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GLOBAL INVERSION VIA THE PALAIS-SMALE CONDITION

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Abstract. Fixing a complete Riemannian metric g on $\mathbb{R}^n$, we show that a local diffeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ is bijective if the height function $f \cdot v$ (standard inner product) satisfies the Palais-Smale condition relative to g for each for each nonzero $v \in \mathbb{R}^n$. Our method substantially improves a global inverse function theorem of Hadamard. In the context of polynomial maps, we obtain new criteria for invertibility in terms of Lojasiewicz exponents and tameness of polynomials.

1. **Introduction.** In this paper we are interested in finding conditions under which a local diffeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ is bijective. This type of question appears in areas as diverse as algebraic geometry and mathematical economics. Good general references on this problem are [16], [19] and [27]. Some related bijectivity results can be found in [6, 17]; in particular, the idea of using Palais-Smale type conditions have been exploited in [5, 22, 23].

Letting $f_v : \mathbb{R}^n \to \mathbb{R}$ denote the height function $f_v(x) = \sum_{i=1}^{n} f_i(x)v_i$ for $v \in \mathbb{R}^n$, we have the following:

Theorem 1.1. *A local diffeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ is bijective if there is a a complete Riemannian metric g on $\mathbb{R}^n$ with respect to which f_v satisfies the Palais-Smale condition for all nonzero $v \in \mathbb{R}^n$.*

Let us unwind the hypothesis of Theorem 1.1. Recall that a continuously differentiable function $h : X \to \mathbb{R}$ (X a Banach space) satisfies the Palais-Smale condition if for each sequence $x_n \in X$ such that $h(x_n)$ is bounded and $|dh(x_n)| \to 0$, there is a convergent subsequence x_{n_k} (see [30], p. 161). If h has no critical points, the Palais-Smale condition for h is equivalent to

$$(*) \inf_{x \in h^{-1}[a,b]} |dh(x)| > 0$$

for all $a < b \in \mathbb{R}$. Indeed, the Palais-Smale condition vacuously holds under the condition $(*)$. Conversely, if the infimum in condition $(*)$ is 0, then there is a sequence $x_n \in h^{-1}[a, b]$ such that $dh(x_n) \to 0$. Assuming the Palais-Smale condition, there is a convergent subsequence x_{n_k} whose limit x satisfies $dh(x) = 0$, contradicting the assumption that h has no critical points.

For a complete Riemannian metric g on $\mathbb{R}^n$ and a smooth function $h : \mathbb{R}^n \to \mathbb{R}$, let $\nabla^{(g)} h$ denote the gradient of h relative to g, so that $g_x(\nabla^{(g)} h, w) = dh_x(w)$ for

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all $w \in \mathbb{R}^n$. Taking $h = f_v$ in Theorem 1.1 above, h has no critical points (because f is a local diffeomorphism), hence by the preceding paragraph the Palais-Smale condition on h is equivalent to

$$\inf_{x \in h^{-1}[a,b]} |\nabla^{(g)} h(x)|_g > 0$$

for all $a < b \in \mathbb{R}$. Here $|\nabla^{(g)} h(x)|_g$ is the norm of $\nabla^{(g)} h(x)$ computed with respect to g and hence we have

$$|\nabla^{(g)} h(x)|_g = \max_{|w|_g = 1} |dh_x(w)|.$$

Another global injectivity result having a Palais-Smale type hypothesis can be found in [5].

An immediate consequence of Theorem 1.1 is the following:

Theorem 1.2. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth map and consider the sets $S_v = \{Df(x)^* v : x \in \mathbb{R}^n\}$ for $v \in \mathbb{R}^n$.*
(a) If $0 \notin S_v$ for each nonzero $v \in \mathbb{R}^n$, then f is locally invertible.
(b) If $0 \notin \overline{S_v}$ for each nonzero $v \in \mathbb{R}^n$, then f is globally invertible.

Part (a) is the usual inverse function theorem and is included for comparison purposes only. Part (b) follows from Theorem 1.1 if we take g to be the standard metric on $\mathbb{R}^n$ and observe that $\nabla f_v(x) = Df(x)^* v$. Part (b) was first proved in the special case where f is a perturbation of the identity by a homogeneous map; see [25], Theorem 6.

It is important to note that the hypothesis of part (b) does *not* ask that $0 \notin \overline{\bigcup_{v \neq 0} S_v}$. This is a much stronger condition which is essentially the hypothesis of Hadamard's well-known theorem (Theorem 1.6 below).

Remark 1.3. Theorem 1.2 has a natural counterpart for local biholomorphisms $F : \mathbb{C}^n \to \mathbb{C}^n$, where $DF(x)^*$ is the adjoint (conjugate transpose) of the complex Jacobian matrix $DF(x)$. This follows from the Cauchy-Riemann equations since the induced real map $\tilde{F} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ satisfies $|D\tilde{F}(\tilde{p})^t \tilde{v}| = |DF(p)^* v|$. The complex version of Theorem 1.2 is well-suited to polynomial maps $F : \mathbb{C}^n \to \mathbb{C}^n$ in algebraic geometry. In this case the sets $\{DF(x)^* v : x \in \mathbb{C}^n\}$ are constructible; in particular, the closures in the Euclidean and Zariski topologies coincide and the hypothesis of Theorem 1.2(b) becomes an algebraic condition (see Remark 3.5).

Theorem 1.1 is a special case of the following more general result:

Theorem 1.4. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a local diffeomorphism and g a complete Riemannian metric on $\mathbb{R}^n$. Suppose that for each $v \neq 0$ the smooth vector field*

$$\frac{\nabla^{(g)} f_v(x)}{|\nabla^{(g)} f_v(x)|_g^2}$$

is complete, i.e. its integral curves are defined for all times. Then f is bijective.

Theorem 1.1 follows from Theorem 1.4 because a vector field which is bounded (or grows at most like the distance to a fixed point) with respect to a complete metric is itself complete. For example, consider the (complete) metric with weight $(1+|x|)^{-1}$ on $\mathbb{R}^n$. The gradient of a function h in this metric has length $(1+|x|)|\nabla h|$, where ∇h is the standard gradient. Boundedness of the vector field in Theorem 1.4 is equivalent to

$$|\nabla h(x)| > c(1 + |x|)^{-1}$$

for some constant $c > 0$. Comparing with the hypothesis of Theorem 1.2, we see that Theorem 1.1 is strictly better; we get more by allowing metrics other than the standard one.

Another consequence of Theorem 1.4 is the following (see Lemma 2.2 for details):

Theorem 1.5. *A local diffeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ is bijective if, $\forall\ v \in \mathbb{R}^n \setminus \{0\}$,*

$$\int_0^\infty \min_{|x|=r} |\nabla f_v(x)| dr = \infty.$$

This invites comparison with the following ([3], 5.1.5 or [21], Thm. 4.2):

Theorem 1.6. (Hadamard, Plastock) *A local diffeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ is bijective if*

$$\int_0^\infty \min_{|x|=r} ||Df(x)^{-1}||^{-1} dr = \infty.$$

Theorem 1.5 is a considerable strengthening of Theorem 1.6. To see this, observe that

$$||Df(x)^{-1}||^{-1} = ||Df(x)^{-1*}||^{-1} = ||Df(x)^{*-1}||^{-1}$$

$$= \min_{|v|=1} |Df(x)^* v| = \min_{|v|=1} |\nabla f_v(x)|.$$

In particular, we may rewrite the condition of Theorem 1.6 as

$$\int_0^\infty \min_{|x|=r} \min_{|v|=1} |\nabla f_v(x)| dr = \infty.$$

Now it is clear that Theorem 1.5 is much stronger than Theorem 1.6. The following example illustrates this point.

Example 1.7. Consider the simple map $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x, y) = (x+y^3, y)$. To apply Theorem 1.6 we need the divergence of the integral

$$\int_0^\infty \min_{|(x,y)|=r} \min_{|v|=1} |\nabla f_v(x, y)| dr.$$

Writing $v = (v_1, v_2)$ we find that

$$|\nabla f_v(x, y)| = \sqrt{v_1^2 + (3y^2 v_1 + v_2)^2}$$

and in choosing $v = ((1 + 9r^4)^{-\frac{1}{2}}, -3r^2(1 + 9r^4)^{-\frac{1}{2}})$ and $(x, y) = (0, r)$ we obtain the value $(1 + 9r^4)^{-\frac{1}{2}}$, which yields a convergent integral.

Note however that $|\nabla f_v(x, y)| = |v_2| > 0$ if $v_1 = 0$ while $|\nabla f_v(x, y)| \geq |v_1| > 0$ otherwise. In either case

$$\int_0^\infty \min_{|(x,y)|=r} |\nabla f_v(x, y)| dr = \infty$$

and Theorem 1.5 applies (in fact, $f^{-1}(u, v) = (u-v^3, v)$). We will see in Proposition 3.6 that Theorem 1.5 applies to any quadratic map, while Theorem 1.6 need not (see Example 3.7).

Theorem 1.6 remains valid for infinite-dimensional Banach spaces. This version of the theorem is quite useful in non-linear analysis. Although in finite dimension Theorem 1.6 is superseded by Theorem 1.5, the situation in infinite dimension remains unclear since the arguments presented here rely on degree theory.

It is tautological that a local diffeomorphism is injective if and only if the pre-images of points are connected. The Palais-Smale condition on the height functions

f_v implies directly that the pull-backs of hyperplanes are connected. We are thus led to:

Conjecture 1.8. A local diffeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ is injective if the pre-image of every affine hyperplane is connected.

This paper is organized as follows. In the second section we prove the main result, Theorem 1.4, using ideas similar to those found in [25] [29]. We also explain how Theorem 1.4 implies Theorem 1.5. Section 3 is devoted to polynomial maps. After a brief discussion on connectedness of preimages in the this case, we interpret our results to give criteria for invertibility of polynomial maps in terms of Lojasiewicz exponents and tameness of polynomials. We then recover some known results on the jacobian conjecture.

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2. **Proof of the main theorem.** We begin with the following lemma, which readily follows from degree theory (see [15]). We give a self-contained proof to illustrate the dynamics of our approach. For notational convenience, let $S_r = \{x \in \mathbb{R}^n : |x| = r\}$ denote the $(n-1)-$sphere of radius r.

Lemma 2.1. *Let $\psi : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map and fix a point $p \in \mathbb{R}^n$. The map ψ is surjective provided that the following hold:*
(a) $\psi(x) = p$ if and only if $x = 0$.
(b) There is $r_0 > 0$ such that $\psi(S_{r_0}) \neq 0$ in $\pi_{n-1}(\mathbb{R}^n - \{p\}) \cong \mathbb{Z}$.
(c) For each $M > 0$, there exists $R > 0$ such that $|\psi(x)| > M$ if $|x| \geq R$.

Proof: After composing ψ with a translation which affects none of the hypotheses, we lose no generality in assuming $p = 0$. We now show that any $w \in \mathbb{R}^n$ lies in $\psi(\mathbb{R}^n)$. This is immediate from condition (a) if $w = 0$, so we may assume $w \neq 0$. By continuity of ψ there exists a small $0 < \delta < r_0$ such that $|\psi(x)| < |w|$ for each $x \in S_\delta$, hence $\psi(S_\delta)$ is zero in $\pi_{n-1}(\mathbb{R}^n - \{w\})$. Using condition (c), we can find $R > r_0 > \delta$ large enough that $|\psi(x)| > |w|$ for any x with $|x| \geq R$. The images $\psi(S_t)$ for $t \in [r_0, R]$ provide a homotopy which shows that $\psi(S_R) \neq 0$ in $\pi_{n-1}(\mathbb{R}^n - \{0\})$, since $0 \notin \psi(S_t)$ for $t \in [r_0, R]$ by condition (a).

We now claim that $w \in \psi(S_t)$ for some $t \in [\delta, R]$. Indeed, if this is not the case, then the homotopy $\psi(S_t)$ shows that $\psi(S_R)$ is trivial in $\pi_{n-1}(\mathbb{R}^n - \{w\})$. Contracting now the closed ball B of radius $|w|$ centered at 0 (which does not intersect $\psi(R)$ by our choice of R) to 0 and to w, we see that $\psi(S_R) = 0$ in $\pi_{n-1}(\mathbb{R}^n - \{0\})$, a contradiction.

Now we prove Theorem 1.4, stated in the introduction. Given a local diffeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ and a complete Riemannian metric g such that the vector field

$$Z_v(x) = \frac{\nabla^{(g)} f_v}{|\nabla^{(g)} f_v|^2}$$

is complete for $v \neq 0$, we wish to show that f is bijective.

To see that f is surjective, let $\beta_v : [0, \infty) \to \mathbb{R}^n$ denote the maximal forward trajectory of Z_v passing through $0 \in \mathbb{R}^n$ at time $t = 0$. The interval of definition of β_v is $[0, \infty)$ by hypothesis. Define a map $\psi : \mathbb{R}^n \to \mathbb{R}^n$ by

$$v \mapsto \begin{cases} f(0) \text{ if } v = 0 \\ f(\beta_v(|v|^2)) \text{ if } v \neq 0. \end{cases}$$

It is clear that ψ is continuous. For $t \geq 0$ and $v \neq 0$ we have

$$\frac{d}{dt} f_v(\beta_v(t)) = \nabla f_v(\beta_v(t)) \cdot \beta_v'(t) = g(\nabla^{(g)} f_v(\beta(t)), Z_v(\beta(t))) = 1.$$

Integration between $t = 0$ and $t = |v|^2$ gives

$$f_v(\beta_v(|v|^2)) - f_v(\beta_v(0)) = |v|^2,$$

so that

$$|\psi(v) - \psi(0)||v| \geq < \psi(v) - \psi(0), v >= |v|^2 \tag{1}$$

(here $<,>$ denotes the standard inner product).

Now apply Lemma 2.1 to ψ, taking $p = \psi(0)$. It is immediate from equation (1) that conditions (a) and (c) hold. To see condition (b), we define a homotopy from $\psi(S_1)$ to the unit sphere centered at p by $H : [0,1] \times S^n \to \mathbb{R}^n$ by $H(s,v) = s(p+v) + (1-s)\psi(v)$ for $v \in S^n$. The important point here is that this homotopy avoids the point p, which we see from Equation (1):

$$< H(s,v) - p, v >=< s(p+v) + (1-s)\psi(v) - p, v >=$$

$$< s(p+v) + (1-s)\psi(0) - p, v > +(1-s)|v|^2 = |v|^2 > 0$$

so that $H(s,v) \neq p$ and hence $\psi(S_1)$ generates $\pi_{n-1}(\mathbb{R}^n - \{p\}) \cong \mathbb{Z}$. It follows from Lemma 2.1 that ψ is surjective. Since $\psi(\mathbb{R}^n) \subset f(\mathbb{R}^n)$, we conclude that f is surjective as well.

We now show that f is injective. Letting $a \neq b \in \mathbb{R}^n$, we want that $f(a) \neq f(b)$. For this, it suffices to find a flow line $\alpha(t)$ from a to b for one of the vector fields

$$Y_v = \frac{\nabla^{(g)} f_v}{|\nabla^{(g)} f_v|}$$

for $v \neq 0$. Indeed, supposing that $\alpha(0) = a$ and $\alpha(\tau) = b$ we may write

$$< f(b) - f(a), v >= f_v(b) - f_v(a) = \int_0^\tau \frac{d}{dt} f_v(\alpha(t)) dt$$

but we have

$$\frac{d}{dt} f_v(\alpha(t)) = g(\nabla^{(g)} f_v(\alpha(t)), \alpha'(t)) = |\nabla f_v^{(g)}(\alpha(t))| > 0$$

so that $f(a) \neq f(b)$.

To obtain a flow line α as above, we begin by establishing the following:

Claim: Let $M > 0$. Then there is a uniform bound $L > 0$ such that for any flow line $\alpha : [0, \tau] \to \mathbb{R}^n$ satisfying $\alpha'(t) = Y_v(\alpha(t))$, $\alpha(0) = a$ and $|\alpha(\tau) - a| \leq M$, the length of the curve α is $\leq L$.

For this we use the complete vector fields Z_v for $v \in S^{n-1}$, which have the same trajectories as Y_v. Let $\gamma_v : [0, \infty) \to \mathbb{R}^n$ be a trajectory for Z_v with $\gamma_v(0) = a$ and $|\gamma(\tau) - a| < M$. As we computed before, $\frac{d}{dt} f_v(\gamma_v(t)) = 1$ and integrating between 0 and τ yields

$$\tau =< f(\gamma_v(\tau)) - f(\gamma_v(0)), v >\leq 2 \max_{|x-a| \leq M} |f(x)| = A(M) \tag{2}$$

by Schwarz's inequality.

Now consider the continuous map $F : [0, A(M)] \times S^{n-1} \to \mathbb{R}^n$ given by $F(t,v) = |Z_v(\gamma_v(t))|$. Having compact domain, F takes on a maximum value U. Now we can uniformly estimate the length of trajectories for Z_v whose right endpoint has distance $\leq M$ from a: for any such path γ_v, Equation (2) gives

$$l(\gamma_v) = \int_0^\tau |\gamma_v'(t)|dt = \int_0^\tau |Z_v(\gamma_v(t))|dt \leq$$
$$A(M) \max_{\substack{|v|=1 \\ |t|\leq A(M)}} |Z_v(\gamma_v(t))| = M(A)U.$$

With the claim on uniform lengths established, we now turn to the flows of the vector fields Y_v. Let $\alpha_v : [0,\infty) \to \mathbb{R}^n$ be the maximal forward trajectory of Y_v which passes through $a \in \mathbb{R}^n$ at time $t = 0$. Consider the continuous map $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$v \mapsto \begin{cases} a \text{ if } v = 0 \\ \alpha_v(|v|) \text{ if } v \neq 0. \end{cases}$$

In order to find a Y_v-trajectory from a to b, it suffices to show that φ is surjective, and for this we apply Lemma 2.1. This time we have

$$\frac{d}{dt} f_v(\alpha_v(t)) = g(\nabla f_v^{(g)}(\alpha_v(t)), Y_v(\alpha_v(t))) = |\nabla f_v^{(g)}(\alpha_v(t))|$$

and integration gives

$$< f(\varphi(v)) - f(a), v >= \int_0^{|v|} |f_v^{(g)}(\alpha(t))|dt > 0 \tag{3}$$

From equation (3) it is immediate that condition (a) of Lemma 2.1 applies to φ if we take $p = a$. Condition (c) follows from the Claim above. Indeed, for $M > 0$, let L be the uniform bound on the length of flow lines which start at a and end within M of a given by the Claim. Since the flow lines to Y_v proceed at unit speed, the condition $|v| > L + 1$ forces $|\alpha_v(|v|)| > M$, as desired.

Finally we verify condition (b) of the lemma. Equation (3) shows that $f(\varphi(v)) = f(a)$ if and only if $v = 0$. It follows that f induces a group homomorphism $\pi_{n-1}(\mathbb{R}^n - \{a\}) \to \pi_{n-1}(\mathbb{R}^n - \{f(a)\})$. Thus to show that $\varphi(S_1)$ has nontrivial image in $\pi_{n-1}(\mathbb{R}^n - \{a\})$ it suffices to prove that $f(\varphi(S_1))$ has nontrivial image in $\pi_{n-1}(\mathbb{R}^n - \{f(a)\})$.

To see this, consider the homotopy $H : [0,1] \times S^{n-1} \to \mathbb{R}^n$ from $f(\varphi(S_1))$ to the unit sphere centered at $f(a)$ given by

$$H(s,v) = s(f(a) + v) + (1-s)f(\varphi(v)).$$

Using equation (3) once again, one calculates that

$$< H(s,v) - f(a), v >= s|v|^2 + \int_0^{|v|} |\nabla f_v(\alpha_v(t))|dt > 0$$

so that $H(s,v) \neq f(a)$ for any pair (s,v). In particular, $f(\varphi(S_1))$ has nontrivial image in $\pi_{n-1}(\mathbb{R}^n - \{f(a)\})$, as desired. Having verified this, Lemma 2.1 implies that φ is surjective, which in turn yields a Y_v-flow from a to b, giving injectivity of f.

Now we prove a lemma which shows how Theorem 1.5 follows from Theorem 1.4.

Lemma 2.2. *Let $V : \mathbb{R}^n \to \mathbb{R}^n$ be a nonvanishing smooth vector field which satisfies*

$$\int_0^\infty \min_{|x|=r} |V(x)|dr = \infty.$$

Then the vector field $\dfrac{V(x)}{|V(x)|^2}$ is complete.

Proof: By a theorem of Plastock ([21], Theorem 4.1), the hypothesis of the lemma implies that $\mathbb{R}^n$ is complete with respect to arc length with weight $|V(x)|$. Now let $\alpha(t)$ be a solution to $\alpha'(t) = Y(\alpha(t))$, where $Y(x) = \dfrac{V(x)}{|V(x)|^2}$ and assume that the maximal interval of definition for α is $[0, \tau)$ with $\tau < \infty$. Observe that the integral

$$\int_0^\tau |V(\alpha(t))||\alpha'(t)||dt = \int_0^\tau 1dt = \tau$$

is finite. Since $\mathbb{R}^n$ is complete with respect to the weight $|V(x)|$, we conclude (see [21], Definition 3.2) that $\lim_{t\to\tau} \alpha(t)$ exists and is finite. Solving the differential equation in a neighborhood of $\alpha(\tau)$, we see that the maximal interval of definition for α is larger than $[0, \tau)$. We conclude that the maximal interval of definition of $\alpha(t)$ is $[0, \infty)$ and hence $Y(x)$ is a complete vector field.

3. **Applications to Polynomial Maps.** A long-standing problem in algebraic geometry is the famous

Conjecture 3.1. (Jacobian Conjecture) *If $F : \mathbb{C}^n \to \mathbb{C}^n$ is a polynomial map such that* $\det DF(x)$ *is a nonzero constant, then F has a polynomial inverse.*

This problem has weathered attacks from algebra, geometry and even differential equations. In various special cases the answer is positive, but there is no general solution (for $n \geq 2$) in spite of some incomplete proofs in the literature. Excellent general references for this problem are [2],[16]. In [16] the reader will find a substantial literature devoted to this problem.

It is known [2] that the Jacobian conjecture (over $\mathbb{C}$) holds if and only if it holds over $\mathbb{R}$ and that in either case it is enough to prove injectivity. A recent example of Pinchuk [20] shows that a polynomial local diffeomorphism of $\mathbb{R}^n$ need not be bijective, thus the condition that the nonzero Jacobian determinant be *constant* is crucial over $\mathbb{R}$. A theorem of Hadamard [10] states that any proper local diffeomorphism of $\mathbb{R}^n$ is a global diffeomorphism, hence properness is a key issue.

Before applying the results of §1, we would like to put forth the philosophy that *connectedness* is also a key issue in the Jacobian problem. Since the hypothesis of 3.1 implies that F is a quasi-finite étale morphism ([2], §1, proof of Theorem 2.1), global injectivity is equivalent to the connectedness of the fibres of F. We ask more generally if the inverse image of every affine linear subspace is connected. The following observation, which one would not expect in a more analytic setting, gives evidence towards a positive answer.

Remark 3.2. Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map with nonzero constant Jacobian determinant for some $n \geq 2$. As noted above, these hypotheses imply that F is a quasi-finite étale morphism. In particular, F is dominant and $\dim F(\mathbb{C}^n) = n$. Since $\mathbb{C}^n$ is (geometrically) irreducible, we may apply Jouanolou's Bertini type theorem ([14], Theorem 6.6) to see that $F^{-1}(L)$ is an irreducible, smooth subvariety of $\mathbb{C}^n$ for a general affine linear subspace L of fixed dimension ≥ 1: in particular, $F^{-1}(L)$ is connected in the Euclidean topology. In view of this fact and Conjecture 1.8, it natural to ask the following question: must the pre-image of *every* affine hyperplane L (real or complex) be connected in the Euclidean topology?

Remark 3.3. We have just seen that the pre-image of general complex lines are connected for $F : \mathbb{C}^n \to \mathbb{C}^n$ satisfying the Jacobian hypothesis. On the other hand, if the pre-image $C = F^{-1}(L)$ of general *real* lines is connected in the Euclidean topology for a polynomial map $F : \mathbb{R}^n \to \mathbb{R}^n$ satisfying the Jacobian hypothesis, then we conjecture 3.1 follows: Since C is connected, we can find a path $\gamma(t)$ between any two inverse images of a point $p \in L$. The image path $F \circ \gamma(t) \subset L$ must evidentally change direction at some point t_0, where we have $DF(\gamma(t_0)) \cdot \gamma'(t_0) = 0$. Since $\gamma'(t_0) \neq 0$, $DF(x)$ is not invertible at $x = \gamma(t_0)$, contradicting the Jacobian hypothesis. Thus the general point would have exactly one pre-image, in which case F is injective, hence invertible by the main theorem in [1].

Now let us apply the results of section 1 to a polynomial map $F : k^n \to k^n$, where $k = \mathbb{R}$ or $\mathbb{C}$ (see Remark 1.3). The hypotheses of these results rely on how close 0 is to the set

$$S_v = \{DF(x)^* v : x \in k^n\}$$

for nonzero v in an asymptotic sense. These can be conveniently phrased in terms of Lojasiewicz exponents. Recall that for a polynomial mapping $g : k^n \to k^n$, the *Lojasiewicz exponent of g at infinity* is the number

$$\mathcal{L}_\infty(g) = \sup\{s \in \mathbb{R} : \exists A, r > 0 : (|x| > r) \Rightarrow (|g(x)| \geq A|x|^s)\}.$$

We now have the following test for invertibility.

Criterion 3.4. *Let $F : k^n \to k^n$ be a polynomial map with nonzero constant Jacobian determinant. If $\mathcal{L}_\infty(DF(x)^* v) > -1$ for each nonzero $v \in k^n$, then F is invertible.*

Proof: If $\mathcal{L}_\infty(DF(x)^* v) > -1$ for each nonzero v, then there exist $A, r_0 > 0$ such that $|DF(x)^* v| \geq A|x|^{-1}$ for $|x| > r_0$ and hence

$$\int_0^\infty \min_{|x|=r} |\nabla f_v(x)| dr \geq \int_{r_0}^\infty A|r|^{-1} dr = \infty.$$

Theorem 1.5 shows that F is invertible.

Remark 3.5. There are lower bounds for the Lojasiewicz exponents in terms of the degrees of the polynomials [7, 12, 13], but in general they are difficult to compute. Theorem 1.2 leads to a weaker criterion for global invertibility of a polynomial map $F : k^n \to k^n$ which may be easier to apply. With the notation above, we need to know that $0 \notin \overline{\{DF(x)^* v : x \in k^n\}}$ for each nonzero $v \in k^n$. Since 0 is not in the set by hypothesis, this condition is equivalent to saying that the polynomial function $F_v : k^n \to k$ is *tame* for each $v \neq 0$ (a polynomial $p(x)$ is tame if there is no sequence z_n with $|z_n| \to \infty$ and $\nabla p(z_n) \to 0$, as defined in [4]). This can be tested with polynomials:

Assume that for each $v \neq 0$ there exists a polynomial $g_v \in k[x_1, \dots, x_n]$ such that
(i) $g_v(DF(x)^* v) = 0$ for all $x \in k^n$.
(ii) $g_v(0) \neq 0$.
Then F is invertible.

Indeed, the closed set $g_v^{-1}(0)$ contains S_v but not zero. When $k = \mathbb{C}$, the conditions above are equivalent to the hypotheses of Theorem 1.2, since the set S_v is constructible.

The fact that the Jacobian conjecture holds for quadratic maps was originally proven by Wang [26] via algebraic methods for maps with *constant* Jacobian determinant. Later it was strengthened to the form below for $k = \mathbb{R}$. There is a short proof of this using an effective mean-value theorem for quadratic maps; see [16] §2.3. The algebraic criteria above also readily yield this result:

Proposition 3.6. *Let $F : k^n \to k^n$ be a quadratic map. If* $\det DF(x) \neq 0$ *for each $x \in k^n$, then F is invertible.*

Proof: For any nonzero $v \in k^n$, the set

$$S_v = \{DF(x)^* v : x \in k^n\}$$

is an affine linear space which misses the origin. Hence we can find a linear polynomial g_v which vanishes on S_v but for which $g_v(0) \neq 0$ and apply Remark 3.5. Alternatively, we may note that $\mathcal{L}_\infty(DF(x)^* v) = 1$ for the nonvanishing affine linear map $x \mapsto DF(x)^* v$ and use Criterion 3.4.

Example 3.7. Consider the quadratic map $f : \mathbb{R}^3 \to \mathbb{R}^3$ given by $f(x, y, z) = (x + y^2, y + z^2, z)$. Theorem 1.6 cannot be applied to this situation because

$$\int_0^\infty \min_{|(x,y,z)|=r} \min_{|v|=1} |\nabla f_v(x, y, z)| dr < \infty.$$

Indeed, the choice $(x, y, z) = (\frac{r}{\sqrt{2}}, \frac{r}{2}, \frac{r}{\sqrt{2}})$ and $v = (\frac{1}{r^4} + \frac{1}{r^2} + 1)^{-\frac{1}{2}} (\frac{1}{r^2}, -\frac{1}{r}, 1)$ shows that the double minimum in the integrand is $\leq (\frac{1}{r^4} + \frac{1}{r^2} + 1)^{-\frac{1}{2}} \frac{1}{r^2} \leq \frac{1}{r^2}$ which gives a convergent integral. On the other hand, f is invertible by Proposition 3.6.

While quadratic maps may seem a rather restricted class, the general Jacobian conjecture reduces to polynomial maps of degree ≤ 3. A polynomial map $F : \mathbb{C}^n \to \mathbb{C}^n$ is said to be *cubic homogeneous* if $F(x) = x + H(x)$ where $H : \mathbb{C}^n \to \mathbb{C}^n$ is a polynomial map whose coordinate functions consist of cubic homogeneous polynomials. At the cost of increasing the number of variables, it suffices to consider the Jacobian conjecture for such maps. More precisely, the Jacobian conjecture holds for each cubic homogeneous map $F : \mathbb{C}^n \to \mathbb{C}^n$ and each $n \geq 2$ if and only if it holds in general [2].

In fact, Drużkowski [8] has further reduced the Jacobian conjecture to the consideration of cubic homogeneous maps of a particular form. A cubic homogeneous map $F(x) = x + H(x) : \mathbb{C}^n \to \mathbb{C}^n$ is called a *cubic linear map* if $H(x) = (L_1^3, L_2^3, \ldots, L_n^3)$ for some linear forms $L_1, L_2, \ldots, L_n$. Letting A be the $n \times n$ matrix of coefficients of the L_i (i.e. $A_{i,j}$ is the coefficient of x_j in L_i), one often uses the notation $H = H_A$ and $F = F_A$ for such maps.

Proposition 3.8. *With the notation above, suppose that* $\det DF_A(x) = 1$ *and*

$$\{(L_1(x)^2, L_2(x)^2, \ldots, L_n(x)^2) : x \in \mathbb{C}^n\}$$

has closed image under any linear self-map of $\mathbb{C}^n$. Then F_A is invertible.

Proof: For $v = (v_1, v_2, \ldots, v_n) \neq 0$, differentiation shows that

$$DF(x)^* v = v + 3A^*(L_1^2 v_1, L_2^2 v_2, \ldots, L_n^2 v_n)$$

and $0 \notin S_v = \{DF(x)^* v : x \in \mathbb{C}^n\}$ by hypothesis. On the other hand, S_v is closed since it is obtained from the set $\{(L_1^2, L_2^2, \ldots, L_n^2) : x \in \mathbb{C}^n\}$ by multiplying by the diagonal matrix D with $3v_i$ on the diagonal, then multiplying by A^*, and finally translating by v. It follows that $0 \notin \overline{S_v}$ and Theorem 1.2 applies.

Remarks 3.9. (a) Rusek proved Proposition 3.8 when the set

$$\{(L_1(x)^2, L_2(x)^2, \dots, L_n(x)^2) : x \in \mathbb{C}^n\}$$

is a full *linear* variety by showing that $\det(DF_A(x)+DF_A(y)) = 2^n$ for all $x, y \in \mathbb{C}^n$. This is known as condition (J) of Jagzhev, who observed that any cubic homogeneous map satisfying this condition must be an automorphism of $\mathbb{C}^n$ ([11], Thm. (ii)) or [24], Prop. 2.1). A simple case to which both results apply occurs when A has rank r and each row is a scalar multiple of one of r fixed rows. Drużkowski's proofs [9] show that these are the only cases to be considered when the corank of A is less than three.

(b) There are certainly non-linear subsets that are closed under any self-map of $\mathbb{C}^n$, for example the image of any polynomial map $p : \mathbb{C} \to \mathbb{C}^n$. It would be interesting to know precisely which subsets of $\mathbb{C}^n$ have closed image under any linear map.

Example 3.10. We consider the following example of Wright (see [28], §3). Let $H : \mathbb{C}^4 \to \mathbb{C}^4$ be given by

$$H(x, y, u, v) = (0, 0, uxy + vy^2, -ux^2 - vxy)$$

and set $F(x, y, u, v) = (x, y, u, v) + H(x, y, u, v)$. As Wright notes, F is an automorphism of $\mathbb{C}^4$ which does not have property (J), hence is not linearly tame. This is of interest because any cubic homogeneous automorphism in three variables *is* linearly tame [28]. For each $v \neq 0$ in the plane $\{x = y = 0\}$ the hypotheses of Remark 3.5 or Theorem 1.2 fail; for example, if $v = (0, 0, a, b)$ with $ab \neq 0$, then $\lim_{n\to\infty} DF(x_n)^* v = 0$ for the sequence $x_n = (an + \frac{1}{2bn}, bn - \frac{1}{2an}, b, -a)$. Nonetheless, we can apply these results after modifying F:

In the spirit of the reduction theorems of [2] and [8], we add four new variables p, q, r, s and define $\widetilde{F} : \mathbb{C}^8 \to \mathbb{C}^8$ by $\widetilde{F}(x, y, u, v, p, q, r, s) =$

$$(F(x, y, u, v) + (0, 0, px^2 + qy^2, rx^2 + sy^2), p, q, r, s).$$

It is easily checked that F is injective if and only if $\widetilde{F}$ is injective and that $\det DF(x, y, u, v) = \det D\tilde{F}(x, y, u, v, p, q, r, s)$. Assuming that $D\widetilde{F} = 1$, one can apply Remark 3.5 by choosing suitable quadratic and linear polynomials depending on the choice of vector v, however we will argue directly with Theorem 1.2. Let $v = (v_1, v_2, \dots, v_8) \neq 0$ and consider the set $S_v = \{D\widetilde{F}^*(x, y, u, v, p, q, r, s)v\}$. If $v_3 = v_4 = 0$, then $S_v = \{v\}$ and it is clear that $0 \notin \overline{S_v}$. On the other hand, if $(v_3, v_4) \neq (0, 0)$ and $0 \in \overline{S_v}$, then there is a sequence $X_n = (x_n, y_n, u_n, v_n, p_n, q_n, r_n, s_n)$ such that $\lim_{n\to\infty} D\widetilde{F}^*(X_n)v = 0$. Examining the variables p, q, r, s, we see that x_n and y_n are bounded in this sequence, so we can find a subsequence for which $\lim x_n = a$ and $\lim y_n = b$. The variables u and v yield the matrix equation

$$\begin{pmatrix} 1 + ab & a^2 \\ b^2 & 1 - ab \end{pmatrix} \begin{pmatrix} v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The matrix on the left is invertible for all (a, b), contradicting our assumption $(v_3, v_4) \neq (0, 0)$. This shows that $0 \notin \overline{S_v}$. We conclude from Theorem 1.2 that $\widetilde{F}$ is bijective and hence so is F. In this example, Jagzhev's condition (J) holds for neither F nor $\widetilde{F}$.

Question 3.11. In view of the reduction theorems, we ask the following question: when can we modify a cubic homogeneous map $F(x) = x + H(x)$ to obtain another such map $\widetilde{F}(x, y)$ for which we can test invertibility as in Example 3.10?

REFERENCES

[1] A. Białynicki-Birula and M. Rosenlicht, *Injective morphisms of real algebraic varieties*, Proc. Amer. Math. Soc. **13** (1962) 200–203.

[2] H. Bass, E. Connell and D. Wright, *The Jacobian conjecture: reduction of degree and formal expansion of the inverse*, Bull. A.M.S. **7** (1982) 287-330.

[3] M. Berger, "Nonlinearity and Functional Analysis", Pure and Applied Mathematics **74**, Academic Press, New York 1977.

[4] S.A. Broughton, *Milnor numbers and the topology of polynomial hypersurfaces*, Invent. Math. **92** (1988) 217–241.

[5] M. Chamberland and G. Meisters, *A Mountain Pass to the jacobian conjecture*, Canad. Math. Bull. **41** (1998) 442–451.

[6] C. Cicone and J. Sotomayor, *On a class of complete polynomial vector fields in the plane*, J. Diff. Eqns. **61** (1986) 398–418.

[7] E. Cygan, T. Krasinski and P. Tworzewski, *Separation of algebraic sets and the Lojasiewicz exponent of polynomial mappings*, Invent. Math. **136** (1999) 75–87.

[8] L. Drużkowski, *An effective approach to Keller's Jacobian conjecture*, Math. Ann. **264** (1983) 303–313.

[9] L. Drużkowski, *The Jacobian conjecture in case of rank or corank less than three*, J. Pure and Appl. Alg. **85** (1993) 233–244.

[10] J. Hadamard, *Sur les transformations ponctuelles*, Bull. Soc. Math. France **34** (1906) 71–84.

[11] A. Jagzev, *On Keller's problem*, Siberian Math. J. **21** (1980) 141–150.

[12] S. Ji, J. Kollár and B. Shiffman, *A global Lojasiewicz inequality for algebraic varieties*, Trans. of Amer. Math. Soc. **329** (1992) 813–818.

[13] J. Kollár, *An effective Lojasiewicz inequality for real polynomials*, preprint, 1999.

[14] J. - P. Jouanolou, "Théorèmes de Bertini et Applications", Progress in Mathematics **42**, Birkhäuser, Boston 1983.

[15] N. G. Lloyd, "Degree Theory", Cambridge Tracts in Mathematics **73**, Cambridge University Press, London 1978.

[16] G. Meisters, *Inverting polynomial maps of n-space by solving differential equations*, in "Delay and Differential Equations, Proceedings in Honor of George Seifert, Ames, Iowa, Oct. 18-19, 1991" (A. Fink, R. Miller and W. Kliemann, Eds.), World Scientific, Singapore (1992).

[17] G. Meisters and Cz. Olech, *A polyflow formulation of the Jacobian conjecture*, Bull. Polish Acad. Sci. Math. **35** (1987) 725–731.

[18] C. F. Palmeira, *Open manifolds foliated by planes*, Ann. of Math. **107** (1978) 109–131.

[19] T. Parthasarathy, "On Global Univalence Theorems", Lecture Notes in Mathematics **977**, Springer-Verlag, Berlin-Heidelberg-New York 1983.

[20] S. Pinchuk, *A counterexample to the strong real Jacobian conjecture*, Math. Z. **217** (1994) 1–4.

[21] R. Plastock, *Homeomorphisms between Banach spaces*, Trans. Amer. Math. Soc. **200** (1974) 169–183.

[22] P.J. Rabier, *On global diffeomorphisms of Euclidean space*, Nonlinear Anal. **21** (1993) 925–947.

[23] P.J. Rabier, *Ehresmann fibrations and Palais-Smale conditions for morphisms of Finsler manifolds*, Ann. of Math. (2) **146** (1997) 647–691.

[24] K. Rusek, *A geometric approach to Keller's jacobian conjecture*, Math. Ann. **264** (1983) 315–320.

[25] B. Smyth and F. Xavier, *Injectivity of local diffeomorphisms from nearly spectral conditions*, J. Diff. Eqs. **130** (1996) 406–414.

[26] S. S.-S. Wang, *A Jacobian criterion for separability*, J. Algebra **69** (1980) 453–494.

[27] A. van den Essen (ed.), "Automorphisms of Affine Spaces", Kluwer Academic Publishers Dordrecht/Boston/London 1995.

[28] D. Wright, *The Jacobian conjecture: linear triangulization for cubics in dimension three*, Linear and Multilinear Algebra **34** (1993) 85–97.

[29] F. Xavier, *Injectivity as a transversality phenomenon in geometries of negative curvature*, Illinois J. Math. (2) **43** (1999) 256–263.

[30] E. Zeidler, "Nonlinear Functional Analysis and its Applications III - Variational Methods and Optimization", Springer-Verlag, New York 1984.

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