so we also ensure that C generates the appropriate subgroup in the completion.

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