A Remark on Connectedness in Hilbert Schemes

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Dedicated to Robin Hartshorne with affection and gratitude

Let $H_{d,g}$ denote the Hilbert scheme of locally Cohen-Macaulay curves in $\mathbb{P}^3$. For any $d > 4$ and $g \leq \binom{d-3}{2}$, $H_{d,g}$ has two well-understood irreducible families: There is a component $E \subset H_{d,g}$ corresponding to extremal curves (see [5]; these are the curves with maximal Rao function) and S, the family of subextremal curves (see [7]; these have the next largest Rao function). In this short note we show that $\overline{S} \cap E \neq \emptyset$ in $H_{d,g}$ by constructing an explicit specialization (Prop. 1). Our construction also works for ACM curves of genus $g = \binom{d-3}{2} + 1$ (Remark 2) and hence $H_{d,g}$ is connected for $g > \binom{d-3}{2}$ (Corollary 3).

Proposition 1 *For each $d \geq 4$ and $g \leq \binom{d-3}{2}$ there exist extremal curves in $H_{d,g}$ which lie in the closure of the family of subextremal curves.*

Proof: Fixing g as in the statement, let (x, y^{d-2}) be the ideal of a planar multiple line of degree $d-2$ with support the line L given by $\{x = y = 0\}$. We define the map

$$(x, y^{d-2}) \xrightarrow{\phi} S_L(-1)$$

by $x \mapsto 1, y^{d-2} \mapsto z^{d-3}$. This map is surjective and the kernel

$$I_V = (x^2, xy, y^{d-1}, xz^{d-3} - y^{d-2}) = (x^2, xy, xz^{d-3} - y^{d-2})$$

is the total ideal of an ACM curve V of degree $d-1$ and arithmetic genus $\binom{d-3}{2}$ supported on L.

Now we construct a multiplicity d-line with support L as follows. We define the map $I_V \xrightarrow{\psi} S_L(b)$ by $x^2 \mapsto 0, xy \mapsto z^{b+2}, xz^{d-3} - y^{d-2} \mapsto w^b + d - 2$, where $b = \binom{d-3}{2} - 1 - g$ (it is easy to check that the kernel of the surjection $S(-d+2) \oplus S(-2)^2 \to I_V$ is $S(-d+1) \oplus S(-3)$ and maps to zero under ψ). Although ψ is not surjective (unless $b = -2$; see Remark 2), its sheafification is. The kernel

$$I_W = (x^2, xy^2, y^{d-1} - xyz^{d-3}, xyw^{b+d-2} - z^{b+2}(y^{d-2} - xz^{d-3})).$$

is the total ideal of a multiple d-line W of genus g as we can see from the exact sequence

$$0 \to \mathcal{I}_W \to \mathcal{I}_V \to \mathcal{O}_L(b) \to 0.$$

This sequence further shows that $H^1_*(\mathcal{I}_W) \cong S/(x, y, z^{b+2}, w^{b+d-2})(b)$ and hence the Rao function achieves the upper bound given in ([7], Thm. 2.11): thus W is a subextremal curve.

Now we deform W by considering the ideal

$$I_t = (x^2, xy^2, ty^{d-1} - xyz^{d-3}, xyw^{b+d-2} - tz^{b+2}(ty^{d-2} - xz^{d-3}))$$

parametrized by $t \in \mathbb{A}^1$. Flattening over $\mathbb{A}^1$, we add to this ideal polynomials p such that $pt \in I_t$ (see [3], III, Example 9.8.4). If A, B, C are the last three generators appearing in I_t, we add

$$D = (w^{b+d-2}B + z^{d-3}C)/t = w^{b+d-2}y^{d-1} - z^{b+d-1}(ty^{d-2} - xz^{d-3})$$

and

$$E = (z^{d-3}A + yB)/t = y^d.$$

Letting $t \longrightarrow 0$ we obtain the limit ideal

$$I_0 = (x^2, xy^2, xyz^{d-3}, xyw^{b+d-2}, y^d, w^{b+d-2}y^{d-1} - z^{2d+b-4}x).$$

The saturation $\overline{I}_0$ contains $(x^2, xy, y^d, w^{b+d-2}y^{d-1} - z^{2d+b-4}x)$, which is the saturated ideal of an extremal curve of degree d and genus g ([5], Prop. 0.6), completing the proof.

Remark 2 The deformation used in the proof above works if we take $b = -2$ (i.e. $g = \binom{d-3}{2} + 1$), but in this case the map $I_V \to S_L(b)$ is surjective and hence $H^1_*(\mathcal{I}_W) = 0$. Thus W is an ACM curve and we have produced extremal curves in the closure of the ACM curves of genus $g = \binom{d-3}{2} + 1$.

Corollary 3 *The Hilbert scheme $H_{d,g}$ of locally Cohen-Macaulay curves in $\mathbb{P}^3$ is connected for $d \geq 4$ and $g > \binom{d-3}{2}$.*

Proof: For $g > \binom{d-3}{2} + 1$ this is easy because $H_{d,g}$ is irreducible (the curves are either extremal or ACM; see [7] Lemma 2.5) or empty. In the remaining case $g = \binom{d-3}{2} + 1$ there are two irreducible components given by the ACM curves and the extremal curves. By Remark 2, we conclude that $H_{d,g}$ is connected.

Remark 4 In his thesis [1], Aït-Amrane proves Proposition 1 above using the theory of triades of families of curves [4], although he doesn't recover the extension of Remark 2. His main result is that $H_{d,g}$ is connected for $d \geq 4, g = \binom{d-3}{2}$. This case is more complicated, as there is another irreducible component to deal with.

References

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