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Bounds on c_3 for threefolds

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Abstract. We bound the third Chern number of a minimal smooth threefold with ample canonical bundle by quadratic functions in the first two Chern numbers.

One of the goals in algebraic geometry is the classification of algebraic varieties. This is usually done by first determining which discrete invariants occur (geography) and then describing the continuous families with fixed invariants (diffeomorphism classes). For example, if we use the geometric genus as discrete invariant for smooth curves, it is well known that the moduli of curves M_g of fixed genus is irreducible. Things are more complicated in higher dimension.

While the geography (Chern numbers) for minimal surfaces is fairly well understood [13, 8], Catanese has shown that the diffeomorphism classes for minimal surfaces with fixed Chern numbers generally have many irreducible components [2] (see also [4]). In his paper [8], Hunt initiates the study of geography for threefolds. He points out that some restriction on the threefolds is necessary to have a good geography (see §2.6 and 3.1 of [8] for a discussion): he considers threefolds which are minimal or have ample canonical bundle.

In the present paper, we are interested in Chern numbers $[c_1^3, c_1c_2, c_3]$ which arise from smooth threefolds with ample canonical bundle. One often studies instead the Chern *ratios* $(\frac{{c_1}^3}{c_1c_2}, \frac{c_3}{c_1c_2}) \in \mathbb{A}^2(\mathbb{Q})$. Hunt [8] and Liu [12] have constructed threefolds whose Chern ratios fill in various triangular shaped regions. For threefolds in $\mathbb{P}^5$, the first author has determined the limit points of the Chern ratios [5] as well as explicitly describing the region corresponding to determinantal threefolds [3]. Further, she has described the

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region corresponding to general threefold complete intersections. Beyond this, little is known about existence.

As to restrictions on the Chern numbers, there are the easy inequalities $c_1^3 < 0$ and $c_1c_2 < 0$. Yau proved the much more difficult inequality $\frac{c_1^3}{c_1c_2} \leq \frac{8}{3}$. In the present note we show that there are quadratic polynomials Q_1, Q_2 in two variables such that $Q_1(c_1^3, c_1c_2) \leq c_3 \leq Q_2(c_1^3, c_1c_2)$ (theorem 7). Unfortunately, the corresponding region of Chern ratios is not bounded. There is a linear upper bound on c_3 due to Van de Ven in the case where the canonical bundle is very ample (remark 8).

The method of proof is to write $\chi(\mathcal{T}_X)$ and $\chi(\Omega_X)$ as linear functions of the three Chern numbers and then find upper bounds on the odd cohomology of $\mathcal{T}_X$ and Ω_X in terms of the first two Chern numbers. The upper bounds on the first cohomology group dimensions are obtained by first noting that there is an effective bound on which multiple r of K_X is spanned and gives a birational map to projective space (one can take $r = 6$ by a recent refinement of Lee [11] on earlier work of Lazarsfeld and Ein [7]) and then using a Castelnuovo–Mumford regularity argument on the tangent and cotangent bundles of a smooth surface section S of rK_X. Our bounds hold more generally for smooth minimal projective threefolds whose intermediate cohomologies $H^i(\Omega_X \otimes mK_X)$ and $H^i(\mathcal{T}_X \otimes mK_X)$ vanish for $0 < i < 3$ and $m << 0$ (see remark 6).

Throughout this paper X denotes a smooth threefold over the complex numbers $\mathbb{C}$ with ample canonical bundle K_X. We let $r > 1$ be an integer such that $(r-1)K_X$ is spanned by global sections and rK_X gives a birational map onto its image in projective space. We set $\mathcal{O}_X(1) = rK_X$ and let $S \subset X$ be a general hyperplane section under the map associated to $\mathcal{O}_X(1)$. By Jouanolou's Bertini theorem ([10], theorem 6.10) S is a smooth irreducible surface. The irregularity is $q = h^1(\mathcal{O}_S) = h^1(\mathcal{O}_X)$ (equality is due to the Kodaira vanishing theorem) and the geometric genus of the surface is $p_g = h^0(K_S)$. The invertible sheaf $\mathcal{O}_S(1)$ gives a morphism from S to a projective space with two dimensional image, hence Jouanolou's Bertini theorem shows that the general hyperplane section of this map is a smooth connected curve C of genus g.

Let c_i denote the Chern classes of X with corresponding Chern numbers c_1^3, c_1c_2 and c_3. The Hirzebruch–Riemann–Roch theorem ([9], Appendix A, theorem 4.1) gives formulas for the Euler characteristic of the tangent sheaf and the sheaf of differentials:

$$\chi\mathcal{T}_X = \frac{1}{2}c_1^3 - \frac{19}{24}c_1c_2 + \frac{1}{2}c_3 \tag{1}$$

$$\chi\Omega_X = \frac{1}{24}c_1c_2 - \frac{1}{2}c_3 \tag{2}$$

We will bound c_3 by finding lower bounds for these Euler characteristics in terms of c_1^3 and c_1c_2.

Lemma 1. *The following bounds on cohomology groups hold.*
(a) $h^1\mathcal{T}_X \leq \sum_{m\geq 0} h^1\mathcal{T}_S(-m) + q + h^1\mathcal{O}_S(1)$
(b) $h^3\mathcal{T}_X \leq q + h^2\mathcal{T}_S$
(c) $h^1\Omega_X \leq \sum_{m\geq 0} h^1\Omega_S(-m)$
(d) $h^3\Omega_X \leq p_g$.

Proof. In considering all nonpositive twists of the exact sequences

$$0 \to \mathcal{T}_X(-1) \to \mathcal{T}_X \to \mathcal{T}_X|_S \to 0 \tag{3}$$

$$0 \to \mathcal{T}_S \to \mathcal{T}_X|_S \to \mathcal{O}_S(1) \to 0 \tag{4}$$

we obtain the inequality $h^1\mathcal{T}_X \leq \sum_{m\geq 0} h^1\mathcal{T}_S(-m) + \sum_{m\geq 0} h^1\mathcal{O}_S(-m+1)$ and the second sum is $h^1\mathcal{O}_S + h^1\mathcal{O}_S(1) = q + h^1\mathcal{O}_S(1)$ by Kodaira vanishing, proving part (a).

Twisting the exact sequence 3 with $\mathcal{O}_X(1)$ gives the inequality $h^3\mathcal{T}_X \leq h^2\mathcal{T}_X|_S(1) + h^3\mathcal{T}_X(1)$. The duality $H^3\mathcal{T}_X(1) \perp H^0\Omega_X(-(r-1)K_X) \subset H^0\Omega_X$ shows that $h^3\mathcal{T}_X(1) \leq h^0\Omega_X = h^1\mathcal{O}_X = q$. The sequence 4 further gives $h^2\mathcal{T}_X|_S(1) \leq h^2\mathcal{T}_S(1)$ (as $H^2\mathcal{O}_S(2) \perp H^0\mathcal{O}_S(-(r-1)K_X|_S) = 0$ by Kodaira vanishing) while the restriction sequence of $T_S(1)$ to the curve C shows that $h^2\mathcal{T}_S(1) \leq h^2\mathcal{T}_S$. Combining, we obtain part (b).

As in part (a), the nonpositive twists of the exact sequences

$$0 \to \Omega_X(-1) \to \Omega_X \to \Omega_X|_S \to 0 \tag{5}$$

$$0 \to \mathcal{O}_S(-1) \to \Omega_X|_S \to \Omega_S \to 0 \tag{6}$$

give part (c). For part (d), we have $h^3\Omega_X = h^1K_X = h^2\mathcal{O}_X$ and the exact sequence

$$0 \to \mathcal{O}_X(-1) \to \mathcal{O}_X \to \mathcal{O}_S \to 0$$

gives $h^2\mathcal{O}_X \leq h^2\mathcal{O}_S = h^0K_S = p_g$.

Lemma 2. *Let S be a smooth connected surface with spanned and ample line bundle $\mathcal{O}_S(1)$ and suppose that C is a smooth connected section of $\mathcal{O}_S(1)$. Let $\mathcal{E}$ be a vector bundle on S such that $h^0(\mathcal{E}|_C(-R)) = 0$ for some $R > 0$. Then*
(a) $h^1\mathcal{E}(-R-1) \leq h^1\mathcal{E}(-R)$
(b) If $h^1\mathcal{E}(l) \neq 0$ *and* $l < -R$ *then* $h^1\mathcal{E}(l-1) < h^1\mathcal{E}(l)$

Proof. Assertion (a) is immediate after twisting the restriction sequence

$$0 \to \mathcal{E}(-1) \to \mathcal{E} \to \mathcal{E}|_C \to 0 \tag{7}$$

by $-R$. For $l \le -R$ we now consider the exact sequences

$$0 \to \mathcal{E}^\vee \otimes K_S(-l) \xrightarrow{\alpha_l} \mathcal{E}^\vee \otimes K_S(-l+1) \xrightarrow{\pi_l} \mathcal{E}^\vee \otimes K_C(-l) \to 0$$

in which the map $H^1(\alpha_l)$ is Serre dual to $H^1(\beta_l)$. Duality on the curve C gives the vanishing $H^1(\mathcal{E}^\vee \otimes K_C(R)) \perp H^0(\mathcal{E}_C(-R)) = 0$. According to the Castelnuovo–Mumford theorem ([14], theorem 2), the multiplication maps

$$H^0(\Omega_S \otimes K_C(-l)) \otimes [s_0, s_1] \to H^0(\Omega_S \otimes K_C(-l+1)$$

are surjective for $l < -R$ and sections $s_0, s_1 \in H^0\mathcal{O}_C(1)$ which span $\mathcal{O}_C(1)$. In particular, these sections may be chosen as the restriction of sections of $H^0\mathcal{O}_S(1)$.

We now see from the commutative diagram

$$\begin{array}{ccc} H^0(\mathcal{E}^\vee \otimes K_S(-l+1)) \otimes H^0(\mathcal{O}_S(1)) & \xrightarrow{\pi_l} & H^0(\mathcal{E}^\vee \otimes K_C(-l)) \otimes [s_0, s_1] \\ \downarrow & & \downarrow \\ H^0(\mathcal{E}^\vee \otimes K_S(-l+2) & \to & H^0(\mathcal{E}^\vee \otimes K_C(-l+1)) \end{array}$$

that if $H^0(\pi_l)$ is surjective for some $l < -R$, then $H^0(\pi_n)$ is surjective for all $n < l$. In other words, if $H^1(\alpha_l)$ is an isomorphism for $l < -R$, then $H^1(\alpha_n)$ is an isomorphism for all $n < l$. This would contradict Serre's vanishing theorem if $H^1(\mathcal{E}^\vee \otimes K_S(-l)) \perp H^1(\mathcal{E}(l)) \neq 0$, proving part (b).

Corollary 3. *Letting $A = h^1\mathcal{T}_S$, $B = h^0\mathcal{O}_C(1)$, $D = h^1\Omega_S$ and $E = g$, we have the following estimates.*
(a) $\sum_{m\ge 0} h^1\mathcal{T}_S(-m) \le \frac{1}{2}(A^2 + 2AB + B^2 + 9A + 7B + 4)$
(b) $\sum_{m\ge 0} h^1\Omega_S(-m) \le \frac{1}{2}(D^2 + 6DE + 9E^2 + 9D + 15E)$.

Proof. The exact sequence

$$0 \to \mathcal{T}_C \to \mathcal{T}_S|_C \to \mathcal{O}_C(1) \to 0 \tag{8}$$

shows that $h^0\mathcal{T}_S|_C(-2) = 0$ ($\mathcal{T}_C(-2)$ and $\mathcal{O}_C(-1)$ have negative degree), hence we may take $R = 2$ in lemma 2. The restriction sequence

$$0 \to \mathcal{T}_S(-1) \to \mathcal{T}_S \to \mathcal{T}_S|_C \to 0$$

yields $h^1\mathcal{T}_S(-1) \le A + h^0\mathcal{T}_S|_C$ and $h^1\mathcal{T}_S(-2) \le A + h^0\mathcal{T}_S|_C + h^0\mathcal{T}_S|_C(-1)$. Since $\mathcal{T}_C$ has negative degree, the sequence 8 shows that $h^0\mathcal{T}_S|_C \le B$ and $h^0\mathcal{T}_S|_C(-1) \le 1$. Applying lemma 2, we deduce part (a).

Similarly $h^0\Omega_S|_C(-3) = 0$ and we will take $R = 3$ in lemma 2. The restriction sequences show that $h^1\Omega_S(-l) \le D + \sum_{p=0}^{p=l-1} h^0\Omega_S|_C(-p)$ and we have the bound $h^0\Omega_S|_C(-p) \le h^0\Omega_S|_C \le h^0K_C = E$. Combining these with lemma 2 gives part (b).

Lemma 4. *We have the following estimates.*
(a) $q \leq g = 1 - \frac{1}{2}(1+2r)r^2c_1^3$
(b) $h^2\mathcal{T}_S \leq 2K_S^2 + q + 1 \leq 2 - (3r^3 + \frac{9}{2}r^2 + 2r)c_1^3$
(c) $h^1\mathcal{T}_S \leq 2 - \frac{5}{6}rc_1c_2 - \frac{1}{6}(16r^3 + 18r^2 + 5r)c_1^3$
(d) $p_g \leq -\frac{1}{12}(rc_1c_2 + (14r^3 + 9r^2 + r)c_1^3)$
(e) $h^1\mathcal{O}_S(1) \leq -r^3c_1^3 + 1$
(f) $h^0\mathcal{O}_C(1) \leq -\frac{1}{2}r^3c_1^3 + 1$
(g) $h^1\Omega_S \leq 2 - (\frac{5}{6}r)c_1c_2 - \frac{1}{6}(16r^3 + 9r^2 - r)c_1^3$.

Proof. The inequality of part (a) comes from the exact sequence

$$0 \to \mathcal{O}_S(-1) \to \mathcal{O}_S \to \mathcal{O}_C \to 0 \tag{9}$$

and Kodaira vanishing. Since $2g - 2 = \deg K_C$, the equality in part (a) follows from the isomorphism $K_C \cong (1+2r)K_X|_C$. Tensoring the sequence 9 by $\mathcal{O}_S(2)$ gives $h^1\mathcal{O}_S(1) \leq h^1\mathcal{O}_S(2) + h^0\mathcal{O}_C(2)$ and $h^1\mathcal{O}_S(2) = h^1K_S(-2) = h^1(-(r-1)K_X|_S) = 0$ by Kodaira's vanishing theorem. To estimate $h^0\mathcal{O}_C(2)$, we note that if $\mathcal{O}_C(2)$ is nonspecial, then $h^0\mathcal{O}_C(2) = (\frac{1}{2}r^2 - r^3)c_1^3$ while if it is special, then Clifford's theorem gives $h^0\mathcal{O}_C(2) \leq -r^3c_1^3 + 1$. In either case, $h^0\mathcal{O}_C(2) \leq -r^3c_1^3 + 1$ and we deduce part (e).

Since $(r+1)K_X$ gives a birational map of X onto its image in projective space, $K_S = (r+1)K_X|_S$ gives a birational map of S to projective space and hence S is minimal of general type and has a smooth canonical curve (as K_S is spanned, Jouanolou's Bertini theorem applies). Following [2], theorem C, we obtain the first inequality of part (b) and $h^0\mathcal{T}_S = 0$. Applying part (a) and calculating the intersection $K_S^2 = -(r+1)^2rc_1^3$ gives the second inequality of part (b). Since $\chi\mathcal{T}_S = h^2\mathcal{T}_S - h^1\mathcal{T}_S$, the computation $\chi\mathcal{T}_S = \frac{1}{6}(5rc_1c_2 - (2r^3 + 9r^2 + 7r)c_1^3)$ and part (b) yield part (c).

A calculation shows that

$$1 - q + p_g = \chi\mathcal{O}_S = -\frac{1}{12}(rc_1c_2 + r(r+1)(2r+1)c_1^3).$$

Combining this with part (a) gives the inequality of part (d). Similarly one can compute that $\chi\Omega_S = (\frac{5}{6}r)c_1c_2 + \frac{1}{6}r(r+1)(4r-1)c_1^3$. Noting that $h^2\Omega_S = h^1K_S = q$ and $h^0\Omega_S = h^1\mathcal{O}_S = q$ we find that $\chi\Omega_S = 2q - h^1\Omega_S$, and applying the inequality of part (a) gives part (g). It is easily checked that $\chi\mathcal{O}_C(1) < 0$ and hence $\mathcal{O}_C(1)$ is special. Applying Clifford's theorem gives part (f).

Corollary 5. *There exist quadratic forms Q_1, Q_2 with coefficients in $\mathbb{Q}[r]$ such that the following statement holds: For each smooth threefold $X/\mathbb{C}$ with K_X ample and each integer $r > 1$ such that $(r-1)K_X$ is spanned and rK_X gives a birational map to projective space, we have*

$$Q_1(c_1^3, c_1c_2) \leq c_3 \leq Q_2(c_1^3, c_1c_2).$$

Proof. Follows from equations 2 and 1, lemma 1, corollary 3 and lemma 4.

Remark 6. We note that the same proof goes through if we only assume the vanishings of the intermediate cohomologies $h^i(\mathcal{T}_X \otimes mK_X) = 0$ and $h^i(\Omega_X \otimes mK_X) = 0$ for $0 < i < 3$ and $m << 0$. Indeed, these vanishings imply that $h^1(\mathcal{T}_X|_S(m)) = 0$ for $m << 0$, which in turn implies that $h^1(\mathcal{T}_S(m)) = 0$ for $m << 0$ via the sequence 4 and the Kodaira vanishing theorem. In particular, we deduce sufficient vanishings for the conclusion of lemma 2 to hold.

Theorem 7. *There exist quadratic forms Q_1, Q_2 with coefficients in $\mathbb{Q}$ such that every smooth threefold $X/\mathbb{C}$ with K_X ample has third Chern number is bounded by*

$$Q_1(c_1^3, c_1c_2) \leq c_3 \leq Q_2(c_1^3, c_1c_2).$$

Proof. Applying [11], theorem 3.1 and proposition 3.8, we may take $r = 6$ in corollary 5.

Remark 8. In the case that K_X is very ample, there is a linear bound on c_3 due to Van de Ven (see [8], introduction):

$$c_3 \leq -2c_1c_2 - 7c_1^3.$$

There is a similar bound when mK_X is very ample.

Remark 9. Of course, one can compute the quadratic forms of theorem 7 explicitly. Setting $x = c_1^3$ and $y = c_1c_2$, we obtain

$$Q_1 = -635209x^2 - 7970xy - 25y^2 + 14750x + \frac{919}{12}y - 48$$

and

$$Q_2 = 1771561x^2 + 13310xy + 25y^2 - 23040x - \frac{1151}{12}y + 58.$$

Recalling that the hyperplane section surface S is minimal of general type (see proof of lemma 4), we may lift Noether's inequality to the threefold X to find that $-6 + 203x \leq y$. In summary, we find that the Chern numbers $x = c_1^3$, $y = c_1c_2$, $z = c_3$ for a smooth threefold with ample canonical bundle satisfy the following inequalities:

$$x < 0,\ y < 0$$

$$-6 + 203x \leq y \leq \frac{3}{8}x$$

$$Q_1 \leq z \leq Q_2$$

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