On Curve Sections of Rank Two Reflexive Sheaves

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Abstract

Let $\mathcal{F}$ be a normalized rank 2 reflexive sheaf on $\mathbf{P}^3$ with Chern classes c_1, c_2, c_3. Let α be the least integer such that $0 \neq H^0\mathcal{F}(\alpha)$ and β be the smallest integer such that $H^0\mathcal{F}(n)$ has sections whose zero scheme is a curve for all $n \geq \beta$. We show that if T_0 is the largest root of the cubic polynomial

$$P(T) = T^3 - (6c_2 + 6\alpha c_1 + 6\alpha^2 + 1)T + 3(2\alpha + c_1)(c_2 + c_1\alpha + \alpha^2)$$

then $\beta \leq \lfloor T_0 \rfloor - \alpha - c_1 - 1$. There are applications to the smallest degree of a surface containing a curves which are the zero schemes of sections of $H^0\mathcal{F}(\alpha)$.

1 Introduction

In this paper we consider rank 2 reflexive sheaves $\mathcal{F}$ on $\mathbf{P} = \mathbf{P}^3_k$, where k is an algebraically closed field of characteristic zero. Suppose that $\mathcal{F}$ has Chern

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classes c_1, c_2, c_3 and let α be the smallest integer such that $H^0\mathcal{F}(\alpha) \neq 0$. Then we have the following bound (see [H2], theorem 0.1):

Theorem 1.1 *(Hartshorne) If $\mathcal{F}$ is a normalized (i.e. $c_1 = 0$ or -1) stable rank two reflexive sheaf on $\mathbf{P}^3$, then*

$$\alpha \leq \sqrt{3c_2 + 1 + \frac{3}{4}c_1} - 1 - \frac{1}{2}c_1.$$

Moreover, this bound is sharp for all pairs (c_1, c_2) (except for $(-1, 2)$ and $(-1, 4)$), equality being realized by bundles with natural cohomology. In this paper we are interested in which twists of $\mathcal{F}$ have sections which are not multiples of some $0 \neq s \in H^0\mathcal{F}(\alpha)$.

In the earlier paper [RV1], the second and third author introduce the invariant $\beta = \min\{n : h^0(\mathcal{F}(n)) > h^0(\mathcal{O}_{\mathbf{P}}(n - \alpha))\}$, the twist of the so called "second relevant section of $\mathcal{F}$". Is is easy to see that the general non-zero section of $\mathcal{F}(n)$ gives rise to a curve if and only if either $n = \alpha$ or $n \geq \beta$ (see [GRV], theorem 0.1). In the case $n = \alpha$, we call this curve (which need be neither reducible nor reduced) a *minimal* curve for $\mathcal{F}$. If a curve C is the scheme of zeros of a non-zero section of $\mathcal{F}(n)$, then the smallest degree r of a surface containing C is $n + \alpha + c_1$ if $n > \alpha$ and $\beta + \alpha + c_1$ if $n = \alpha$. Thus β determines the minimal degree of a surface containing a minimal curve for $\mathcal{F}$ as well as determining which twists of $\mathcal{F}$ give rise to curves which are not minimal.

In the papers [RV1] and [RV2], the second and third authors give some upper bounds for β in terms of α, c_1 and c_2. In particular they show that if $\mathcal{F}$ is unstable (we say that $\mathcal{F}$ is stable if $\alpha > 0$, semistable if $\alpha + c_1 \geq 0$, unstable otherwise) and T_0 is the largest root of the cubic polynomial

$$P(T) = T^3 - (6c_2 + 6\alpha c_1 + 6\alpha^2 + 1)T + 3(2\alpha + c_1)(c_2 + c_1\alpha + \alpha^2),$$

then $\beta \leq T_0 - \alpha - c_1 - 1$. For semistable sheaves other and higher bounds have been proved in the same papers. The aim of the present paper is to show that the bound above ($\beta \leq \lfloor T_0 \rfloor - \alpha - c_1 - 1$) also holds in the semistable case.

We emphasize that the cubic polynomial $P(T)$ above arises naturally if one looks at the Euler characteristic of $\mathcal{F}$. Indeed, we have that $P(T) = 6\chi(\mathcal{F}/\mathcal{O}(-\alpha)(T - \alpha - c_1 - 2)) - 3c_3$ and in the case that $\mathcal{F}$ is a vector bundle ($c_3 = 0$) our theorem states that $H^0(\mathcal{F}/\mathcal{O}(-\alpha)(t)) \neq 0$ for t greater than the last root of $\chi(\mathcal{F}/\mathcal{O}(-\alpha)(t))$. For vector bundles $\mathcal{F}$, Hartshorne's original theorem can be phrased this way as well: it states that for t greater than the last root of $\chi(\mathcal{F}(t))$, we have $H^0(\mathcal{F}(t)) \neq 0$.

In section two we prove the main theorem of the paper. The principal tool of our proof is a reduction step which allows us to induct on c_2. This reduction step was introduced in [H2], although we use the strengthened version found in [RV2]. In section three we give an application to the minimal degree of surfaces containing a curve and some examples in which our result is sharp.

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2 The Main Theorem

In this section, we adopt the following conventions. The invariants α and β have already been defined in the introduction. We let a and b denote the analogous notions for the restriction $\mathcal{F}_H$ of $\mathcal{F}$ to a general plane $H \subset \mathbf{P}$, that is $a = \min\{t : H^0\mathcal{F}_H(t) \neq 0\}$ and $b = \min\{t : H^0\mathcal{F}_H/\mathcal{O}_H(-a)(t) \neq 0\}$. The symbol $*$ denotes the k-vector space dual. We further set $\delta = c_2(\mathcal{F}(\alpha)) = c_2 + c_1\alpha + \alpha^2$, the degree of a minimal curve for $\mathcal{F}$. In terms of δ, the cubic polynomial mentioned in the introduction takes on the simpler form

$$P(T) = T^3 - (6\delta + 1)T + 3(2\alpha + c_1)\delta.$$

We begin with some lemmas that allow us to make a reduction step in the proof of the main theorem. The first two of these lemmas can essentially be found in [RV2].

Lemma 2.1 *Let $\mathcal{F}$ be a semistable rank 2 reflexive sheaf on $\mathbf{P}^3$, $0 \neq x \in H^2\mathcal{F}(t)^*$ and $f \in H^0\mathcal{O}(d)$ an annihilator for x of minimal degree. If $r = t + 4 - d > 0$, then the surface X defined by $f = 0$ is an unstable surface for $\mathcal{F}$ of order r and there is an exact sequence*

$$0 \longrightarrow G(-\epsilon) \longrightarrow \mathcal{F} \longrightarrow I_{Z,X}(-r) \longrightarrow 0$$

where G is a normalized rank 2 reflexive sheaf and Z has codimension ≥ 1 in X. Moreover, if G has Chern classes c_1', c_2', c_3', and k is the degree of the curve part of Z, then

$$2\epsilon = d - c_1 + c_1'$$

$$c_2' = c_2 - d(t+4+c_1-d) - \frac{1}{4}(d-c_1)^2 - \frac{1}{4}c_1' - k < c_2$$
$$2\alpha + c_1 - d \le 2\alpha' + c_1' \le 2\alpha + c_1 + d \text{ where } \alpha' = \alpha(G)$$
$$2\beta + c_1 - d \le 2\beta' + c_1' \text{ where } \beta' = \beta(G)$$

Proof: The first part is proved in [H2], example 1.0.5 and proposition 1.1. The proof of the inequalities is found in [RV2], lemma 2.1.

Lemma 2.2 *Let $\mathcal{F}$ be a semistable rank 2 normalized reflexive sheaf and assume that $H^2\mathcal{F}(t) \neq 0$ for some integer t. Then*

(a) There is $x \in H^2\mathcal{F}(t)^*$ whose image $x_H \in H^1\mathcal{F}_H(-t-c_1-4)$ is $\neq 0$ for a general plane H.

(b) Taking x as in part (a), let d be the smallest integer such that x_H is annihilated by an element of $H^0\mathcal{O}_H(d)$ for all H. If we further have

(i) $t+3 > a+d$ and

(ii) $t+4 > 2d-a-c_1$,

then x has only one (up to a unit) annihilator f in $H^0\mathcal{O}_{\mathbf{P}^3}(d)$.

Proof: Part (a) is [RV2], proposition 3.3. For part (b), we first add the inequalities of (i) and (ii) and divide by 2 to obtain $-t-c_1-4+\frac{1}{2}(3d+c_1+1) < 0$ and (ii) can be written $-t-c_1-4+2d-a < 0$. Taking $l = -t-c_1-4$, the hypotheses of [H2], proposition 3.1 are met and we deduce conclusion (b).

Lemma 2.3 *Let t be an integer such that $t+\alpha+c_1+2$ is greater than the largest root of $P(T)$ and assume that $H^2\mathcal{F}(t) \neq 0$. Let $0 \neq s \in H^2\mathcal{F}(t)^*$ whose image $s_H \in H^1\mathcal{F}_H(-t-c_1-4)$ is $\neq 0$ for a general plane H and let d be the smallest degree of an annihilator of s_H. Then we have:*

(1) $d \le a+b+c_1$ and $d \le 2a+c_1+1$ if $a < \alpha$;

(2) $t+3 > d+a$

(3) $d \le \sqrt{2c_2 - 2a^2}$

(4) $t+4 > 2d-a-c_1$.

Proof: Statement (1) is [RV1], lemma 2.1 (ii) and [H2], corollary 4.2. For (2) it is sufficient to prove that $t+\alpha+c_1+2 \ge \sqrt{3c_2+1+\frac{3}{4}c_1} + \alpha + \frac{1}{2}c_1$ by [H2], proposition 4.3. Substituting $T = \sqrt{3c_2+1+\frac{3}{4}c_1} + \alpha + \frac{1}{2}c_1$ into $P(T)$, we check that the value obtained is not positive:

$$P(\sqrt{3c_2+1+\frac{3}{4}c_1} + \alpha + \frac{1}{2}c_1) =$$

$$= -(\sqrt{3c_2 + 1 + \frac{3}{4}c_1} - \alpha - \frac{1}{2}c_1)((\sqrt{3c_2 + 1 + \frac{3}{4}c_1} - \alpha - \frac{1}{2}c_1)^2 - 1) \leq 0$$

by theorem 1.1.

With the inequality of part (2) proved, part (3) follows immediately from [RV2], lemma 2.4. In the case $a < \alpha$ part (4) follows from the second inequality of part (1) and the inequality of part (2): $t+4 > a+d+1 \geq 2d-a-c_1$. Thus we may assume that $a = \alpha$.

First suppose $\alpha \geq \sqrt{\frac{4}{5}c_2}$. From part (3) we have $d \leq \sqrt{\frac{2}{5}c_2}$ and hence $t+\alpha+c_1+2 \geq \sqrt{3\delta+1} > \sqrt{\frac{8}{5}c_2} > 2d-2$. (To see that $t+\alpha+c_1+2 \geq \sqrt{3\delta+1}$ it suffices that $P(\sqrt{3\delta+1}) \leq 0$ and this is the case:

$$P(\sqrt{3\delta+1}) = \sqrt{3\delta+1}(-3\delta) + 3(2\alpha+c_1)\delta = 3\delta(2\alpha+c_1-\sqrt{3\delta+1}) \leq 0).$$

We can obtain (4) in the remaining case ($a = \alpha < \sqrt{\frac{4}{5}c_2}$) by proving that $t+4 > b+d$, since $d+b \geq 2d-a-c_1$ by part (1).

If not, then $t+3-b < d$ and hence (as in [RV1], Prop. 2.3)

$$\frac{(t+4-b)(t+5-b)}{2} \leq h^1E(-b-c_1-1) = \delta - \frac{(a+b+c_1)^2}{2} + \frac{a+b+c_1}{2}.$$

By easy calculations this may be rewritten

$$\frac{1}{2}(X^2 - 4\delta) + \frac{1}{2}(X-2Y)^2 + 3(X-2Y) + 2X + 6 \leq 0$$

where $X = t+a+c_1+2$ and $Y = a+b+c_1$. This last inequality is clearly false because $X \geq \sqrt{4\delta}$ and $\frac{1}{2}(X-2Y)^2 + 3(X-2Y) \geq -\frac{9}{2}$ (To see that $X \geq \sqrt{4\delta}$, it suffices that $P(\sqrt{4\delta+1}) = (\sqrt{4\delta+1} - 3\alpha - \frac{3}{2}c_1)(-2\delta) \leq 0$, but this is easily checked using the hypothesis that $\alpha < \sqrt{\frac{4}{5}c_2}$).

Corollary 2.4 *If $t+\alpha+c_1+2$ is larger than the largest root of $P(T)$ and $H^2\mathcal{F}(t) \neq 0$, then there is a reduction step for $\mathcal{F}$ as described in lemma 2.1.*

Theorem 2.5 *Let $\mathcal{F}$ be a semistable rank 2 normalized reflexive sheaf and T_0 the largest root of the polynomial $P(T)$. Then $\beta \leq \lfloor T_0 \rfloor - \alpha - c_1 - 1$.*

Proof: We induct on c_2. By [RV1], theorem 4.3(1), the theorem holds for unstable reflexive sheaves, which proves the theorem when $c_2 < 0$. Thus we may assume that $c_2 > 0$ and proceed to the induction step.

Let t be an integer such that $T_1 = t + \alpha + c_1 + 2$ is greater than T_0; Riemann-Roch gives:

$$\chi\mathcal{F}(t) = 2\binom{t+3}{3} - c_1\binom{t+2}{2} - c_2(t+2) + \frac{1}{2}(c_3 - c_1c_2).$$

A calculation shows that $P(T_1) = 6\chi\mathcal{F}/\mathcal{O}(-\alpha)(t) - 3c_3$. In particular, we have that $\chi\mathcal{F}(t) > 0$. If $H^2\mathcal{F}(t) = 0$, then we conclude immediately that $H^0\mathcal{F}(t) > 0$.

So assume that $H^2\mathcal{F}(t) \neq 0$. By induction hypothesis we assume that the statement is true for every rank 2 semistable normalized reflexive sheaf with second Chern class $< c_2$. By corollary 2.4 there is a reduction step giving a new sheaf G with $c_2' = c_2(G) < c_2$ (see lemma 2.1). We will prove that $t' = t - \frac{1}{2}(d - c_1 + c_1')$ satisfies the same conditions relative to G. This will imply the claim for $\mathcal{F}$, because the claim is true for G and moreover ([RV2] lemma 2.1) $t = t' + \frac{1}{2}(d - c_1 + c_1') \geq \beta' + \frac{1}{2}(d - c_1 + c_1') \geq \beta$.

We have to prove that $P'(T) = T^3 - (6\delta' + 1)T + 3(2\alpha' + c_1')\delta'$ is positive when $T = T_2 = t' + \alpha' + c_1' + 2 = T_1 + \alpha' - \alpha + \frac{1}{2}(d - c_1 + \frac{1}{2}c_1') - d$.

The derivative of $Q(T_2) = P'(T_2) + T_2$ with respect to α' is:

$$3T_2^2 - 6T_2(2\alpha' + c_1') + 3(2\alpha' + c_1')^2 = 3(T_2 - 2\alpha' - c_1')^2$$

and so $Q(T_2)$ always increases as a function of α'. Moreover, the coefficient of c_2' in $Q(T)$ is $-3(2t + 4 - d + c_1)$, which is negative by lemma 2.3.

So we have $Q(T_2) \geq R$ where R is obtained by replacing α' with $\alpha - \frac{1}{2}(d - c_1 + c_1')$ (recall that $\alpha - \frac{1}{2}(d - c_1 + c_1') \leq \alpha' \leq d + \alpha - \frac{1}{2}(d - c_1 + c_1')$ by lemma 2.1) and c_2' with $c_2 - d(T_1 + 2 - \alpha - d) - \frac{1}{4}(d - c_1)^2 - \frac{1}{4}c_1'$ (see lemma 2.1) in $Q(T_2)$.

Thus

$$P'(T_2) = Q(T_2) - T_2 \geq R - T_1 =$$
$$P(T_1) + 3d(t + 4 - d + \frac{1}{2}c_1)^2 + d(3c_2 - d^2 + 6d + \frac{3}{4}c_1 - 12)$$

is positive, because of the hypothesis on T_1 and lemma 2.3.

3 Applications to Curves and Examples

Let C be a locally Cohen-Macaulay generic complete intersection curve and e_C' be the largest integer such that $\omega_C(-e_C')$ has a section which generates the sheaf almost everywhere. This is not to be confused with e_C, the largest integer such that $\omega_C(-e_C)$ has a nonzero section, however note that $e' \leq e$ and they are equal when, for example, C is an integral curve. By Serre's correspondence (see [H1], theorem 4.1) C is the scheme of zeros of a non-vanishing section of a reflexive sheaf $\mathcal{F}$ with first Chern class $c_1 = 4 + e'_C$. Normalizing $\mathcal{F}$ and applying theorem 2.5 we obtain the following result on the minimal degree of a surface containing C.

Theorem 3.1 *Let C be a locally Cohen-Macaulay generic complete intersection curve on $\mathbf{P}^3$ of degree d and assume that the minimal degree $s_0(C)$ of a surface containing C satisfies $s_0(C) \geq e'_C + 4$. Then $s_0(C)$ is less than or equal to $1+$ the largest root of the polynomial*

$$p(z) = \binom{z+3}{3} - d(z - \frac{e'_C}{2}).$$

Proof. A section of $\omega_C(-e'_C)$ which generates this sheaf almost everywhere gives rise to a rank two reflexive sheaf $\mathcal{E}$ with Chern classes $c_1 = c_1(\mathcal{E}) = e'_C + 4, c_2 = d, c_3 = 2p_a(C) - 2 + d(4 - c_1)$, a nonzero section $s \in H^0\mathcal{E}$ and an exact sequence

$$0 \to \mathcal{O}(-c_1) \to \mathcal{E}(-c_1) \to \mathcal{I}_C \to 0.$$

The hypothesis $e'_C+4 \leq s_0(C)$ shows that C is a minimal curve for $\mathcal{E}$. Writing $\mathcal{E} = \mathcal{F}(\alpha_{\mathcal{F}})$, where $\mathcal{F}$ is the normalization of $\mathcal{E}$, we see from the exact sequence that $s_0(C) = \beta_{\mathcal{F}}+\alpha_{\mathcal{F}}+c_1(\mathcal{F})$. By theorem 2.5, this last quantity is $\leq \lfloor T_0 \rfloor - 1$, where T_0 is the largest root of $P(T)$. Equivalently, $s_0(C) \leq 1+$ the largest root of $P(z+2)$. However, interpreting this as an Euler characteristic gives $P(z+2) = 6\chi\mathcal{F}/\mathcal{O}(-\alpha)(z-\alpha-c_1(\mathcal{F})) - 3c_3 = 6\chi\mathcal{E}(-c_1)/\mathcal{O}(-c_1)(z) - 3c_3 = 6\chi\mathcal{I}_C(z) - 3c_3$. Computing this last polynomialand dividing by 6 gives the theorem statement.

Remark 3.2 In the case $s_0(C) < e'_C+4$, C is the scheme of zeros of a section of $\mathcal{F}(n)$ for $n \geq \beta$ and the exact sequence gives that $s_0(C) = \alpha_{\mathcal{F}} + n + c_1$. In this case bounds on $s_0(C)$ can be found in [RV1], propositions 5.3 and 5.8 and [RV2], theorem 4.3.

Remark 3.3 In the special case $e'_C = e_C$ (e.g. C integral), theorem 3.1 can be easily proven by looking at the Euler characteristic of $\mathcal{I}_C$ (in fact, one gets a slightly stronger bound). Many examples of such integral curves are given in the paper [LR], where it is shown that a general high degree embedding of a smooth curve into $\mathbf{P}^3$ satisfies $e+4<s$ (here this condition implies both that C is a minimal curve section of a rank two reflexive sheaf *and* that C is minimal in its even liaison class).

On the other hand, this condition does *not* hold for minimal curve sections of a (stable) rank two reflexive sheaf $\mathcal{E}$ with fixed Chern classes and many choices of the spectrum (see [H1], §7 for the definition). Indeed, e_C is determined by the spectrum of $\mathcal{E}$, which can vary linearly with c_2, while e'_C is determined by $\alpha_{\mathcal{E}}$, which can vary as a square root of c_2 by theorem 1.1: thus for most spectra we expect that $e'_C < e_C$. For several examples of this behavior when $\alpha = 1$, see any of the examples constructed by Hartshorne and Rao in [HR], §2 with spectrum $\neq \{0^{c_2}\}$.

Example 3.4 As an example in which $e' << e$, let C be the disjoint union of a plane curve of degree d and d lines in general position. Then we see that $e'_C = -2$ while $e_C = d-3$. A general section $\xi \in H^0\omega_C(-2)$ generates the sheaf at all but d^2-5d points and exhibits C as a minimal curve for a stable rank 2 reflexive sheaf on $\mathbf{P}^3$ with $c_1 = 2, c_2 = 2d$ and $c_3 = d^2-5d$. Theorem 3.1 predicts that $s_0(C) \leq \frac{1}{2}(-3+\sqrt{25+48d})$ (the polynomial $p(z)$ has $z+1$ as a factor). On the other hand, the d lines in general position are of maximal rank ([HH], theorem 0.1), and from this we see that the actual value is $s_0(C) = \frac{1}{2}(-3+\sqrt{25+24d})$. For example, if $d = 100$, then the theorem predicts that $s_0(C) \leq 33$. The true value is $s_0(C) = 23$.

Example 3.5 Let L be the skew union of $r \geq 3$ smooth complete intersections of the same type (m,m) and assume that no surface of degree $2m$ contains L. In this case $\omega_L \cong \mathcal{O}_L(2m-4)$, $e'_L = e_L = 2m-4$ and the unit section of $\omega_L(4-2m)$ generates this sheaf. Thus L is a section of a stable rank two vector bundle $\mathcal{E}(m)$ on $\mathbf{P}^3$ with $c_1 = 0$ and $c_2 = (r-1)m^2$. The hypothesis $s_0(L) > 2m$ shows that L is in fact a minimal section of $\mathcal{E}$.

Of course L lies on a surface of degree rm, but theorem 3.1 shows that L lies on surfaces of degree $\leq 1+$ the largest root of the polynomial

$$p(z) = \binom{z+3}{3} - rm^2(z-m+2).$$

This is in general a much better bound, for example when $r = m = 10$ we find that $s_0(L) \leq 70$ instead of 100. When $m = 1$ and L is a union of lines in general position, then L is of maximal rank by the main theorem of [HH], and hence the bound given above is sharp. If we knew that the general union of r complete intersections of type (m, m) was of maximal rank, then we would obtain sharpness in theorem 2.5 for some mid-range values of α

Note that counting the surfaces of degree z containing $zm^2 + 1$ points of each irreducible component of L does not yield a better bound: This estimate would show that $s_0(L) \leq 1+$ the last root of

$$q(z) = \binom{z+3}{3} - r(zm^2 + 1).$$

However it is easily checked that $q(z) > p(z)$ for all $m > 1$ (for $m = 1$ we have $p(z) = q(z)$. When $r = m = 10$ this estimate shows that $s_0(L) \leq 75$.

Example 3.6 For fixed c_1, c_2 and α, Hartshorne's theorem and the definition of stability give that $0 < \alpha \leq \sqrt{3c_2 + 1 + \frac{3}{4}c_1} - 1 - \frac{1}{2}c_1$. The sharpness of the bound given in theorem 2.5 has already been shown for the extreme cases $\alpha = 1$ and $\alpha = \sqrt{3c_2 + 1 + \frac{3}{4}c_1} - 1 - \frac{1}{2}c_1$ (see [RV1] and [RV2]: in the case $\alpha = 1$, it can be shown that the bound of theorem 2.5 and the bound in [RV2] agree). In this example we show that for $c_1 = 0$, $c_2 > 4$ and $\alpha = 2$ the bound of theorem 2.5 is sharp (when $c_2 \leq 3$, this already follows from the cases mentioned above).

For $c_2 > 4$, let C be a generic elliptic curve of degree $\delta = c_2 + 2c_1 + 4$. By the main theorem of [BE], C has maximal rank. Since it is evident that $h^1\mathcal{O}_C(l) = 0$ for $l > 0$, the maximal rank condition implies that both $h^1(I_C(l))$ and $h^0(I_C(l))$ are determined by $\chi(I_C(l))$ for $l > 1$ (since one of the two is zero). In particular, $h^0(I_C(l))$ becomes positive exactly when the Euler characteristic of I_C becomes positive for good.

Now we use the nowhere vanishing section $1 \in \mathcal{O}_C$ to define a rank two bundle $\mathcal{E}$ with corresponding exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}^3}(-4) \to \mathcal{E}(-2) \to I_C \to 0.$$

The cubic polynomial of theorem 4.1 is precisely 6 times

$$\chi(\mathcal{E}(t-2)/O_{\mathbf{P}^3}(t-4))$$

which from the exact sequence is the same as $\chi(I_C(t))$. The same sequence (which is exact on global sections for every twist) shows that the second section of $\mathcal{E}$ occurs precisely when the cubic polynomial of theorem 4.1 becomes positive for good.

Question 3.7 While the above example shows that theorem 2.5 is sharp when $c_1 = 0$ and $\alpha = 2$, we do not know about sharpness for when $2 < \alpha < \lfloor \sqrt{3c_2 + 1 + \frac{3}{4}c_1} - 1 - \frac{1}{2}c_1 \rfloor$. In fact, it is not immediate that every value of α is possible: Given $c_1 = -1$ or 0 and $c_2 > 0$, does there exist a (stable) rank two reflexive sheaf achieving each value of α allowed by theorem 1.1?

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