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HILBERT POLYNOMIALS OVER ARTINIAN RINGS

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ABSTRACT. This paper characterizes Hilbert functions and Hilbert polynomials of standard algebras over an Artinian ring R_0.

Introduction

Let R_0 be an Artinian ring. A standard algebra over R_0 is a graded ring S, finitely generated as $R_0 = S_0$-algebra by elements of degree 1. That is, $S = R/I$, where R is a polynomial ring with coefficients in R_0 and I is a homogeneous ideal. The Hilbert function of S, denoted by H_S, is given by $H_S(n) = \lambda_{R_0}(S_n)$, where λ stands for length. For $n \gg 0$ it holds $H_S(n) = P_S(n)$ where P_S is a polynomial, the Hilbert polynomial of S.

The purpose of this paper is to describe the possible Hilbert functions and Hilbert polynomials of such standard algebras. In the case of a field, these questions were initially addressed in Macaulay's pioneering work [9]. His results were strengthened and extended by Sperner [11], Hartshorne [8], Gotzmann [6], and Stanley [12]. More recently, Green's remarkable paper [7] has stimulated new interest in the subject. A number of papers generalizing these results to settings other than standard k-algebras have appeared over the last few years.

The present paper completes work begun in [1], where Hilbert functions and polynomials are characterized over Artinian local rings R_0 which contain a field. The proofs of necessity there use hyperplane section arguments. These are not easy to find without a base field, so we use a different method here: the quotients associated to a composition series for R_0 allow us to reduce the questions to the case of a field. This method also gives analogs to Gotzmann's regularity and persistence theorems (see [6], [7]).

The paper is divided into two sections. The first section describes the Hilbert polynomials over an Artinian ring and includes an analog to Gotzmann's regularity theorem. The second section characterizes the Hilbert functions and gives a generalization of Gotzmann's persistence theorem.

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1. Hilbert polynomials

The question of which polynomials occur as Hilbert polynomials for a proper subscheme of $\mathbb{P}^r_k$ over a field k has been studied since Macaulay ([9]; see also [1], [8], [11]). The answer can be stated as follows.

PROPOSITION 1.1. *Fix an integer $r > 0$ and let $p(z) \in \mathbb{Q}[z]$. Let k be a field. Then the following conditions are equivalent.*

(1) *$p(z)$ is the Hilbert polynomial of a proper subscheme $X \subset \mathbb{P}^r_k$.*
(2) *There exist integers $m_0 \geq m_1 \geq \cdots \geq m_{r-1} \geq 0$ such that*

$$p(z) = \sum_{t=0}^{r-1} \left[\binom{z+t}{t+1} - \binom{z+t-m_t}{t+1} \right].$$

(3) *There exist integers $r > c_1 \geq c_2 \geq \cdots \geq c_s \geq 0$ such that*

$$p(z) = \sum_{i=1}^{s} \binom{z + c_i - (i-1)}{c_i}.$$

(4) *There exist integers $0 \leq q \leq r-1$ and $1 \leq a_0 \leq a_1 \leq \cdots \leq a_q$ such that*

$$p(z) = \binom{z+r}{r} - \sum_{t=0}^{q} \binom{z - a_t + r - t}{r-t}.$$

Proof. Conditions (1) and (2) are equivalent by [8], Corollary 5.7. Conditions (1) and (3) are equivalent by [1], Theorem 4.5, where this was more generally proved for subschemes over an Artinian local ring containing a field. The equivalence of (1) and (4) occurs due to Macaulay's characterization of the Hilbert polynomials for homogeneous ideals, however Green interpreted condition (4) as condition (3) in [7].

Remark 1.2. In the proposition above, let $d = \dim X$. Then the expressions for the Hilbert polynomial $p(z)$ are related by the following formulas:

(a) Set $m_r = 0$. Then $d = \max\{i\colon m_i > 0\}$ and for $0 \leq i < r$ we have

$$m_i - m_{i+1} = \#\{j\colon c_j = i\}.$$

(b) We have $q = r - \min\{i\colon m_i < m_0\}$, $a_q = m_{r-q-1}$ and for $0 \leq i < q$,

$$a_i = m_{r-i-1} + 1.$$

The equivalent notions of Proposition 1.1 have some importance in the study of homogeneous ideals and projective varieties. Motivated by Gotzmann's results, we give these conditions the following name.

Definition 1.3. We say that $p(z) \in \mathbb{Q}[z]$ *admits a Gotzmann development* if $p(z)$ satisfies any of the equivalent conditions of proposition 1.1 for some integer r. In this case, the *Gotzmann development* for $p(z)$ is the expression given in condition (3).

LEMMA 1.4. *Let $p(z), q(z) \in \mathbb{Q}[z]$ be polynomials which admit a Gotzmann development. Then*:

(*a*) *The polynomial $r(z) = p(z) + q(z)$ admits a Gotzmann development.*
(*b*) *Assume that the Gotzmann developments for $p(z)$, $q(z)$ and $r(z)$ are*

$$p(z) = \textstyle\sum_{i=1}^{s} \binom{z+a_i-(i-1)}{a_i},\ q(z) = \sum_{i=1}^{t} \binom{z+b_i-(i-1)}{b_i},\ r(z) = \sum_{i=1}^{u} \binom{z+c_i-(i-1)}{c_i}.$$

Let $s_i = \#\{j\colon a_j \geq i-1\}$, $t_i = \#\{j\colon b_j \geq i-1\}$ and $u_i = \#\{j\colon c_j \geq i-1\}$. Then for each $i \geq 1$ we have $u_i \geq s_i + t_i$.

Proof. Let $p(z)$ and $q(z)$ be polynomials admitting a Gotzmann development. By Proposition 1.1 above, $p(z)$ (resp. $q(z)$) is the Hilbert polynomial of a subscheme $X \subset \mathbb{P}_k^n$ (resp. $Y \subset \mathbb{P}_k^m$) over some field k. Embedding $\mathbb{P}_k^n$ and $\mathbb{P}_k^m$ as disjoint linear subspaces of a common projective space $\mathbb{P}_k^N$, the union of the images of X and Y yield a closed subscheme with Hilbert polynomial $r(z) = p(z) + q(z)$, which proves statement (a) via Proposition 1.1.

Now we prove the statement about the Gotzmann development for $r(z) = p(z) + q(z)$. We proceed by induction on the degree of $r(z)$. The result is trivial when $\deg r(z) = 0$ (all three polynomials are constant positive integers), so assume $\deg r(z) = d > 0$. Notice that the Gotzmann coefficients for $\Delta r(z) = r(z) - r(z-1)$ are $c_1 - 1, \ldots, c_{u_2} - 1$. Since $\Delta r = \Delta p + \Delta q$, the induction hypothesis shows that $u_i \geq s_i + t_i$ for all $i \geq 2$. Now consider

$$p'(z) = \sum_{i=1}^{s_2} \binom{z + a_i - (i-1)}{a_i} \quad \text{and} \quad q'(z) = \sum_{i=1}^{t_2} \binom{z + b_i - (i-1)}{b_i}.$$

By part (a), $p' + q'$ has a Gotzmann development. Since $r = p' + q' + (s_1 - s_2) + (t_1 - t_2)$, the uniqueness of Gotzmann developments shows that $u_1 - u_2 \geq (s_1 - s_2) + (t_1 - t_2)$. Therefore $u_1 - s_1 - t_1 \geq u_2 - s_2 - t_2 \geq 0$, as required.

Remark 1.5. The same argument gives the stronger inequalities

$$\#\{j\colon c_j = i-1\} \geq \#\{j\colon a_j = i-1\} + \#\{j\colon b_j = i-1\} \quad \text{for all } i \geq 1.$$

In what follows, $\lambda(R_0)$ will denote the length of an Artinian ring R_0.

LEMMA 1.6. *Let R_0 be an Artinian ring of length $\lambda = \lambda(R_0)$. Let $R = R_0[x_0, x_1, \ldots, x_r]$, $I \subset R$ a homogeneous ideal, and $S = R/I$. Then there exist λ surjections of graded R-algebras*

$$S = S_0 \xrightarrow{\psi_0} S_1 \xrightarrow{\psi_1} \cdots \xrightarrow{\psi_{\lambda-1}} S_\lambda = 0$$

such that the kernels $T_i = \ker \psi_i$ are principal, generated in degree 0, and annihilated by a maximal ideal. In particular, $T_i \cong k_i[x_0, x_1, \dots, x_r]/J_i$ for some residue field k_i of R_0, and some homogeneous ideal $J_i \subset k_i[x_0, \dots, x_r]$.

Proof. Let $(0) = N_0 \subset N_1 \subset \cdots \subset N_\lambda = R_0$ be a composition series for R_0. There are exact sequences

$$0 \to N_{i+1}/N_i \to R_0/N_i \to R_0/N_{i+1} \to 0$$

where $N_{i+1}/N_i \cong k_i$. Tensoring these sequences with S gives the sequence of R-algebras $S_i = S \otimes_{R_0} R_0/N_i$ along with surjections $\psi_i \colon S_i \to S_{i+1}$ whose kernels are images of $k_i \otimes_{R_0} S$, generated in degree 0.

Remark 1.7. (a) In the construction above, let $R_i = R_0/N_i$. Then there are ideals I_i such that $S_i \cong R_i[x_0, x_1, \dots, x_r]/I_i$. The snake lemma shows that there are short exact sequences of graded R-modules

$$0 \to J_i \to I_i \to I_{i+1} \to 0.$$

(b) Let us make explicit the first surjection of Lemma 1.6. There is $a \in R_0$ such that $(0 : a)$ is a maximal ideal m_0 and $N_1 = (a)$. Setting $\overline{R} = (R_0/(a))[x_0, \dots, x_r]$ and $\overline{I} = (I + (a))/(a)$, we get $S_1 = \overline{R}/\overline{I}$ and there is an exact sequence

$$0 \to R/(I : a) \xrightarrow{a} R/I \to \overline{R}/\overline{I} \to 0.$$

Since $\mathrm{m}_0 R \subset (I : a)$, we see that $J_0 = (I : a)/\mathrm{m}_0 R$.

THEOREM 1.8. *Let R_0 be an Artinian ring and $p(z) \in \mathbb{Q}[z]$. Then the following statements are equivalent.*

- (*a*) *There is a closed subscheme $X \subset \mathbb{P}^r_{R_0}$ such that $p(z) = p_X(z)$ is the Hilbert polynomial for X.*
- (*b*) *We may write $p(z) = q\binom{z+r}{r} + r(z)$, where $0 \le q \le \lambda(R_0)$ is an integer and $r(z) \in \mathbb{Q}[z]$ is a polynomial of degree $< r$ admitting a Gotzmann development such that if $q = \lambda(R_0)$ then $r(z) = 0$.*

Proof. First suppose that $p(z) = p_X(z)$ is the Hilbert polynomial for $X \subset \mathbb{P}^r_{R_0}$, and hence is the Hilbert polynomial for a graded ring $S = R_0[x_0, x_1, \dots, x_r]/I$, where I is a homogeneous ideal. By Lemma 1.6, we obtain successive quotients S_i with kernels T_i. Let $p_i(z)$ denote the Hilbert polynomial of T_i. For each $1 \le i \le \lambda(R_0)$, note that either $p_i(z) = \binom{z+r}{r}$ or $\deg p_i(z) < r$ and $p_i(z)$ admits a Gotzmann development. Letting q be the number of i such that $p_i(z) = \binom{z+r}{r}$, it is clear from Lemma 1.4 and the fact that $p(z) = \sum_i p_i(z)$ that $p(z)$ may be written in the form above.

Conversely, if $p(z)$ can be written in the above form, then $p(z)$ satisfies the sufficiency conditions of [1], Theorem 4.5. The constructive part of the proof (see [1], proof of Theorem 2.9) makes no use of the local equicharacteristic hypothesis (it only uses a filtration of R_0), hence there exists an ideal I such that $S = R/I$ has Hilbert polynomial $p(z)$ and we may take $X = \text{Proj}(S)$.

We shall now state and prove the promised analogue of Gotzmann's regularity theorem.

THEOREM 1.9. *Let R_0 be an Artinian ring, $X \subset \mathbb{P}^r_{R_0}$ a closed subscheme and $p(z) = p_X(z)$ the Hilbert polynomial of X. Write*

$$p(z) = q\binom{z+r}{r} + r(z) \quad \textit{with} \quad r(z) = \sum_{i=1}^{s} \binom{z + a_i - (i-1)}{a_i}$$

as in Theorem 1.8 (set $s = 0$ if $r(z) = 0$). Let $s_t = \#\{j\colon a_j \geq t-1\}$. Then

$$H^t(\mathcal{I}_X(n-t)) = 0 \textit{ for } t > 0 \textit{ and } n \geq s_t.$$

In particular, the ideal sheaf $\mathcal{I}_X$ is s-regular.

Proof. Let $I = H^0_*(\mathcal{I}_X) \subset R = R_0[x_0, x_1, \ldots, x_r]$ be the homogeneous ideal for X and let $S = R/I$ be the homogeneous coordinate ring for X. Recalling the construction from Lemma 1.6, we have exact sequences

$$0 \to T_i \to S_i \to S_{i+1} \to 0$$

where $T_i \cong k_i[x_0, x_1, \ldots, x_r]/J_i$ for homogeneous ideals J_i. From Remark 1.7 (a), we have exact sequences of ideals

$$0 \to J_i \to I_i \to I_{i+1} \to 0$$

where $I_i \subset R_i[x_0, x_1, \ldots, x_r]$ and $I_{\lambda(R_0)+1} = 0$. Note that $\tilde{J}_i \subset \mathcal{O}_{\mathbb{P}^r_k}$ is the ideal sheaf for $\text{Proj}(T_i) \subset \mathbb{P}^r_k$. Let $p_i(z)$ be the Hilbert polynomial for T_i.

Now we note some vanishings of higher cohomology. If $p_i(z) = \binom{z+r}{r}$, then $\tilde{J}_i = 0$ and hence $H^t_*(\tilde{J}_i) = 0$ for all $t > 0$. If $p_i(z) = 0$, then $\tilde{J}_i \cong \mathcal{O}_{\mathbb{P}^r_{k_i}}$ and hence all the intermediate cohomology vanishes and $H^r(\tilde{J}_i(n-r)) = 0$ for $n \geq 0$. In particular, $\tilde{J}_i$ is 0-regular. Finally, if $0 \neq p_i(z)$ and $\deg p_i < r$, let $p_i(z) = \sum_{j=1}^{s^i} \binom{z+a^i_j-(j-1)}{a^i_j}$ be the Gotzmann development for p_i. If we define $s^i_t = \#\{j\colon a^i_j \geq t-1\}$, then by Green's interpretation of Gotzmann's vanishing theorem [7], we have $H^t(\tilde{J}_i(n-t)) = 0$ for $n \geq s^i_t$.

In considering the long exact cohomology sequence associated to the sequences

$$0 \to \tilde{J}_i \to \tilde{I}_i \to \tilde{I}_{i+1} \to 0$$

and the vanishings above, we conclude that $H^t(\tilde{I}_0(n-t)) = 0$ for all $t > 0$ and $n \geq \max_i\{s_t^i\}$, where this maximum is taken over i such that $\deg p_i(z) < r$. On the other hand, $r(z)$ is the sum of such $p_i(z)$, so by repeated application of Lemma 1.4, we conclude that

$$\max_i\{s_t^i\} \leq \sum_i s_t^i \leq s_t$$

and hence $H^t(\tilde{I}_0(n-t)) = 0$ for all $t > 0$ and $n \geq s_t$. Noting that $\tilde{I}_0 = \mathcal{I}_X$, we conclude the proof.

Remark 1.10. The same kind of proof can be carried out using long exact sequences of local cohomology to prove a similar result over a polynomial ring (see [1], Theorem 3.3).

Remark 1.11. The proof actually gives the stronger regularity bound $\sum_i s_t^i$, where s_t^i are defined by the filtration of S_X induced by lemma 1.6. For general subschemes $X \subset \mathbb{P}^r_{R_0}$, this bound is much stronger than the bound given in the statement of 1.9, because the Gotzmann development of a sum of polynomials generally has many more terms than the sum of the Gotzmann developments of the polynomials (see proof of Lemma 1.4).

For example, consider $R_0 = \mathbf{Z}/p^2\mathbf{Z}$ with residue field $k = \mathbf{Z}/p\mathbf{Z}$, where $p \in \mathbf{Z}$ is prime. Let $I_1 = (x_0, x_1)(x_2, x_3) \subset (x_0, x_1) = I_0 \subset R_0[x_0, x_1, x_2, x_3]$ (over a field, these are the ideals of a pair of skew lines and of one of the lines, respectively) and consider the ideal $I = (I_1, pI_0)$. This defines a scheme X, which is the disjoint union of a line and a double line. Using the standard composition series $(0) \subset (p) \subset R_0$, we see that J_0 is the image of I_0 in $k[x_0, x_1, x_2, x_3]$ under the natural surjection, while J_1 is the image of I_1. The Gotzmann development for X has 6 terms, so the theorem says that the ideal sheaf is 6-regular. However, Gotzmann regularity for the individual ideal sheaves suggests that $\mathcal{I}_X$ is only 3-regular. In fact, the actual ideal sheaf of two skew lines is 2-regular, so this is also true of $\mathcal{I}_X$.

2. Hilbert functions

In this section, we extend Macaulay's criterion for Hilbert functions to arbitrary Artinian rings. As in the previous section, the key is to reduce to the case when R_0 is a field by using Lemma 1.6.

We first recall Macaulay's criterion (see [9], [12]), for which we must define certain binomial transformations. For any integers $h, n \geq 1$, there exist unique integers $k_n > k_{n-1} > \cdots > k_\delta \geq \delta \geq 1$ such that

$$h = \binom{k_n}{n} + \binom{k_{n-1}}{n-1} + \cdots + \binom{k_\delta}{\delta}.$$

This gives the *n-binomial expansion of h*. Since this expression is unique, we may define

$$(h_n)_+^+ = \binom{k_n+1}{n+1} + \binom{k_{n-1}+1}{n} + \cdots + \binom{k_\delta+1}{\delta+1}.$$

By convention, $(0_n)_+^+ = 0$ for all $n \geq 0$.

With this definition, Macaulay proved that a function $H\colon \mathbf{N} \to \mathbf{N}$ is the Hilbert function of a standard k-algebra if and only if $H(0) = 1$ and $H(n+1) \leq (H(n)_n)_+^+$ for all $n \geq 1$.

PROPOSITION 2.1 (ELIAS). *Let $a, b, r, n \geq 0$ be integers such that $a, b < \binom{n+r}{r}$. Then the following inequalities hold.*

(*a*) *If $a + b < \binom{n+r}{r}$, then*

$$(a_n)_+^+ + (b_n)_+^+ \leq ((a+b)_n)_+^+.$$

(*b*) *If $a + b \geq \binom{n+r}{r}$, then*

$$(a_n)_+^+ + (b_n)_+^+ < \binom{n+r+1}{r} + \left(\left(a+b-\binom{n+r}{r}\right)_n\right)_+^+.$$

Proof. Part (a) is [4], Corollary 2.7 (iii) with the choices $t_1 = t_2 = n, s = n+1$, and $h = r+1$. For (b), in [4], Corollary 2.7 (ii) with the same choices, Elias writes in the proof that

$$\begin{aligned}
(a_n)_+^+ + (b_n)_+^+ &= \sum_{i=0}^{r} (a_{<n>(i)} + b_{<n>(i)}) \\
&\leq \sum_{i=0}^{r} \left(\binom{n+r}{r}_{<n>(i)} + \left(a+b-\binom{n+r}{r}\right)_{<n>(i)} \right) \\
&= \left(\binom{n+r}{r}_n\right)_+^+ + \left(\left(a+b-\binom{n+r}{r}\right)_n\right)_+^+.
\end{aligned}$$

Note that the inequality is strict in the $i = r$ term of the summation: since a, b and $(a+b-\binom{n+r}{r})$ are strictly less than $\binom{n+r}{r}$, we have $a_{<n>(r)} = b_{<n>(r)} = (a+b-\binom{n+r}{r})_{<n>(r)} = 0$, while $\binom{n+r}{r}_{<n>(r)} = 1$.

PROPOSITION 2.2. *Let $H_1, \ldots, H_t\colon \mathbf{N} \to \mathbf{N}$ be functions and $H = \sum_{i=1}^t H_i$. Consider the functions q_i, r_i and q, r defined for all $n \geq 0$ by the Euclidean divisions*

$$H_i(n) = q_i(n)\binom{n+r}{r} + r_i(n) \quad \textit{and} \quad H(n) = q(n)\binom{n+r}{r} + r(n).$$

Assume that, for all $n \geq 0$, $H_1, \ldots, H_t$ satisfy the condition

$$(*)_i \quad H_i(n+1) \leq q_i(n)\binom{n+1+r}{r} + (r_i(n)_n)_+^+.$$

Then H also satisfies $(*)$: $H(n+1) \leq q(n)\binom{n+1+r}{r} + (r(n)_n)_+^+$ for all $n \geq 0$. Moreover, if $(*)_i$ are equalities for all $n \geq d$ and $H(d+1) = q(d)\binom{d+1+r}{r} + (r(d)_d)_+^+$, then $(*)$ is an equality for all $n \geq d$.

Proof. By induction, it is enough to show that H verifies $(*)$ in the case $t = 2$. Let $n \geq 0$. By $(*)_1$ and $(*)_2$ we have

$$H(n+1) \leq \binom{n+1+r}{r}(q_1(n) + q_2(n)) + (r_1(n)_n)_+^+ + (r_2(n)_n)_+^+.$$

Now we consider two cases. If $r_1(n) + r_2(n) < \binom{n+r}{r}$, then $r(n) = r_1(n) + r_2(n)$, $q(n) = q_1(n) + q_2(n)$ and condition $(*)$ is immediate from Proposition 2.1 (a). On the other hand, if $\binom{n+r}{r} \leq r_1(n) + r_2(n) < 2\binom{n+r}{r}$, then $q(n) = q_1(n) + q_2(n) + 1$, $r(n) = r_1(n) + r_2(n) - \binom{n+r}{r}$ and $(*)$ follows from Proposition 2.1 (b).

To prove the second part, let $H' = \sum_{i=2}^{t} H_i$ and define q' and r' as usual. We have just seen that

$$\begin{aligned} H(d+1) &= H_1(d+1) + H'(d+1) \\ &\leq \binom{d+1+r}{r}q_1(d) + (r_1(d)_d)_+^+ + \binom{d+1+r}{r}q'(d) + (r'(d)_d)_+^+ \\ &\leq \binom{d+1+r}{r}q(d) + (r(d)_d)_+^+ = H(d+1). \end{aligned}$$

Then $H'(d+1) = \binom{d+1+r}{r}q'(d) + (r'(d)_d)_+^+$. By induction hypothesis $H'(n+1) = \binom{n+1+r}{r}q'(n) + (r'(n)_n)_+^+$ for all $n \geq d$, and hence it will be enough again to prove the case $t = 2$.

For $t = 2$, notice that the strict inequality in Proposition 2.1 (b) assures that the case $r_1(d) + r_2(d) \geq \binom{d+r}{r}$ cannot occur; thus $q_1(n) + q_2(n) = q(n)$ for all $n \geq d$, and these values remain constant. Replacing H_1, H_2 and H by r_1, r_2 and r respectively, we may assume that H_1 and H_2 satisfy the conditions of the classical Macaulay's theorem: by [2], Theorem 4.2.10, there exist homogeneous ideals $I \subseteq R = k[x_0, \ldots, x_r]$ and $J \subseteq S = k[y_0, \ldots, y_r]$ (where k is any field) such that $H_1 = H_{R/I}$ and $H_2 = H_{S/J}$.

Let $T = k[x_0, \ldots x_r, y_0, \ldots y_r]$ and $Q = (x_0, \ldots, x_r) \cdot (y_0, \ldots, y_r)$. One can easily show that $(IT + (y_0, \ldots, y_r)) \cap (JT + (x_0, \ldots, x_r)) = IT + JT + Q$ and $(IT + (y_0, \ldots, y_r)) + (JT + (x_0, \ldots, x_r)) = (x_0, \ldots, x_r, y_0, \ldots, y_r)$. Hence, letting

$K = IT + JT + Q$ we have an exact sequence of graded k-algebras

$$0 \to T/K \to R/I \oplus S/J \to k \to 0$$

which gives $H_{T/K}(n) = H_1(n) + H_2(n) = H(n)$ for all $n \geq 1$.

First assume $d = 1$. Then a straightforward computation shows that $((H_1(1) + H_2(1))_1)^+_+ = (H_1(1)_1)^+_+ + (H_2(1)_1)^+_+$ holds only when $H_1 = 0$ or $H_2 = 0$, in which case the result is obvious. Thus we may assume that $d \geq 2$. From the equalities $H_i(n+1) = (H_i(n)_n)^+_+$ for $n \geq d$ and [5], Corollary 2.6 (b), we see that I and J (and hence also K) are generated in degrees $\leq d$. By Gotzmann's persistence theorem (see [7]), we get $H(n+1) = (H(n)_n)^+_+$ for all $n \geq d$, as required.

Notice that the construction in the above proof is an algebraic version of the proof of Lemma 1.4(a).

THEOREM 2.3. *Let R_0 be an Artinian ring, $R = R_0[x_0, \ldots, x_r]$, and $H\colon \mathbf{N} \to \mathbf{N}$ be a function. Define the functions q and r by the Euclidean division $H(n) = \binom{n+r}{r}q(n) + r(n)$. Then $H = H_{R/I}$ for a homogeneous ideal $I \subset R$ if and only if*

(*a*) $H(0) \leq \lambda(R_0)$ *and*
(*b*) $H(n+1) \leq \binom{n+1+r}{r}q(n) + (r(n)_n)^+_+$ *for all* $n \geq 0$.

Proof. Suppose that $H = H_{R/I}$. When R_0 is a field, condition (b) follows straightforwardly from the classical Macaulay's theorem. Thus we may assume $t = \lambda(R_0) \geq 2$. From Lemma 1.6 we obtain for $1 \leq i \leq t$, graded k_i-algebras $k_i[x_0, \ldots, x_r]/J_i$ with respective Hilbert functions $H_1, \ldots, H_t$, such that $H_{R/I} = \sum_{i=1}^{t} H_i$. Then (b) follows from Proposition 2.2.

Conversely, if the function H satisfies conditions (a) and (b), then we may use the construction in [1], Theorem 2.9, of an ideal I such that $H = H_{R/I}$, since that construction does not use the fact that R_0 is local equicharacteristic.

Notice that as a corollary of this theorem we obtain the straightforward translation of the usual Macaulay's theorem to the Artinian coefficient case. See [1], Corollary 2.11; the same proof works here.

We now give the generalization of Gotzmann's persistence theorem.

THEOREM 2.4. *Let R_0 be an Artinian ring, $R = R_0[x_0, x_1, \ldots, x_r]$ and $I \subset R$ a homogeneous ideal generated in degrees $\leq d$. With the notations of theorem 2.3, assume that $H_{R/I}(n+1) = \binom{n+1+r}{r}q(n) + (r(n)_n)^+_+$ for $n = d$. Then the same holds for all $n \geq d$. Equivalently, if $r(d)$ has d-binomial expansion $\sum_{i=1}^{s}\binom{d+c_i-(i-1)}{d-(i-1)}$, then*

$$H_{R/I}(n) = q(d)\binom{n+r}{r} + \sum_{i=1}^{s}\binom{n+c_i-(i-1)}{c_i}$$

for all $n \geq d$.

Proof. We may assume that $R_0 = (R_0, \mathrm{m}, k)$ is local, because the Hilbert function of R/I is the sum of the Hilbert functions of its localizations at the maximal ideals of R_0, and we apply Proposition 2.2.

We may also assume that I is generated in degree exactly d. We will show by induction on $\lambda(R_0)$ that

(a) $H(n+1) = q(n)\binom{n+1+r}{r} + (r(n)_n)_+^+$ for all $n \geq d$ and
(b) $\mathrm{Tor}_1^R(I, k) = 0$ in degrees $> d$.

When $R_0 = k$ is a field, part (a) is a direct consequence of Gotzmann's persistence theorem as it appears in [7]; since I is generated in degree d, either $I = 0$ or $q(d) = 0$. Since this gives the Hilbert polynomial, the ideal sheaf is s-regular by Gotzmann's regularity theorem. Moreover, since the Hilbert function and polynomial coincide for $n \geq d \geq s$, we conclude that the R-module I is d-regular. Applying [3], Theorem 1.2, we get (b).

For the general case let us first notice the following fact:

Claim. Let $\mathfrak{a} \subset R_0$ an ideal, $\overline{R_0} = R_0/\mathfrak{a}$ and $\overline{R} = \overline{R_0}[x_0, \ldots, x_r]$. Let $\overline{M}$ be a graded $\overline{R}$-module generated in degrees $\leq d$. Then there is a surjection of graded R-modules

$$\mathrm{Tor}_1^R(\overline{M}, k) \to \mathrm{Tor}_1^{\overline{R}}(\overline{M}, k)$$

which is an isomorphism in degrees $> d$.

For the induction step, consider the exact sequence

$$0 \to k[x_0, \ldots, x_r]/J \xrightarrow{a} R/I \to \overline{R}/\overline{I} \to 0$$

from Remark 1.7(b). Note that $\overline{R}$ is a polynomial ring over $\overline{R}_0$, which has length $\lambda(R_0) - 1$. Let $\overline{H}$ and H' denote the Hilbert functions of $\overline{R}/\overline{I}$ and $k[x_0, \ldots, x_r]/J$. As in the proof of Proposition 2.2, Macaulay's bound for $\overline{H}$ and H' is an equality when $n = d$. By induction and Proposition 2.2, part (a) will follow once we know that $\overline{I}$ and J are generated in degree d. This being obvious for $\overline{I}$, we prove it for J.

Consider the exact sequence of graded R-modules

$$\mathrm{Tor}_1^R(\overline{I}, k) \to J \otimes_R k \to I \otimes_R k \to \overline{I} \otimes_R k \to 0$$

derived from the exact sequence of Remark 1.7 (a). Induction hypothesis and the claim about the Tor modules show that $\mathrm{Tor}_1^R(\overline{I}, k) = 0$ in degrees $> d$. Since I is generated in degree d, $I \otimes_R k = 0$ in degrees $> d$, and hence J is generated in degrees $\leq d$; since $J \hookrightarrow I$, it is generated in degree exactly d.

To prove (b), consider the exact sequence of graded R-modules

$$\mathrm{Tor}_1^R(J, k) \to \mathrm{Tor}_1^R(I, k) \to \mathrm{Tor}_1^R(\overline{I}, k).$$

By induction, both $\overline{I}$ and J satisfy (b); since they are generated in degree d, we are done by the claim .

To prove the claim, notice that the property of being a minimal system of generators does not depend on whether we consider $\overline{M}$ as an R or $\overline{R}$-module. It follows that a given $\overline{R}$-minimal free surjection $\overline{F}_0 \to \overline{M}$ lifts to an R-minimal free surjection $F_0 \to \overline{M}$ such that $F_0/\mathfrak{a}F_0 = \overline{F}_0$. We get a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \to & K & \to & F_0 & \to & \overline{M} & \to & 0 \\ & & \downarrow & & \downarrow & & =\downarrow & & \\ 0 & \to & \overline{K} & \to & \overline{F}_0 & \to & \overline{M} & \to & 0. \end{array}$$

The snake lemma gives an exact sequence of graded R-modules

$$0 \to \mathfrak{a}F_0 \to K \to \overline{K} \to 0.$$

By minimality of the surjections, one has $\operatorname{Tor}_1^R(\overline{M}, k) = K \otimes_R k$ and $\operatorname{Tor}_1^{\overline{R}}(\overline{M}, k) = \overline{K} \otimes_{\overline{R}} k \cong \overline{K} \otimes_R k$. Tensoring the exact sequence with k gives an exact sequence

$$(\mathfrak{a}F_0) \otimes_R k \to \operatorname{Tor}_1^R(\overline{M}, k) \to \operatorname{Tor}_1^{\overline{R}}(\overline{M}, k) \to 0$$

which proves the claim because $\mathfrak{a}F_0$ is generated in degrees $\leq d$.

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