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Moishezon's theorem and degeneration

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ABSTRACT

Let L be a tensor product of two very ample line bundles on the smooth projective complex threefold X. Under the hypothesis that $H^0(X, K_X(L)) \neq 0$, we show that the restriction map $r : \operatorname{Pic} X \to \operatorname{Pic} Y$ is an isomorphism for very general Y in the linear system $|L|$. For such L this result recovers the Noether-Lefschetz theorem of Moishezon, who extended the original topological and Hodge-theoretic arguments of Lefschetz. We give a degeneration argument which is almost completely algebraic in nature.

Introduction

Lefschetz [22] achieved a high point in algebraic geometry in the 1920s when he proved M. Noether's classic statement [26] from the 1880s: For very general surfaces $S \subset \mathbb{P}^3_{\mathbb{C}}$ of degree $d \geq 4$, the restriction map $r : \operatorname{Pic} \mathbb{P}^3_{\mathbb{C}} \to \operatorname{Pic} S$ is an isomorphism. This began Noether-Lefschetz theory [4], which seeks to determine the line bundles L on varieties X for which the restriction $r : \operatorname{Pic} X \to \operatorname{Pic} Y$ is an isomorphism for general Y in the linear system $|L| = \mathbb{P}H^0(X, L)$. The results are very strong if $\dim X > 3$, when the

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Grothendieck-Lefschetz theorem asserts that if X is smooth and $Y \subset X$ is *any* effective ample divisor, then $r : \operatorname{Pic} X \to \operatorname{Pic} Y$ is an isomorphism [13,14]. Recent results include variations for class groups of normal varieties [27] and linear systems with base locus [5].

The problem is more subtle when $\dim X = 3$. One must take $Y \in |L|$ to be *very general*, avoiding a countable union of proper subvarieties $V_i \subset |L|$ called the *Noether-Lefschetz components*, which are dense in the Euclidean topology [6,8]; moreover, the conclusion fails without additional positivity assumptions on L. Since 1980 there have been many variations on the theorem for smooth complex threefolds X using a variety of methods: Carlson, Green, Griffiths and Harris [7] used infinitesimal variations of Hodge structures; Green [10,11] used Koszul cohomology to give sharp lower bounds on the codimension of the Noether-Lefschetz components, which were subsequently refined by Voisin [30,31]; Ein [9] computed Picard groups of general dependency loci of sections of vector bundles; Joshi [19] used an infinitesimal approach due to Mohan Kumar and Srinivas involving formal completions. These results all require that $K \otimes L$ is generated by its global sections or that the multiplication map $H^0(K \otimes L) \otimes H^0(L) \to H^0(K \otimes L \otimes L)$ is surjective. Our main theorem requires only that $H^0(K \otimes L) \neq 0$ and that L decomposes as a tensor product:

Theorem 1. *Let X be a smooth projective complex threefold. If $A, B \in \operatorname{Pic} X$ are very ample and $H^0(K_X(A \otimes B)) \neq 0$, then $r : \operatorname{Pic} X \xrightarrow{\sim} \operatorname{Pic} Y$ for very general $Y \in |A \otimes B|$.*

The most important special case is the following.

Corollary 2. *Let $X \subset \mathbb{P}^n_{\mathbb{C}}$ be a smooth projective threefold. If $d > 1$ and $H^0(K_X(d)) \neq 0$, then the restriction $r : \operatorname{Pic} X \to \operatorname{Pic} Y$ is an isomorphism for very general $Y \in |\mathcal{O}_X(d)|$.*

Remark. For $L = A \otimes B$ in Theorem 1, the non-vanishing $H^0(K_X(L)) \neq 0$ is equivalent to $h^0(K_X) < h^0(K_X(L))$ (see Lemma 2.1), but in view of the exact sequence

$$0 \to H^0(K_X) \to H^0(K_X(L)) \to H^0(K_Y) \to H^1(K_X) \to 0$$

arising from Kodaira vanishing, the condition that $h^0(K_X) < h^0(K_X(L))$ is equivalent to $h^0(K_Y) > h^1(K_X)$, which is equivalent by Serre duality to the Hodge condition $h^{2,0}(Y) > h^{2,0}(X)$. This shows the connection to Moishezon's theorem [24, 7.5], which says that for L very ample on X, the restriction $r : \operatorname{Pic} X \to \operatorname{Pic} Y$ is an isomorphism for very general $Y \in |L|$ if and only if (a) $b_2(Y) = b_2(X)$ or (b) $h^{2,0}(Y) > h^{2,0}(X)$. When $X = \mathbb{P}^3$ and $\deg Y = d$, (b) recovers the classical theorem for $d \geq 4$, while (a) picks up the missing case $d = 1$. Since $b_2(Y) > b_2(X)$ for sufficiently ample L [4, 1.10], the more important case is (b), which Moishezon achieved by extending the topological and Hodge-theoretic ideas of Lefschetz. Voisin takes a similar path, assuming $H^2_v(Y, \mathbb{C}) \cap H^{2,0}(Y) \neq 0$, [32, 3.33], but this is equivalent to $h^{2,0}(Y) > h^{2,0}(X)$ [4, 1.11]. Thus Theorem 1 recovers the theorem of Moishezon-Voisin for line bundles L that are decomposable as a tensor product of very ample line bundles.

Our proof of Theorem 1 follows the outline of the argument for $X = \mathbb{P}^3$ by Griffiths and Harris [12], who carefully studied the degeneration of a degree d surface Y to a reducible surface $T \cup P$ in which $\deg T = d - 1$ and P is a plane. The general case is complicated by the lack of handy planes $P \subset X$ for which $\operatorname{Pic} X \to \operatorname{Pic} P$ is an isomorphism and also by the possibility that $\operatorname{Pic}^0 X \neq 0$. The key observation is that for a reducible union $P \cup T$ with $P \in |A|$, $T \in |B|$ and $D = P \cap T$ smooth, the non-vanishing $H^0(K_X(A \otimes B)) \neq 0$ implies $\dim \operatorname{Pic}^0 D > \dim \operatorname{Pic}^0 P$ and $\dim \operatorname{Pic}^0 D > \dim \operatorname{Pic}^0 T$. This difference in dimension gives enough leverage to calculate the Picard group of the central fiber of a desingularization of a linear deformation from Y to $P \cup T$, from which we deduce the isomorphism $\operatorname{Pic} X \cong \operatorname{Pic} Y$ for very general Y. We sketch the proof in three steps, which correspond to the three sections of the body of the paper.

1. The Noether-Lefschetz locus For $A, B \in \operatorname{Pic} X$ as in Theorem 1, set $L = A \otimes B$ and let $\mathcal{Y} \subset X \times |L| \to |L|$ be the universal family of surfaces. For any morphism $Z \to |L|$, let $\mathcal{Y}_Z \subset X \times Z \to Z$ be the pullback to Z and, for the sake of lightening notation, denote by $\mathbf{Hilb}_Z$ the relative Hilbert scheme $\mathbf{Hilb}_{\mathcal{Y}_Z/Z}$ of curves for the family $\mathcal{Y}_Z \to Z$ of surfaces and $h_Z : \mathbf{Hilb}_Z \to Z$ the structural map, which is locally projective over Z.

For any fixed embedding

$$j : X \hookrightarrow \mathbb{P}^N, \tag{1}$$

the Hilbert scheme decomposes into a disjoint union $\mathbf{Hilb}_{|L|} = \coprod \mathbf{Hilb}^{\varphi}_{|L|}$ indexed by the Hilbert polynomial $\varphi \in \mathbb{Q}[z]$ with respect to Embedding (1) and for each φ carries a universal flat family of curves

$$\begin{array}{ccccc} \mathcal{C}^{\varphi} & \subset & \mathcal{Y}_{\mathbf{Hilb}^{\varphi}_{|L|}} & \subset & X \times \mathbf{Hilb}^{\varphi}_{|L|} \\ & & & & \downarrow \\ & & & & \mathbf{Hilb}^{\varphi}_{|L|} \end{array} \tag{2}$$

The subscheme $\mathbf{Div}^{\varphi}_{|L|} \subset \mathbf{Hilb}^{\varphi}_{|L|}$ corresponding to pairs (C, Y) with C Cartier on Y is open (when Y is smooth this means C has no isolated or embedded points) and we prove that over the open locus $U \subset |L|$ of smooth surfaces Y for which the restriction $r : \operatorname{Pic}^0 X \to \operatorname{Pic}^0 Y$ is an isomorphism, $\mathbf{Div}^{\varphi}_U \subset \mathbf{Hilb}^{\varphi}_U$ is also closed. It follows that the locus $W^{\varphi} \subset \mathbf{Div}^{\varphi}_U$ of pairs (C, Y) with $\mathcal{O}_Y(C)$ not in the image of $\operatorname{Pic} X \to \operatorname{Pic} Y$ is closed and hence has finitely many irreducible components W^{φ}_i, whose images $\Sigma^{\varphi}_i = h_U(W^{\varphi}_i) \subset U$ are closed subvarieties. There are countably many choices of Hilbert polynomial $\varphi \in \mathbb{Q}[z]$, so there are countably many Noether-Lefschetz components.

2. Degeneration Consider a linear pencil $\ell \cong \mathbb{P}^1$ inside of $|A \otimes B|$ passing through a smooth surface S and having reducible central fiber $P \cup T$ at $0 = u \in \mathbb{P}^1$ for generally chosen $P \in |A|$, $T \in |B|$ so that $D = P \cap T$ is smooth. That is, if P, T, S are defined by respective sections $f_P \in H^0(A), f_T \in H^0(B), f_S \in H^0(A \otimes B)$, the total family $M \subset X \times \mathbb{P}^1$ of the pencil has equation (in an affine coordinate t about 0) $tf_S - f_P f_T = 0$. This local equation shows that M is singular at the points $S \cap P \cap T$ in the central fiber. We resolve them with the family

$$\tilde{M} \subset \tilde{X} \times \mathbb{P}^1 \tag{3}$$

of strict transforms in the blow-up $\tilde{X} \to X$ at $S \cap T$. Family (3) agrees with the original family for $t \neq 0$, but the new central fiber is $\tilde{P} \cup T$, where $\tilde{P} \to P$ is the blow-up at the points $S \cap T \cap P$. For very general S, T, P we prove that

$$\operatorname{Pic} \tilde{M}_0 \cong \operatorname{Pic} X \oplus \mathbb{Z} \tag{4}$$

with the second summand generated by $\tilde{P}|_{\tilde{M}_0}$. The key point is an argument using the relative Picard scheme to show that the natural diagram

$$\begin{tikzcd} \operatorname{Pic} X \arrow[r] \arrow[d] & \operatorname{Pic} P \arrow[d] \\ \operatorname{Pic} T \arrow[r] & \operatorname{Pic} D \end{tikzcd} \tag{5}$$

is Cartesian.

3. Properness To show that any Noether-Lefschetz component $h_U(W) = \Sigma \subset U$ is a proper closed set, we show that $\Sigma \cap \mathbb{P}^1$ is a proper closed subset of $\mathbb{P}^1$, where $\mathbb{P}^1 \subset |A \otimes B|$ is a pencil determined by surfaces S, T, P as above. The idea is that if the projection $W \to T$ is dominant, then there is a curve $f : E \subset W$ dominating $\mathbb{P}^1$. After we normalize to make E smooth, the universal family of curves associated to E corresponds to a family of line bundles which at a point $p \in f^{-1}(0)$ lies in the image of $\operatorname{Pic} X$ modulo the vertical component $\tilde{P}|_{\tilde{M}_0}$, and therefore the nearby line bundles also lie in the image of $\operatorname{Pic} X$. This is complicated by the possibility that $f : E \to \mathbb{P}^1$ may be ramified over $0 \in \mathbb{P}^1$; resolving the resulting singularities creates more vertical components in the Picard group, but the outcome remains the same.

We work over $k = \mathbb{C}$ throughout, but we only use the complex hypothesis for one monodromy argument and the characteristic-zero hypothesis to apply Kawamata-Viehweg vanishing and results on the relative Picard scheme [21]. Except for the monodromy argument, our proof is algebraic.

Ongoing work Before proceeding with our proof of Theorem 1, we remark that we expect the method outlined here applies more broadly to give a result in the spirit of the theorem of Ravindra and Srinivas [28], but with slightly different (and incomparable) hypotheses:

Conjecture 3. *Let X be a complex projective normal threefold, $f : X \to \mathbb{P}^n$ a finite morphism given by a complete linear system $H^0(L)$ (so that $L = f^*\mathcal{O}(1)$), and assume that L can be expressed as $A \otimes B$ where A and B define birational maps onto their images in projective space. If $H^0(K_X(L)) \neq 0$, then the general hyperplane section Y of the image of X under f is normal, and for very general such Y the restriction $\mathrm{Cl}\, X \to \mathrm{Cl}\, Y$ is an isomorphism.*

The main stumbling block is proving that the analog to Diagram (5) is Cartesian, since in general class groups do not have the same nice functorial properties as Picard groups.

1. The Noether-Lefschetz locus

Fix a smooth projective variety $X \subset \mathbb{P}^N$ and $L \in \mathrm{Pic}\, X$. For a family $U \subset |L|$ of smooth divisors, the *Noether-Lefschetz locus* is the set $\mathrm{NL}(U) \subset U$ of divisors $Y \in U$ for which the restriction $\mathrm{Pic}\, X \to \mathrm{Pic}\, Y$ is not surjective. Equivalently, $Y \in \mathrm{NL}(U)$ if there is an effective Cartier divisor $D \subset Y$ for which $\mathcal{O}_Y(D)$ is not in the image of $\mathrm{Pic}\, X$. We will use Hilbert schemes to show that within the open set $U \subset |L|$ consisting of smooth Y for which $\mathrm{Pic}^0 X \to \mathrm{Pic}^0 Y$ is an isomorphism, $\mathrm{NL}(U)$ is a countable union of closed subvarieties of U.

Recall that for a morphism $Z \to |L|$ we denote by $\mathbf{Hilb}_Z$ the relative Hilbert scheme for the associated flat family $\mathcal{Y}_Z \to Z$. It is well known that $\mathbf{Hilb}_Z$ decomposes as the disjoint union of the $\mathbf{Hilb}_Z^\varphi$, where for each numerical polynomial $\varphi \in \mathbb{Q}[t]$,

$$\mathbf{Hilb}_Z^\varphi = \{(D, Y) : Y \in Z \text{ and } D \subset Y \text{ is a closed subscheme with Hilbert polynomial } \varphi\}.$$

Each $\mathbf{Hilb}_Z^\varphi$ is locally projective over Z, as proved by Grothendieck (Nitsure gives a complete exposition [25]), and contains the subset $\mathbf{Div}_Z^\varphi$ corresponding to pairs (D, Y) with D Cartier on Y.

Proposition 1.1. *Let $U \subset |L|$ be a family of smooth hypersurfaces on X and fix a Hilbert polynomial φ. Then $\mathbf{Div}_U^\varphi \subset \mathbf{Hilb}_U^\varphi$ is open and closed.*

Proof. It is known that $\mathbf{Div}_U \subset \mathbf{Hilb}_U$ is open in general [21, 9.3.7]. If $p \in \mathbf{Hilb}_U^\varphi$ lies in the closure of $\mathbf{Div}_U^\varphi$, then there is an integral curve $T \subset \mathbf{Hilb}_U^\varphi$ with $p \in T$ and $T' = T - \{p\} \subset \mathbf{Div}_U^\varphi$. Replace T by its normalization and remove all but one preimage of p; then base extension gives flat families $\mathcal{D} \subset \mathcal{Y} \subset X \times T \xrightarrow{\pi} T$ with $D_t \subset Y_t$ Cartier for $t \in T'$, hence $D_{T'} \subset Y_{T'}$ is an effective Cartier divisor [21, 9.3.4]. The closure $E = \overline{D_{T'}} \subset Y$ is an effective Weil divisor on Y, hence is Cartier because Y is smooth. Since D is flat over T, no components map to p and therefore $D = \overline{D_{T'}} = E$. It follows that $D_p = E_p \subset Y_p$ is Cartier [21, 9.3.4], hence $p \in \mathbf{Div}_U^\varphi$. $\square$

Example 1.2. Proposition 1.1 fails without the smoothness hypothesis. For example, the family of twisted cubic curves (t^2-1, t^3-t, at) in $\mathbb{A}^3$ parametrized by $a \in \mathbb{A}^1$ lies on the family of surfaces $xz - ay$; the flat limit is a nodal cubic curve in the xy-plane with an embedded point [15, III, 9.8.4], which is not Cartier on the limit surface xz, since it is not Cohen-Macaulay at the origin and thus not a local complete intersection.

Proposition 1.3. *Let X be a smooth projective variety and $\mathcal{Z} \subset X \times U \to U$ be a flat family of closed subschemes with reduced connected fibers Z_u such that the restriction maps $r : \operatorname{Pic}^0 X \to \operatorname{Pic}^0 Z_u$ are isomorphisms for all $u \in U$. Fix $\mathcal{L} \in \operatorname{Pic} \mathcal{Z}$. Then*

(a) *The set $A_{\mathcal{L}} = \{u \in U : \mathcal{L}_u \in \operatorname{Pic}^0 Z_u\} \subset U$ is open in U.*
(b) *If a subgroup $G \subset \operatorname{Pic} \mathcal{Z}$ contains $f^*(\operatorname{Pic}^0 X)$ under $f : \mathcal{Z} \to X \times U \xrightarrow{\pi_1} X$, then $G_{\mathcal{L}} = \{u \in U : \mathcal{L}_u \in G_u\}$ is open.*

Proof. The group scheme $\operatorname{Pic}^0 X$ is smooth since we are working over $\mathbb{C}$. In particular, the $\operatorname{Pic}^0 Z_u$ are smooth of constant dimension, and since the fibers Z_u are reduced and connected the relative Picard scheme $\mathbf{Pic}_{\mathcal{Z}/U}$ exists and represents the relative Picard functor in the étale topology [21, 9.4.18.1] and hence also in the fppf topology [21, 9.4.1]. It follows that $\mathbf{Pic}_{\mathcal{Z}/U}$ contains $\mathbf{Pic}^0_{\mathcal{Z}/U}$ as an open group subscheme of finite type whose fibers are the $\operatorname{Pic}^0 Z_u$ [21, 9.5.20]. The invertible sheaf $\mathcal{L} \in \operatorname{Pic} \mathcal{Z}$ defines a continuous section $\sigma : u \to \mathbf{Pic}_{\mathcal{Z}/U}$ by $u \mapsto \mathcal{L}_u$ and $A_{\mathcal{L}} = \sigma^{-1}(\mathbf{Pic}^0_{\mathcal{Z}/U})$, proving part (a). For part (b), observe that $G_{\mathcal{L}} = \bigcup_{\mathcal{M} \in G} A_{\mathcal{L}-\mathcal{M}}$ is a union of open sets. □

For $U \subset |L|$ a family of smooth hypersurfaces and $\varphi \in \mathbb{Q}[z]$, the universal family

$$\begin{array}{ccccc} \mathcal{D}^\varphi & \subset & \mathcal{Y}^\varphi & \subset & X \times \mathbf{Div}^\varphi_U \\ & & & & \downarrow \\ & & & & \mathbf{Div}^\varphi_U \end{array} \tag{6}$$

gives rise to the invertible sheaf $\mathcal{O}_{\mathcal{Y}^\varphi}(\mathcal{D}^\varphi) \in \operatorname{Pic} \mathcal{Y}^\varphi$ defined on the fibers by $\mathcal{O}_{Y_t}(D_t)$ via the Abel map [21, 9.4.6]. At the level of sets we can write

$$\mathrm{NL}(U) = \bigcup_{\varphi \in \mathbb{Q}[z]} h_U(W^\varphi) \tag{7}$$

where $h_U : \mathbf{Hilb}_U \to U$ is the structural map and

$$W^\varphi = \{(D, Y) \in \mathbf{Div}^\varphi_U : \mathcal{O}_Y(D) \text{ is not in the image of } \operatorname{Pic} X \to \operatorname{Pic} Y\}.$$

Proposition 1.4. *Let $U \subset |L|$ be a family of smooth hypersurfaces $Y_t \subset X$ for which the restrictions $\operatorname{Pic}^0 X \to \operatorname{Pic}^0 Y_t$ are isomorphisms. Then $h_U(W^\varphi) \subset U$ is closed for each $\varphi \in \mathbb{Q}[z]$, hence $\mathrm{NL}(U)$ is a countable union of closed subvarieties $\Sigma_i \subset U$.*

Proof. Fix a polynomial φ, and let G be the image of $\operatorname{Pic} X$ in $\operatorname{Pic} \mathcal{Y}^\varphi$. Applying Proposition 1.3 with the line bundle $\mathcal{L} = \mathcal{O}_{\mathcal{Y}^\varphi}(\mathcal{D})$ on $\mathcal{Y}^\varphi$ shows that the set

$$\{t \in \mathbf{Div}_U^\varphi : \mathcal{O}_{Y_t}(D_t) = M|_{Y_t} \text{ for some } M \in \operatorname{Pic} X\}$$

is open in $\mathbf{Div}_U^\varphi$, so its complement $W^\varphi \subset \mathbf{Div}_U^\varphi$ is closed in $\mathbf{Div}_U^\varphi$, hence closed in $\mathbf{Hilb}_U^\varphi$ by Proposition 1.1. Since $\mathbf{Hilb}_U^\varphi$ is projective over U, the image $h_U(W^\varphi)$ is closed and thus a finite union of irreducible closed sets. Taking the union over the countably many choices of $\varphi \in \mathbb{Q}[z]$ expresses $\mathrm{NL}(U)$ as a countable union of irreducible closed subvarieties of U. $\square$

Remark 1.5. The components of $h_U(W^\varphi)$ need not be proper subsets of U in general. For example, if $X = \mathbb{P}^3$ and $L = \mathcal{O}(2)$, then $\operatorname{Pic} \mathbb{P}^3 \to \operatorname{Pic} Q$ is not surjective for any smooth $Q \in |L|$. The point is to show that these sets are proper in the setting of Theorem 1.

2. Degeneration

We establish the claims made in Subsection 2 of the introduction. In the setting of Theorem 1, we show in Theorem 2.8 that for very general $P \in |A|, T \in |B|$ with $D = P \cap T$, Diagram (5) is Cartesian with injective restriction maps. In Corollary 2.13 we show that the central fiber in Family (3) has Picard group given in Isomorphism (4).

2.1. Consequences of the hypothesis $H^0(K_X(A \otimes B)) \neq 0$

Here we show what the condition $H^0(K_X(A \otimes B)) \neq 0$ in Theorem 1 delivers for a smooth surface $S \in |A \otimes B|$ degenerating to a reducible surface $T \cup P$ with $T \in |B|$ and $P \in |A|$.

Lemma 2.1. *Let L, M be line bundles on a variety X of dimension greater than 0 such that $h^0(M) \geq 2$ and $h^0(L \otimes M) \neq 0$. Then $h^0(X, L) < h^0(X, L \otimes M)$.*

Proof. Since X is irreducible, the pairing $H^0(L) \times H^0(M) \to H^0(L \otimes M)$ is non-degenerate, so $h^0(L \otimes M) \geq h^0(L) + h^0(M) - 1 \geq h^0(L) + 1$ [16, 5.1]. $\square$

Lemma 2.2. *Let X be a smooth complex threefold, A and B base-point free, big, and nef line bundles on X, $P \in |A|$ and $T \in |B|$ smooth and intersecting in a smooth connected curve $D = P \cap T$. Then*

$$H^0(K_P \otimes B) \neq 0 \iff H^0(K_X(A \otimes B)) \neq 0 \iff H^0(K_T \otimes A) \neq 0.$$

Proof. For P and T defined by nonzero sections $f_P \in H^0(A)$ and $f_T \in H^0(B)$, adjunction gives the commutative diagram

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K_X \arrow[r, "\cdot f_P"] \arrow[d, "\cdot f_T"] & K_X \otimes A \arrow[r] \arrow[d, "\cdot f_T"] & K_P \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K_X \otimes B \arrow[r, "\cdot f_P"] \arrow[d] & K_X \otimes A \otimes B \arrow[r] \arrow[d] & K_P \otimes B \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K_T \arrow[r] \arrow[d] & K_T \otimes A \arrow[r] \arrow[d] & K_D \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{8}
$$

with the middle column and middle row both exact on global sections due to Kawamata-Viehweg vanishing [29]. Now apply Lemma 2.1, noting that $h^0(A)$ and $h^0(B)$ are both ≥ 2, since each line bundle defines a non-constant morphism to projective space. □

Remark 2.3. For our applications here we could have proved Lemma 2.2 with A, B very ample via the Kodaira vanishing theorem, but we have future applications in mind with the more general statement.

Example 2.4. Both possibilities in Lemma 2.2 are illustrated by surfaces in $X = \mathbb{P}^3$.

(a) A smooth quartic $Y \subset \mathbb{P}^3$ degenerates to the union of a smooth cubic T and a plane P intersecting in a smooth elliptic curve $D = T \cap P$. Here $H^0(K_{\mathbb{P}^3}(4)) \neq 0$ and $H^0(K_P(3)) \neq 0$ and $H^0(K_T(1)) \neq 0$.
(b) A smooth cubic $Y \subset \mathbb{P}^3$ degenerates to a union of a quadric T and plane P meeting in a conic $D = T \cap P$ and we have the vanishings $h^0(K_{\mathbb{P}^3}(3)) = h^0(K_T(1)) = h^0(K_P(2)) = 0$.

Remark 2.5. To interpret the condition in Lemma 2.2, observe that Kawamata-Viehweg vanishing applies to the nef and big line bundles A and $A \otimes B$, and the long exact sequence coming from the middle row of Diagram (8) shows that $H^1(K_P \otimes B) = 0$, giving the exact sequence

$$0 \to H^0(K_P) \to H^0(K_P \otimes B) \to H^0(K_D) \to H^1(K_P) \to 0$$

coming from the right column. Then

$$
\begin{aligned}
H^0(K_P \otimes B) \neq 0 \quad &\Longleftrightarrow \quad h^1(\mathcal{O}_D) > h^1(\mathcal{O}_P) \\
&\Longleftrightarrow \quad h^0(K_D) > h^1(K_P) \\
&\Longleftrightarrow \quad \dim \mathrm{Pic}^0 D > \dim \mathrm{Pic}^0 P,
\end{aligned}
$$

where the first equivalence is by Lemma 2.1, the second by Serre duality, and the third by the fact that $H^1(\mathcal{O}_V)$ is isomorphic to the tangent space at the origin of $\operatorname{Pic}^0 V$ for any variety V [21, 9.5.11].

2.2. A Cartesian diagram

For the remainder of this section, assume that A and B are very ample on a smooth complex threefold X as in Theorem 1. For general surfaces $P \in |A|$ and $T \in |B|$ on X, the intersection curve $D = P \cap T$ is smooth and connected. Assuming $H^0(K_X(A \otimes B)) \neq 0$, we will show that Diagram (5) is Cartesian with injective restriction maps for very general choices of P and T.

Proposition 2.6. *Let $S \subset \mathbb{P}^n$ be a smooth projective surface satisfying $H^0(K_S(1)) \neq 0$.*

(a) *The general pencil in $|\mathcal{O}_S(1)|$ consists of irreducible curves, and if $L \in \operatorname{Pic} S$ satisfies $L|_D \cong \mathcal{O}_D$ for general D in such a pencil, then $L \cong \mathcal{O}_S$.*
(b) *For nonsingular (connected) $D \in |\mathcal{O}_S(1)|$, the restriction $\operatorname{Pic}^0 S \to \operatorname{Pic}^0 D$ on Picard varieties is a closed immersion, identifying $\operatorname{Pic}^0 S$ with a proper closed subvariety of $\operatorname{Pic}^0 D$.*
(c) *For very general $D \in |\mathcal{O}_S(1)|$, the restriction $r : \operatorname{Pic} S \to \operatorname{Pic} D$ is injective.*

Proof. First note that $H^0(K_S(1)) \neq 0$ is equivalent to $h^1(\mathcal{O}_D) > h^1(\mathcal{O}_S)$ for $D \in |\mathcal{O}_S(1)|$ by Remark 2.5. This implies that S is not a Veronese surface embedded by the linear system of conics on $\mathbb{P}^2$ nor its generic (smooth) projection in $\mathbb{P}^4$, because then D is rational and $h^1(\mathcal{O}_D) = 0$; neither is S ruled by lines, since then S is ruled over a general hyperplane section D and $\operatorname{Pic} S \cong \operatorname{Pic} D \oplus \mathbb{Z}$, but then $h^1(\mathcal{O}_S) = \dim \operatorname{Pic}^0 S = \dim \operatorname{Pic}^0 D = h^1(\mathcal{O}_D)$. Therefore the reducible sections in $|\mathcal{O}_S(1)|$ have codimension ≥ 2 [23, II.2.4] so that a general pencil $\mathbb{P}^1 \cong \ell \subset |\mathcal{O}_S(1)|$ consists entirely of integral curves, and hence part (a) holds by [23, II.2.3] or [3, Proof of 3.4 (a)].

The restriction map $r : \operatorname{Pic}^0 S \to \operatorname{Pic}^0 D$ is a homomorphism of smooth projective group schemes [21, 9.5.4 and 9.5.14]. The map $H^1(\mathcal{O}_S) \to H^1(\mathcal{O}_D)$ is injective by Kodaira vanishing, but this map is identified with the differential on Zariski tangent spaces at the origin [21, 9.5.11], so r is a closed immersion and the image is a proper closed subvariety by Remark 2.5 and [20, Lemma 3.11 and Rmk. 3.12]. This proves (b).

Part (c) is equivalent via diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Pic}^0 S & \longrightarrow & \operatorname{Pic} S & \longrightarrow & \operatorname{Pic} S/\operatorname{Pic}^0 S & \longrightarrow & 0 \\
 & & \| & & \downarrow r & & \downarrow \overline{r} & & \\
0 & \longrightarrow & \operatorname{Pic}^0 S & \longrightarrow & \operatorname{Pic} D & \longrightarrow & \operatorname{Pic} D/\operatorname{Pic}^0 S & \longrightarrow & 0
\end{array}
\tag{9}
$$

to showing that $\overline{r}$ is injective for very general $D \in |\mathcal{O}_S(1)|$. Let $U \subset |\mathcal{O}_S(1)|$ be the Zariski open locus of smooth curves with total family $\mathcal{D} \subset S \times U \to U$ having fibers D_u for $u \in U$. Observe that for each $L \in \operatorname{Pic} S$, the subset

$$K(L) = \{u \in U : L|_{D_u} \in \operatorname{Im} \operatorname{Pic}^0 S\}$$

depends only on the image of L in the countable Néron-Severi group $\operatorname{Pic} S/\operatorname{Pic}^0 S = \operatorname{NS}(S)$ and that $K(L) = U$ if $L \in \operatorname{Pic}^0 S$. To show that $\overline{r}$ is injective it suffices to prove that

$$L \notin \operatorname{Pic}^0 S \Rightarrow K(L) \subset U \text{ is a closed proper subset} \tag{10}$$

because then the union $\bigcup K(L)$ taken over $0 \neq L \in \operatorname{NS}(S)$ leaves plenty of curves $D \in U$ for which $\overline{r}$ is injective.

To see that $K(L) \subset U$ is closed, fix $L \in \operatorname{Pic} S$. There is an identification of relative Picard schemes $\mathbf{Pic}_{S/\mathbb{C}} \times U \cong \mathbf{Pic}_{S \times U/U}$ [21, Ex. 9.4.4], where $\mathbf{Pic}_{S/\mathbb{C}} = \operatorname{Pic} S$ is the usual Picard group and the inclusion $\mathbf{Pic}^0_{S\times U/U} \subset \mathbf{Pic}_{S\times U/U}$ is naturally identified with $\operatorname{Pic}^0 S \times U \subset \operatorname{Pic} S \times U$. The restriction morphism $\mathbf{Pic}_{S\times U/U} \to \mathbf{Pic}_{\mathcal{D}/U}$ appears as r after these identifications in diagram

$$\begin{tikzcd} \operatorname{Pic}^0 S \times U \arrow[r, hook, "j"] \arrow[d] & \operatorname{Pic} S \times U \arrow[r, "r"] \arrow[d, "p_2"] & \mathbf{Pic}_{\mathcal{D}/U} \arrow[d, "\pi"] \\ U \arrow[r, equal] & U \arrow[r, equal] & U \end{tikzcd} \tag{11}$$

The composition $r \circ j$ is a homeomorphism onto a closed subset: It injects because it is a closed immersion on each fiber by part (b); the image is closed because $\operatorname{Pic}^0 S \to \operatorname{Spec} \mathbb{C}$ is proper [21, 9.5.20], hence so is the base extension $\operatorname{Pic}^0 S \times U \to U$ and therefore universally closed. Let σ_L be the section to p_2 defined by $\sigma_L(u) = (L, u)$. Then $\tau_L = r \circ \sigma_L$ is a section to π which is a homeomorphism onto a closed subset. Indeed, $\mathbf{Pic}_{\mathcal{D}/U}$ is a disjoint union of open and closed quasi-projective subschemes $\mathbf{Pic}^{\varphi}_{\mathcal{D}/U}$ indexed by the Hilbert polynomial [21, 9.6.20]. Since $\tau_L(U)$ is irreducible, it lies in a fixed $\mathbf{Pic}^{\varphi}_{\mathcal{D}/U}$, so τ_L is a section to a projection $\mathbb{P}^N \times U \to U$ for some N; such a section is a closed immersion [15, Exercise 4.8 (e)]. It follows that $K(L) = \tau_L^{-1}(r(\operatorname{Pic}^0 S \times U)) \subset U$ is closed.

Now assume $K(L) = U$, meaning that $L_{D_u} \in \operatorname{Im} \operatorname{Pic}^0 S$ for all $u \in U$; it remains to show that $L \in \operatorname{Pic}^0 S$. Our strategy is to work over a pencil where the relative Picard scheme represents its functor to construct another constant section $u \mapsto M_{D_u}$ with $M \in \operatorname{Pic}^0 S$ and use part (a) to argue that $L = M$. As in part (a), a general pencil $\ell \subset |\mathcal{O}_S(1)|$ consists of irreducibles, let $V = \ell \cap U$ and base extend Diagram (11) by $V \subset U$. Since $\tau_L(V) \subset r(j(\operatorname{Pic}^0 \times V))$ lies in the image of the closed embedding $r : \operatorname{Pic}^0 S \times V \hookrightarrow \mathbf{Pic}_{\mathcal{D}/V}$, we obtain a section to $\operatorname{Pic}^0 S \times V \to V$, which we view as a section σ_2 to $\operatorname{Pic} S \times V \to V$. By [21, 9.2.5 and 9.4.3] there is a line bundle $\tilde{M} \in \operatorname{Pic}(S \times V)$

which defines σ_2. Viewing $V \subset \mathbb{P}^1$ as a proper open subset we may write $V \subset \mathbb{A}^1$ and if V is obtained from $\mathbb{A}^1$ by removing m points, there is an exact sequence [15, II.6.5]

$$0 \to \mathbb{Z}^m \to \operatorname{Pic}(S \times \mathbb{A}^1) \to \operatorname{Pic}(S \times V) \to 0$$

but the pullback map $\operatorname{Pic} S \to \operatorname{Pic}(S \times \mathbb{A}^1)$ is an isomorphism and the image of $\mathbb{Z}^m$ above is zero, so $\tilde{M} = p_1^* M$ for some $M \in \operatorname{Pic} S$. Moreover, $M \in \operatorname{Pic}^0 S$ because our section was in the image of $\operatorname{Pic}^0 S \times V$, so we may safely call this section σ_M for $M \in \operatorname{Pic}^0 S$. By construction $M|_{D_u} \cong L|_{D_u}$ for $u \in V$, hence $L = M$ by part (a) and so $L \in \operatorname{Pic}^0 S$. This proves Statement (10) and hence part (c). $\square$

Proposition 2.7. *If $H^0(K_X(A \otimes B)) \neq 0$, then both restriction maps $\operatorname{Pic} T \to \operatorname{Pic} D$ and $\operatorname{Pic} P \to \operatorname{Pic} D$ in Diagram (5) are injective for very general $(P,T) \in |A| \times |B|$.*

Proof. Let $W \subset |A| \times |B|$ be the Zariski open locus of pairs (P,T) where P,T and $D = P \cap T$ are smooth and connected. Let $Q \subset W$ be the subset consisting of pairs (P,T) for which there exists a nontrivial $L \in \operatorname{Pic} T$ with $L|_D \cong \mathcal{O}_D$; we claim that Q is a countable union of proper subvarieties of W. Since A is very ample, $L + mA \sim C \subset T$ is effective for some $m > 0$ so that each $L \in \operatorname{Pic} T$ has the form $L = \mathcal{O}_T(C)(-mA)$ for some $m > 0$ and $C \subset T$. We may therefore write $Q = \bigcup_{m,\varphi} Q_m^\varphi$ where $Q_m^\varphi \subseteq Q$ is the subset of pairs (P,T) for which there exists a curve $C \subset T$ with Hilbert polynomial φ with respect to A and $m > 0$ such that $\mathcal{O}_T(C)(-mA)$ is nontrivial on T but has trivial restriction to D. Then Q is a countable union $\bigcup_{m,\varphi} Q_m^\varphi$. It then suffices to show that each $Q_m^\varphi \subset W$ is a finite union of proper closed subvarieties of W.

To this end we fix $m > 0$ and $\varphi \in \mathbb{Q}[z]$. Let $\mathcal{T}$ be the universal family of divisors on X corresponding to $|B|$ and $\mathcal{T}_W$ its pullback to W. Let $\mathbf{Div}^\varphi \subset \mathbf{Hilb}^\varphi_{\mathcal{T}_W/W}$ denote the corresponding relative Hilbert scheme of Cartier divisors with Hilbert polynomial φ. Thus there is a morphism $\mathbf{Div}^\varphi \to W$ and universal families of curves and surfaces

$$\begin{array}{ccccc} \mathcal{D}^\varphi, \mathcal{C}^\varphi & \subset & \mathcal{T}_{\mathbf{Div}^\varphi} & \subset & X \times \mathbf{Div}^\varphi \\ & & & & \downarrow \\ & & & & \mathbf{Div}^\varphi \end{array} \tag{12}$$

A point in $\mathbf{Div}^\varphi$ corresponds to a triple (P,T,C) with $C \subset T$ an effective Cartier divisor having Hilbert polynomial φ. The fibers over such a point in the respective families $\mathcal{D}^\varphi, \mathcal{C}^\varphi \subset \mathcal{T}_{\mathbf{Div}^\varphi} \to \mathbf{Div}^\varphi$ are simply $D = P \cap T, C, T$. We claim that the subset $V \subset \mathbf{Div}^\varphi$ corresponding to triples (P,T,C) satisfying $\mathcal{O}_T(C)(-mA) \otimes \mathcal{O}_D \cong \mathcal{O}_D$ is closed. Since $D \in |\mathcal{O}_T(A)|$, the degree d of the restriction $\mathcal{O}_T(-C)(mA) \otimes \mathcal{O}_D \in \operatorname{Pic} D$ is the intersection number $(mA - D) \cdot A$ on T and depends only on the leading coefficient of φ. If $d \neq 0$, then $\mathcal{O}_T(-C)(mA) \otimes \mathcal{O}_D \not\cong \mathcal{O}_D$ and the corresponding $V \subset \mathbf{Div}^\varphi$ is empty and hence closed. If $d = 0$, then $\mathcal{O}_T(-C)(mA) \otimes \mathcal{O}_D \cong \mathcal{O}_D$ holds if and only if $H^0(\mathcal{O}_T(-C)(mA) \otimes \mathcal{O}_D) \neq 0$ [15, IV, 1.2]. The inclusion $\mathcal{C}^\varphi \subset \mathcal{T}_{\mathbf{Div}^\varphi}$ induces the invertible sheaf $\mathcal{I}_{\mathcal{C}^\varphi}(mA) \in \operatorname{Pic} \mathcal{T}_{\mathbf{Div}^\varphi}$, and restricting to the family $\mathcal{D}^\varphi$ gives a line bundle $\mathcal{I}_{\mathcal{C}^\varphi}(mA) \otimes \mathcal{O}_{\mathcal{D}}$, which is flat over $\mathbf{Div}^\varphi$ because $\mathcal{D}^\varphi$ is. The set of $(P, T, C) \in \mathbf{Div}^\varphi$ where $H^0(D, \mathcal{O}_T(-C)(mA) \otimes \mathcal{O}_D) \neq 0$ is closed by semicontinuity [15, III, 12.8], so again V is closed.

Let V_i be the irreducible components of V and consider the image $W_i \subset W$ of V_i under the composite map

$$V_i \subset \mathbf{Div}^\varphi \xrightarrow{\alpha} \mathbf{Pic}_{\mathcal{T}/W} \to W$$

where α is the Abel map [21, 9.4.6]. Each $W_i \subset W$ is closed because $V_i \subset \mathbf{Hilb}^\varphi_{\mathcal{T}_W/W}$ is closed (Proposition 1.1) and $\mathbf{Hilb}^\varphi_{\mathcal{T}_W/W} \to W$ is a projective morphism. The fact that $\mathbf{Pic}^0_{\mathcal{T}/W}$ is both open and closed in $\mathbf{Pic}_{\mathcal{T}/W}$ [21, 9.5.20] leaves two possibilities. If $\alpha(V_i) \subset \mathbf{Pic}^0_{\mathcal{T}/W}$, then injectivity of the maps $\operatorname{Pic}^0 T \to \operatorname{Pic}^0 D$ (Proposition 2.6 (b)) shows that $\mathcal{O}_T(-C)(m) = 0$ for all $(P, T, C) \in V_i$, so V_i does not contribute to Q^φ_n. Otherwise $\alpha(V_i)$ lies in the complement of $\mathbf{Pic}^0_{\mathcal{T}/W}$, in which case all corresponding tuples (P, T, C) satisfy $\mathcal{O}_T(-C)(m) \neq 0$ in $\operatorname{Pic} T$, so that $W_i \subset Q^\varphi_n$; here $W_i \subset W$ is proper; because for any fixed T there exist P with $\operatorname{Pic} T \to \operatorname{Pic} D$ injective by Proposition 2.6 (c). It follows that Q^φ_n is the finite union of such W_i and that $S = \bigcup_{n,\varphi} Q^\varphi_n \subset W$ is a countable union of proper subvarieties W^B_j.

Similarly, there are proper closed subvarieties $W^A_j \subset W$ corresponding to pairs (P, T) for which $\operatorname{Pic} P \to \operatorname{Pic} D$ is not injective. It follows that $\operatorname{Pic} T \to \operatorname{Pic} D$ and $\operatorname{Pic} P \to \operatorname{Pic} D$ are both injective for all $(T, P) \in W$ avoiding $\bigcup W^B_i \cup \bigcup W^A_j$. $\square$

Theorem 2.8. *Assume $H^0(K_X(A \otimes B) \neq 0$. For (P, T_0) as in Proposition 2.7, there is $T \in |B|$ such that Diagram (5) is Cartesian and all restriction maps inject.*

Proof. The restrictions $\operatorname{Pic} X \to \operatorname{Pic} T$ and $\operatorname{Pic} X \to \operatorname{Pic} P$ in Diagram (5) are injective for Zariski general (P, T) by a theorem of Ravindra and Srinivas [27], so all restrictions are injective for very general (P, T) by Proposition 2.7. Selecting (P, T_0) from this locus, the same holds of (P, T) for very general $T \in |B|$. It remains to prove the Cartesian property: setting $D = P \cap T$, we will show for very general $T \in |B|$ that if $L \in \operatorname{Pic} P$ and $M \in \operatorname{Pic} T$ satisfy $L|_D \cong M|_D$, then there exists $N \in \operatorname{Pic} X$ with $N|_P = L$ and $N|_T = M$.

Let $U \subset |B|$ be the Zariski open subset corresponding to T for which T and $D = T \cap P$ are both connected and smooth. The restrictions $\operatorname{Pic}^0 X \to \operatorname{Pic}^0 T$ and $\operatorname{Pic}^0 X \to \operatorname{Pic}^0 P$ are isomorphisms by an argument similar to that in Proposition 2.6(b), because Kodaira vanishing gives isomorphisms $H^1(\mathcal{O}_X) \to H^1(\mathcal{O}_T)$ and $H^1(\mathcal{O}_X) \to H^1(\mathcal{O}_P)$.

Note that for $L \in \operatorname{Pic} P$, the subset

$$K(L) = \{u \in U : L|_{D_u} \in \operatorname{Im} \operatorname{Pic} T_u\} \subset U$$

depends only on the class of L in $\operatorname{Pic} P / \operatorname{Pic} X$: Indeed, if $L - L' \in \operatorname{Pic} X$ and $M \in \operatorname{Pic} T$ satisfies $M|_{D_u} \cong L_{D_u}$, then $N = M - (L - L') \in \operatorname{Pic} T$ satisfies $N|_{D_u} \cong L'|_{D_u}$. Note also that $\operatorname{Pic} P/\operatorname{Pic} X$ is a quotient of the finitely generated group $\mathrm{NS}(P) = \operatorname{Pic} P/\operatorname{Pic}^0(P)$ so is countable. We will show that for each L, $K(L)$ is contained in a countable union of proper closed subvarieties of U; allowing L to vary over a set of representatives for $\operatorname{Pic} P/\operatorname{Pic} X$ will show that the very general $u \in U$ lies in no $K(L)$, proving Theorem 2.8.

Let $\mathcal{D} \subset \mathcal{T} \subset X \times U \to U$ be the family with fibers $D \subset T$ and $\mathcal{P} = P \times U \subseteq X \times U$ be the constant family. By [21, 9.4.8], all three relative Picard schemes exist and are sheaves in the étale topology; in fact $\mathbf{Pic}_{\mathcal{P}/U} \cong \mathbf{Pic}_{P/\mathbb{C}} \times U = \operatorname{Pic} P \times U$ by [21, 9.4.4] and represents the relative Picard functor $\operatorname{Pic}_{\mathcal{P}/U}$ by the Comparison Theorem [21, 9.2.5], since $\operatorname{Pic} P \times U \to U$ has a constant section. Restriction maps induce the morphisms shown:

$$\begin{tikzcd} \mathbf{Pic}_{\mathcal{T}/U} \arrow[r, "r_1"] \arrow[d, "\pi"] & \mathbf{Pic}_{\mathcal{D}/U} \arrow[d] & \operatorname{Pic} P \times U \arrow[l, "r_2"'] \arrow[d] \\ U \arrow[r, equal] & U \arrow[r, equal] & U \end{tikzcd}$$

The constant section $U \to \operatorname{Pic} P \times U$ determined by L composes with r_2 to give a section $\sigma : U \to \mathbf{Pic}_{\mathcal{D}/U}$. As in the proof of Proposition 2.6(c), σ is a homeomorphism onto the closed set $\sigma(U)$, hence $F = r_1^{-1}(\sigma(U)) \subset \mathbf{Pic}_{\mathcal{T}/U}$ is closed. Notice that $K(L) = \pi(F)$ is precisely the image of F.

Let $\eta \in U$ be the generic point and assume by way of contradiction that $r_1^{-1}(\sigma(\eta))$ contains two points $\xi_1 \neq \xi_2$. Then via the projection π the irreducible sets $F_i = \overline{\{\xi_i\}} \subset F$ both dominate U. Since $\mathbf{Pic}_{\mathcal{T}/U}$ decomposes into disjoint quasi-projective subschemes $\mathbf{Pic}^{\varphi}_{\mathcal{T}/U}$ via the Hilbert polynomial φ [21, 9.6.20], each F_i is contained in some $\mathbf{Pic}^{\varphi}_{\mathcal{T}/U}$ and hence is quasi-projective over U, therefore the images of $F_i \to U$ are constructible by Chevalley's theorem [15, II, Exercise 3.19] and contain open subsets $U_i \subset U$ because they contain the generic point η. Then the projection $F \to U$ has fibers with at least two elements over the open subset $U_1 \cap U_2$, contradicting the fact that $\operatorname{Pic} T_t \to \operatorname{Pic} D_t$ is injective for very general $T \in U$. Therefore $r_1^{-1}(\sigma(\eta))$ is either empty or consists of a single point ξ.

Case 1: $r_1^{-1}(\sigma(\eta)) = \emptyset$. Each connected component F_i of the closed set $F \subset \mathbf{Pic}_{\mathcal{T}/U}$ lies in some $\mathbf{Pic}^{\varphi}_{\mathcal{T}/U}$, hence, as above, $F_i \to U$ is quasi-projective. Since η does not lie in the image, the constructible set $\pi(F_i)$ lies in a finite union of proper subvarieties of U. Taking the union over all Hilbert polynomials $\varphi \in \mathbb{Q}[z]$ shows that $\pi(F)$ lies in a countable union of proper closed subvarieties of U.

Case 2: $r_1^{-1}(\sigma(\eta)) = \{\xi\}$. Let $F_1 = \overline{\{\xi\}} \subset F$; as above $F_1 \to U$ is quasi-projective and is also dominant. Since the very general fiber has size at most 1 by Proposition 2.7, we conclude that $F_1 \to U$ is birational; hence, there is an open subset $U' \subset U$ and a section $\tau : U' \to F_1 \cap Q$ to the projection. Now let $\ell \cong \mathbb{P}^1$ be a general pencil in $|B|$ meeting U' and let $V = \mathbb{P}^1 \cap U'$. ℓ is determined by a general pair of surfaces $T_0, T_1 \in |B|$ that intersect in a smooth connected curve C, because $|B|$ is a very ample linear system, and C is the base locus of the pencil. Any fixed base point $p \in C$ gives a section to the corresponding family $\mathcal{T}_V \to V$ of surfaces, hence the Comparison Theorem [21, 9.2.5] applies and $\mathbf{Pic}_{\mathcal{T}_V/V}$ represents the relative Picard functor $\mathrm{Pic}_{\mathcal{T}_V/V}$. In particular, the section τ constructed above gives rise to a line bundle M on $\mathcal{T}_V$ such that $M_{D_t} \cong L|_{D_t}$ for each $t \in V$ [21, 9.4.3]. Extend M to the family $\mathcal{T}_{\mathbb{P}^1}$ over the all of $\mathbb{P}^1$ [15, Prop. 6.5] and continue to call this bundle M. The total family $\mathcal{T}_{\mathbb{P}^1}$ is isomorphic to the blowup $\tilde{X} \to X$ at C [2, 1.3], hence $\mathrm{Pic}\,\tilde{X} \cong \mathrm{Pic}\,X \oplus \mathbb{Z} \cdot E$, where E is the exceptional divisor and we write $M = M' + kE$ with $M' \in \mathrm{Pic}\,X$ and $k \in \mathbb{Z}$. The restriction to T is $M'|_T + kE|_T = M'|_T + k\mathcal{O}_T(C)$, which is the image of $M'' = M' + k\mathcal{O}_X(T) \in \mathrm{Pic}\,X$ because C is the intersection of two divisors in $|B|$. Finally since $M''|_D \cong L|_D$ and $\mathrm{Pic}\,P \to \mathrm{Pic}\,D$ is injective, it follows that $M''|_P = L$ and $L \in \mathrm{Pic}\,P$ is in the image of $\mathrm{Pic}\,X$. $\square$

Remark 2.9. In the case $X = \mathbb{P}^3$ and $A = \mathcal{O}(1)$ considered by Griffiths and Harris [12], Theorem 2.8 is immediate because $P = \mathbb{P}^2$ is an actual plane and restriction gives an isomorphism $\mathrm{Pic}\,X \cong \mathrm{Pic}\,P$.

Remark 2.10. It is interesting to note how ubiquitous pencils are in Noether-Lefschetz arguments. Lefschetz started it with his famous Lefschetz pencil [22], in which the surfaces possessed at worst a single $\mathbf{A}_1$ singularity. Later the argument of Griffiths and Harris [12] for $X = \mathbb{P}^3$ used a general pencil of degree d surfaces with special member a reducible union of a plane curve P and a degree $d-1$ surface T. Above we use two more aspects of pencils, namely the geometric description of their total families as blowups and the fact that the total family of a pencil of ample divisors has a section and that hence the relative Picard scheme represents the relative Picard functor for the Zariski topology.

Remark 2.11. For smooth irreducible T, P and $D = T \cap P$ we have an isomorphism $\mathrm{Pic}(T \cup P) \cong \mathrm{Pic}\,T \times_{\mathrm{Pic}\,D} \mathrm{Pic}\,P$ [17, 5.1]; in other words,

$$\begin{array}{ccc} \mathrm{Pic}(T \cup P) & \longrightarrow & \mathrm{Pic}\,P \\ \downarrow & & \downarrow \\ \mathrm{Pic}\,T & \longrightarrow & \mathrm{Pic}\,D \end{array} \tag{13}$$

is Cartesian, so the restriction $\mathrm{Pic}\,X \to \mathrm{Pic}(T \cup P)$ is an isomorphism by Theorem 2.8. While this may suggest that Theorem 1 is true, it is not a proof, because Picard groups of surfaces can shrink in the limit; for example, smooth quadrics in $\mathbb{P}^3$ degenerate to the union of two planes, whose Picard group is $\mathbb{Z}$ [17, Ex. 5.2]. To make the limiting idea rigorous, we need a family in which line bundles extend to the reducible fiber.

2.3. Resolution of a pencil of surfaces

We continue to work within the setting of Theorem 1, with A, B very ample line bundles on the smooth threefold X. For surfaces $P \in |A|, T \in |B|, S \in |A \otimes B|$ defined by equations f_P, f_T, f_S, we can form the family $M \subset X \times \mathbb{P}^1$ with (local) equation $tf_S - f_P f_T = 0$. For P and T satisfying the conclusion of Theorem 2.8 above and Zariski general S, the intersection curve $D = P \cap T$ is a smooth, connected curve and the fibers M_t are smooth for general $t \neq 0$. The total family M is visibly singular at the intersection $S \cap P \cap T$ in the reducible central fiber $t = 0$ by the Jacobian criterion and the natural map $\operatorname{Pic} X \to \operatorname{Pic} M_0$ is an isomorphism for very general T and P by Remark 2.11.

To resolve the singularities in the central fiber, let $\tilde{X} \to X$ be the blow-up at $S \cap T$ and let $\tilde{M} \subset \tilde{X} \times \mathbb{P}^1$ be the strict transform of $M \subset X \times \mathbb{P}^1$, creating Family (3). For general choices of S, T, P, the curve $S \cap T \subset M_t$ is a constant Cartier divisor for general $t \neq 0$ so that $\tilde{M}_t \cong M_t$. Moreover $S \cap T$ is Cartier on T at the central fiber but meets P transversely in m distinct points, so we have $\tilde{T} \cong T$ and $\tilde{P} \to P$ is the blow up along $S \cap T \cap P$. Just as distinct lines through a point $p \in \mathbb{P}^2$ are separated when a point p is blown up, so surfaces containing $S \cap T$ are separated in $\tilde{X}$ and in particular the intersection $\tilde{S} \cap \tilde{P} \cap \tilde{T}$ is empty in the central fiber, so the total family $\tilde{M}$ is nonsingular near $t = 0$. If the intersection $S \cap D$ consists of m distinct points $q_1, \ldots q_m$ and $S \cap T$ is a smooth connected curve, then $\operatorname{Pic} \tilde{X} \cong \operatorname{Pic} X \oplus \mathbb{Z}$ with second summand generated by the exceptional divisor E and $\operatorname{Pic} \tilde{P} \cong \operatorname{Pic} P \oplus \mathbb{Z}^m$ with the latter summands generated by the exceptional divisors E_k. With these observations the following commutative diagrams can be identified

$$\text{(a)}\quad \begin{array}{ccc} \operatorname{Pic} \tilde{X} & \longrightarrow & \operatorname{Pic} \tilde{P} \\ \big\downarrow{\scriptstyle r_1} & & \big\downarrow{\scriptstyle r_2} \\ \operatorname{Pic} \tilde{T} & \longrightarrow & \operatorname{Pic} \tilde{D} \end{array} \qquad \text{(b)}\quad \begin{array}{ccc} \operatorname{Pic} X \oplus \mathbb{Z} & \longrightarrow & \operatorname{Pic} P \oplus \mathbb{Z}^m \\ \big\downarrow{\scriptstyle r_1} & & \big\downarrow{\scriptstyle r_2} \\ \operatorname{Pic} T & \longrightarrow & \operatorname{Pic} D \end{array} \tag{14}$$

where in Diagram (14b) the restriction maps are given by $r_1(\mathcal{L}, a) = \mathcal{L}|_T + a\mathcal{O}_X(S)|_T$ and $r_2(\mathcal{A} + \sum b_k E_k) = \mathcal{A}|_D + \sum b_k q_k$ and the top horizontal map is $(\mathcal{L}, a) \mapsto (\mathcal{L}|_P, a \sum E_k)$.

We interpret the line bundle $N = \mathcal{O}_{\tilde{M}}(\tilde{P})|_{\tilde{M}_0}$ on $\tilde{M}_0 = \tilde{T} \cup \tilde{P}$. Its restriction to $\tilde{T}$ is $\mathcal{O}_{\tilde{T}}(\tilde{D})$ by intersecting divisors. Noting that $\tilde{P} + \tilde{T}$ is linearly equivalent to divisors M_t disjoint from $\tilde{P}$ for $t \neq 0$, we see that $\mathcal{O}_{\tilde{P}}(\tilde{P}) = \mathcal{O}_{\tilde{P}}(-\tilde{T}) = \mathcal{O}_{\tilde{P}}(-\tilde{D})$. We can also see N as the restriction of $E - \mathcal{O}_X(T) \in \operatorname{Pic} \tilde{X}$: $E \cap \tilde{T}$ is identified with $S \cap T \subset T$ so the corresponding restriction to $\tilde{T} \cong T$ is $\mathcal{O}_{\tilde{X}}(\tilde{S} - \tilde{T})|_{\tilde{T}} = \mathcal{O}_{\tilde{X}}(\tilde{P})|_{\tilde{T}} = \mathcal{O}_{\tilde{T}}(\tilde{D})$. Similarly the total transform of $D \subset P$ in $\tilde{P}$ can be written $\tilde{D} + \sum E_k$, so the restriction of $-(\mathcal{O}_X(T) - E)$ to $\tilde{P}$ is $-(\mathcal{O}_X(D)|_{\tilde{P}} - \sum E_k) = \mathcal{O}_{\tilde{P}}(\tilde{D})$. We conclude that

$$N = \mathcal{O}_{\tilde{M}}(\tilde{P})|_{\tilde{M}_0} \cong E - \mathcal{O}_X(T)|_{\tilde{M}_0}. \tag{15}$$

Proposition 2.12. *Diagram (14) is Cartesian for very general $S \in |A \otimes B|$.*

Proof. We vary $S \in |A \otimes B|$. Let $W \subset |A \otimes B|$ denote the open subset of smooth surfaces S for which $S \cap D$ consists of m distinct points and form

$$J = \{(S, q_1, \ldots, q_m) \in W \times D^m : S \cap D = \{q_1, \ldots, q_m\}\} \xrightarrow{\pi_1} W.$$

For each $\mathcal{B} \in \operatorname{Pic} T, \mathcal{A} \in \operatorname{Pic} P$ and $b_1, \ldots, b_m \in \mathbb{Z}$ consider the set

$$\begin{aligned} K &= K(\mathcal{B}, \mathcal{A}, b_1, \ldots, b_m) \\ &= \{(S, q_1, \ldots, q_m) \in J : \mathcal{B}|_D - \mathcal{A}|_D - \sum b_k q_k \in \operatorname{Pic}^0 X \subset \operatorname{Pic}^0 D\}. \end{aligned}$$

Clearly K depends only on the classes of $\mathcal{B}$ and $\mathcal{A}$ modulo $\operatorname{Pic}^0 X$, which we regard as a common subgroup of the groups $\operatorname{Pic} T, \operatorname{Pic} P$ and $\operatorname{Pic} D$. Since $\operatorname{Pic}^0 X = \operatorname{Pic}^0 P$ and $\operatorname{Pic}^0 X = \operatorname{Pic}^0 T$ and the quotients $\operatorname{Pic} T / \operatorname{Pic}^0 X$ and $\operatorname{Pic} P / \operatorname{Pic}^0 X$ are finitely generated, there are countably many distinct such subsets K to consider. Notice also that K is closed: for most choices the divisor $\mathcal{B}|_D - \mathcal{A}|_D - \sum b_k q_k \in \operatorname{Pic} D$ has nonzero degree so that K is empty; otherwise, $\operatorname{Pic}^0 X \subset \operatorname{Pic}^0 D$ is a projective Abelian variety and K is its preimage under the morphism $(S, q_1, \ldots q_m) \mapsto \mathcal{B}|_D - \mathcal{A}|_D - \sum b_k q_k$.

Now suppose that $K = J$. The projection $\pi : J \to W$ is étale of degree $m!$ and the monodromy group acts on the fibers of π_1 as the full symmetric group S_m [1, p. 111], so $(b_i - b_j)(q_i - q_j) \in \operatorname{Pic}^0 X$ for each i and j. For fixed $d_{ij} = b_i - b_j$, the set of tuples $(S, q_1, \ldots, q_m) \in J$ with $d_{ij}(q_i - q_j) \in \operatorname{Pic}^0 X$ is closed because $\operatorname{Pic}^0 X$ is projective. It is also a proper subset because we can vary S to take (q_i, q_j) to any pair of points $p, q \in D$, but $d_{ij}(p - q) \notin \operatorname{Pic}^0 X$ for general $p \neq q \in D$ because the subgroup generated by all $d_{ij}(p - q)$ is the image of the (d_{ij})th-power map $\psi_{d_{ij}} : \operatorname{Pic}^0 D \to \operatorname{Pic}^0 D$, which is surjective in characteristic zero[1]: the image is not contained in $\operatorname{Pic}^0 X$ because $\dim \operatorname{Pic}^0 X = \dim \operatorname{Pic}^0 P < \dim \operatorname{Pic}^0 D$ by Remark 2.5. Thus $\pi(K)$ is a proper closed subset of W if the b_k are not all equal.

Choose S to avoid the countable union of proper closed subsets $K(\mathcal{B}, \mathcal{A}, b_1, \ldots, b_m)$ with nonconstant b_k. Then for $\mathcal{B} \in \operatorname{Pic} T$ and $\mathcal{A} + \sum b_k E_k \in \operatorname{Pic} \tilde{P}$ with the same restriction in $\operatorname{Pic} D$ there is $b \in \mathbb{Z}$ with $b_k = b$ for $1 \leq k \leq m$ and so $\sum b_k q_k = b \sum q_k = b\mathcal{O}_P(S)|_D$ is in the image $\operatorname{Pic} P \to \operatorname{Pic} D$. Theorem 2.8 tells us that there is a unique $\mathcal{L} \in \operatorname{Pic} X$ with $\mathcal{L}|_T = \mathcal{B}$ and $\mathcal{L}|_P = \mathcal{A} + b\mathcal{O}_P(S)$. It follows that $\mathcal{M} = (\mathcal{L} - b\mathcal{O}_X(S)) + bE \in \operatorname{Pic} X \oplus \mathbb{Z}$ satisfies $\mathcal{M}|_T = \mathcal{B}$ and $\mathcal{M}|_{\tilde{P}} = \mathcal{A} + b \sum E_k$; moreover, $\mathcal{M}$ is unique because b is equal to the b_k and $\operatorname{Pic} X \to \operatorname{Pic} T$ is injective. □

Corollary 2.13. *In the setting of Proposition 2.12,* $\operatorname{Pic} \tilde{M}_0 = \operatorname{Pic} X \oplus \mathbb{Z}$*, with the second summand generated by* $N = \mathcal{O}_{\tilde{M}}(\tilde{P})|_{\tilde{M}_0}$ *for very general* S, T, P.

Proof. This follows immediately from Proposition 2.12 and Identification (15). □

3. Properness of components

In Section 2 we showed that the Noether-Lefschetz components are closed in the open set $U \subset |A \otimes B|$ consisting of smooth surfaces S for which the restriction $\operatorname{Pic}^0 X \to \operatorname{Pic}^0 S$ is an isomorphism. To finish the proof of Theorem 1, we need only prove the following:

[1] Over $\mathbb{C}$ it amounts to multiplication by d_{ij} on a product of tori.

Proposition 3.1. *Each Noether-Lefschetz component $\Sigma \subset U$ is proper.*

Proof. Assume that some Noether-Lefschetz component $\Sigma \subset U$ is not proper. Then by Proposition 1.4, $\Sigma = h_U(W)$ for some irreducible component $W \subset W^\varphi \subset \mathbf{Div}_U^\varphi$ with W^φ as in Decomposition (7) and some Hilbert polynomial $\varphi \in \mathbb{Q}[z]$, so that the projection $h_U : W \to U$ is dominant and hence $h_U(W)$ contains an open set V. Choosing surfaces $P \in |A|$, $T \in |B|$, and $S \in U \subseteq |A \otimes B|$ as in Corollary 2.13, we obtain a pencil $\ell \cong \mathbb{P}^1$ in $|A \otimes B|$ passing through $P \cup T$ and S. Then, with the identification of ℓ with $\mathbb{P}^1$, the restriction $W \cap h_U^{-1}(\mathbb{P}^1) \to \mathbb{P}^1$ is dominant, so there is an integral curve $E_0 \subset \overline{W} \subset \mathbf{Hilb}^\varphi_{|A\otimes B|}$ for which the projection $f : E_0 \to \mathbb{P}^1$ is onto. Let $E \to E_0$ be the normalization and $\pi : Z \to E$ the base extension of Family (3), giving the following diagram.

$$\begin{array}{ccc} Z & \longrightarrow & \tilde{M} \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow \\ E & \xrightarrow{f} & \mathbb{P}^1 \end{array}$$

We obtain a flat family $C \subset Z \subseteq \tilde{X} \times f^{-1}(\mathbb{P}^1 \cap V)$ where $C_w \subset Z_w$ is Cartier for each $w \in f^{-1}(\mathbb{P}^1 \cap V)$, giving rise to $\mathcal{L} = \mathcal{O}_Z(C) \in \operatorname{Pic}(\pi^{-1}(f^{-1}(\mathbb{P}^1 \cap V)))$. Note that $f^{-1}(\mathbb{P}^1 \cap V)$ contains no pre-image of $0 \in \mathbb{P}^1$ by our choice of $V \subset U$, since $\tilde{M}_0$ is reducible.

If f is unramified at some point $p \in f^{-1}(0)$, we can apply [15, II, 6.5] twice to extend $\mathcal{L}$ to a neighborhood of $Z_p \cong \tilde{P} \cup_D T \cong \tilde{M}_0$ because Z is smooth near p: Corollary 2.13 now shows that $\mathcal{L}_p = R + a\mathcal{O}_Z(\tilde{P})|_{Z_p}$ with $R \in \operatorname{Pic} X$ and $a \in \mathbb{Z}$. Since the restriction of $\mathcal{L} - R - a\mathcal{O}_Z(\tilde{P})$ to Z_p is trivial, the line bundles $\mathcal{L}_w - R - a\mathcal{O}_Z(\tilde{P})$ lie in the image of $\operatorname{Pic}^0 X$ for w in an open neighborhood of p by Proposition 1.3, but $\mathcal{O}_Z(\tilde{P})$ is supported at p, so $\mathcal{L}_w - R \in \operatorname{Pic}^0 X$ and hence $\mathcal{L}_w \in \operatorname{Pic} X$ for w near p, showing that $\Sigma \subset U$ is a proper closed subset.

If f is ramified at each $p \in f^{-1}(0)$, choose one such p. We will desingularize Z and then extend $\mathcal{L}$ and argue as above. Up to unit the local homomorphism $\mathcal{O}_{\mathbb{P}^1,0} \to \mathcal{O}_{E,p}$ sends t to u^s for some $s > 0$, where t and u generate the respective maximal ideals in $\mathcal{O}_{\mathbb{P}^1,0}$ and $\mathcal{O}_{E,p}$. Since $\tilde{M}$ is locally defined in $\tilde{X} \times \mathbb{A}^1$ by an equation $f_{\tilde{P}} f_T - t f_S = 0$, the base extension Z is locally defined in $\tilde{X} \times E$ by $f_{\tilde{P}} f_T - u^s f_S = 0$. Since $f_{\tilde{P}}, f_T$ and f_S have no common zeroes in $\tilde{X}$, Z is singular where $f_{\tilde{P}} = f_T = u = 0$ and the total family Z has $\mathbf{A}_{s-1}$ singularities along $\tilde{D} = \tilde{P} \cap T$ in the central fiber Z_p.

The $\mathbf{A}_{s-1}$ singularities have a standard resolution $\tilde{Z} \to Z$ [18, 5.1 and 5.3]: One successively blows up curves $D_i \cong \tilde{D}$ to obtain a linear chain of $\mathbb{P}^1$-bundles I_i over $\tilde{D}$, giving the description

$$\tilde{Z}_p = \tilde{P} \cup_{D_0} I_1 \cup_{D_1} \cup \cdots \cup I_{s-1} \cup_{D_{s-1}} T$$

(see [12, p. 38] for a picture). Each I_i is a ruled surface over both D_i and D_{i-1}, which are disjoint curves in I_i. Defining line bundles $N_0 = \mathcal{O}_Z(\tilde{P})|_{\tilde{Z}_p}$ and $N_i = \mathcal{O}_Z(I_i)|_{\tilde{Z}_p}$ for $0 < i < s$, we now claim that

$$\operatorname{Pic} \tilde{Z}_p = \langle \operatorname{Pic} X, N_0, N_1, \ldots, N_{s-1} \rangle. \tag{16}$$

Assuming Claim (16), proceed as in the unramified case. Since $\tilde{Z} \to Z \to E$ is smooth in a neighborhood of $\tilde{Z}_p$, we can extend the line bundle $\mathcal{L}$ to an open neighborhood of $\tilde{Z}_p$. By Claim (16) we can write $\mathcal{L}_p = R + a\tilde{P}|_{\tilde{Z}_p} + \sum a_i I_i|_{\tilde{Z}_p}$ for some $R \in \operatorname{Pic} X$ and $a \in \mathbb{Z}$ and $a_i \in \mathbb{Z}$ so that $\mathcal{L} - R - a\tilde{P} - \sum a_i I_i$ is trivial at p. Therefore $\mathcal{L}_w - R - a\tilde{P} - \sum a_i I_i|_{\tilde{Z}_p} \in \operatorname{Pic}^0 X$ for w near p by Proposition 1.3, hence $\mathcal{L}_w - R \in \operatorname{Pic}^0 X$, since $\tilde{P}$ and I_i are vertical components contained in $\tilde{Z}_p$. Therefore $\mathcal{L}_w \in \operatorname{Pic} X$ for w in an open neighborhood of p and again $\Sigma \subset U$ is proper.

It remains to prove Claim (16). For this, first note that just as in Remark 2.11,

$$\operatorname{Pic} \tilde{Z}_p \cong \operatorname{Pic} \tilde{P} \times_{\operatorname{Pic} D_0} \times \operatorname{Pic} I_1 \times_{\operatorname{Pic} D_1} \times \cdots \times \operatorname{Pic} I_{s-1} \times_{\operatorname{Pic} D_{s-1}} \operatorname{Pic} T.$$

Therefore, $\operatorname{Pic} \tilde{Z}_p$ can be thought of as the set of $(s+1)$-tuples $(\alpha_0, \alpha_1, \ldots, \alpha_s)$, where $\alpha_0 \in \operatorname{Pic} \tilde{P}, \alpha_s \in \operatorname{Pic} T$, and $\alpha_i \in \operatorname{Pic} I_i$ for $1 \le i \le s-1$, with the requirement that $\alpha_{i-1}|D_i = \alpha_i|_{D_i}, 1 \le i \le s$. Each successive blowup has analytic form $XY - u^r Z$ blown up at (X, Z, u), and a local calculation shows that the total transform of (X, Y, u) contains one copy of the exceptional divisor, so each I_i appears just once in the total transform of $Z_p = \tilde{P} \cup T$. Thus we see that the divisor $\tilde{Z}_0 = \pi^* f^*(0)$ is the sum of the (reduced) divisor $\tilde{Z}_p$ and the other components disjoint from $\tilde{Z}_p$. Moreover, $\tilde{Z}_0$ is linearly equivalent to a divisor disjoint from $\tilde{Z}_0$, so in particular $\mathcal{O}_{\tilde{Z}}(\tilde{Z}_p)|_{\tilde{Z}_p}$ is trivial. We can use this to calculate the N_i explicitly as tuples. For example, $\mathcal{O}_Z(\tilde{P})|_{\tilde{P}} = -D_0$: $\tilde{Z}_p$ has trivial restriction to $\tilde{Z}_p$ and hence trivial restriction to $\tilde{P}$, but the restriction of $\tilde{Z}_p - \tilde{P}$ to $\tilde{P}$ is D_0, because I_1 is the only component that intersects $\tilde{P}$: it follows that $N_0|_{\tilde{P}} = -D_0$ and that as a tuple $N_0 = (-D_0, D_0, 0, \ldots, 0)$. Similarly,

$$N_1 = (D_0, -D_0 - D_1, D_1, 0, \ldots, 0), N_2 = (0, D_1, -D_1 - D_2, D_2, 0, \ldots, 0), \text{ and so on.}$$

Now let $\alpha \in \operatorname{Pic} \tilde{Z}_p$, and express it as a tuple $(\alpha_0, \alpha_1, \ldots, \alpha_s)$ as above. Successively add multiples of the divisors

$$\sum_{i=0}^{k} N_i = (0, 0, \ldots, \underbrace{-D_i}_{k}, \underbrace{D_i}_{k+1}, 0, \ldots, 0)$$

to reduce to the case that each α_i with $0 < i < s$ is the pullback of a divisor on D_{i-1} (or, equivalently, D_i) under the ruled surface projection from I_i. Agreement on the respective D_{i-1} and D_i gives

$$\alpha_0|_{D_0} = \alpha_1|_{D_0} = \alpha_1|_{D_1} = \cdots = \alpha_s|_{D_{s-1}},$$

so via the identification $D_0 \cong D_{s-1}$, Proposition 2.12 and Corollary 2.13 give $L \in \operatorname{Pic} X$ and $a \in \mathbb{Z}$ with $\alpha_0 = L + aD_0$ and $\alpha_s = L - aD_{s-1}$. Now add

$$a(sN_0 + (s-1)N_1 + \cdots + N_{s-1}) = a(-D_0, D_0 - D_1, D_1 - D_2, \ldots, D_{s-2} - D_{s-1}, D_{s-1})$$

to reduce to the case in which α is of the form $(L, \beta_1, \ldots, \beta_{s-1}, L)$ where each β_i is a pullback of a divisor on D_{i-1} to I_i. Since $\operatorname{Pic} X \to \operatorname{Pic} D_i$ is injective by Proposition 2.7, we have $\beta_i|_{D_i} = L|_{D_i}$ for each i, so α is the restriction of $L \in \operatorname{Pic} X$, and we have shown that every divisor $\alpha \in \operatorname{Pic} \tilde{Z}_p$ is the restriction of a divisor in $\operatorname{Pic} X$ modulo the restrictions of $N_0, N_1, \ldots, N_{s-1}$. $\square$

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