3/12/ Disc 2: Relation

$R \subset A \times A$

$R$ is a partial order if

reflexive, antisymm, transitive

Notation $aRb$ : $a \preceq_R b$

$a \leq b$

$(A, \preceq)$ poset

" partially ordered set

$x$ comparable to $y$ if

$x \leq y$ or $y \leq x$

Total order: If $\forall x, y$

$x$ comparable to $y$

min element / max element

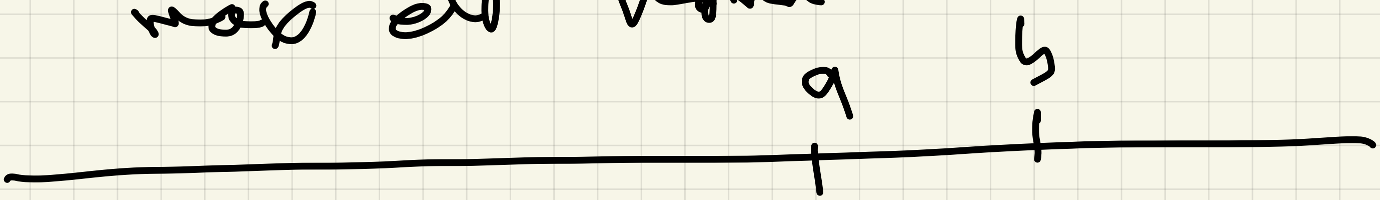
(

$y$ min if $\nexists\, z \in A$ :
$z \leq y$ and $z \neq y$

Ex1 $A = \mathbb{R}$, $x \leq y$
$(\mathbb{R}, \leq)$ is a poset.

Total order ✓ $\forall a, b \in \mathbb{R}$
$a \geq b$ or $b \geq a$

min elt none
max elt none




$a \leq b$ if $a$ left of $b$

Ex2 $A = \mathbb{N}$, $x \leq y$ usual defn

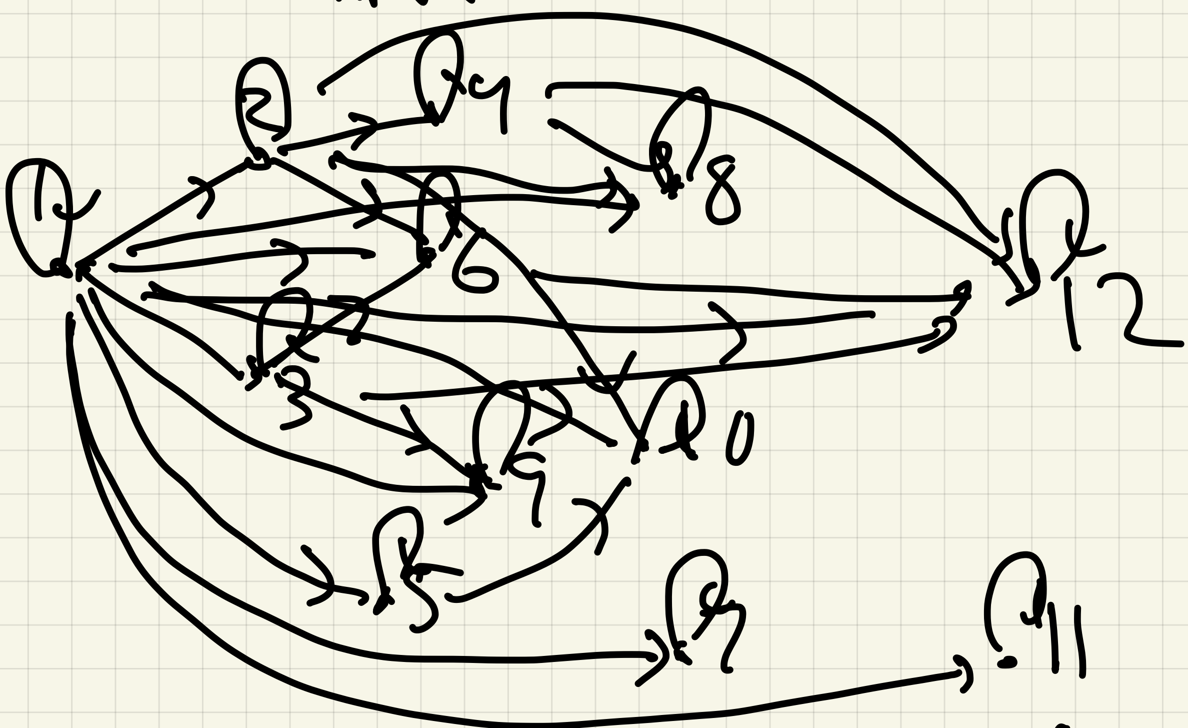
Total order
min elt is $0$
no max element

**Ex3** $A = \{1, 2, 3, 4, 5, \ldots\ 12\}$

$x \leq y$ if $x \mid y$

$2 \leq 4$ but $2 \leq 3$

true false




**MESS!**

**Total order?** **NO** $5 \nmid 7$

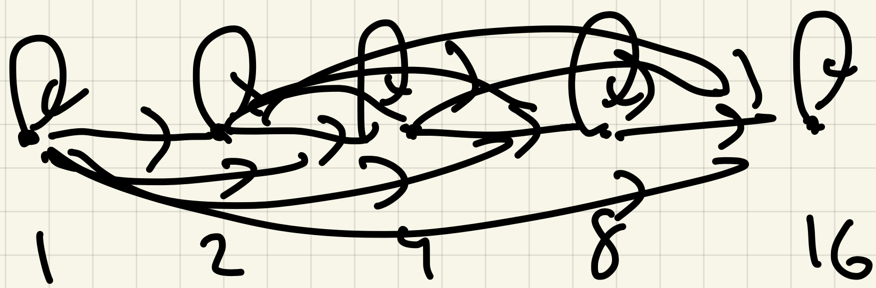
$7 \nmid 5$

**min elts**: 1

**max elts**: 12, 10, 8, 9, 7, 11

Ex 4  $A = \{1, 2, 4, 8, 16\}$

$xRy$  $x \preceq y$ if $x | y$



Total order  ✓

min  1

max  16

Less clutter:

Hasse diagram:

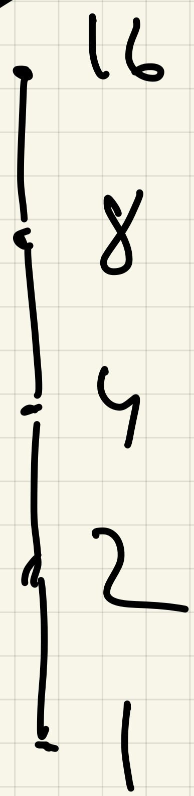
① If $x \preceq y$, make $x$ vertically lower than $y$

② Draw edge from $x$ to $y$ if

$x \le y$ and $\nexists z : x \le z \le y$

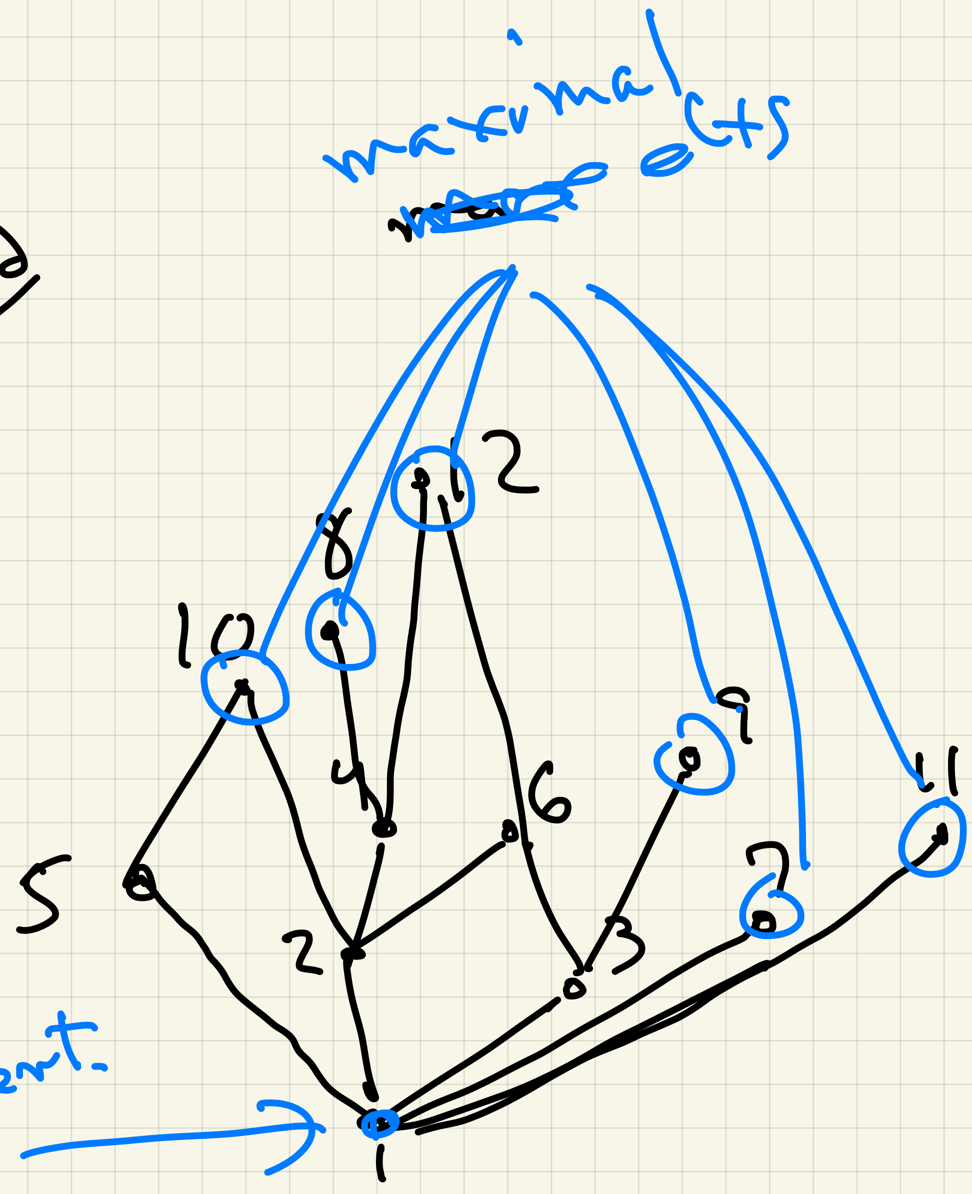
**Ex 4 revisited** $A = \{1, 2, 4, 8, 16\}$

$x \le y$ if $x \mid y$




**Ex 3 revisited**

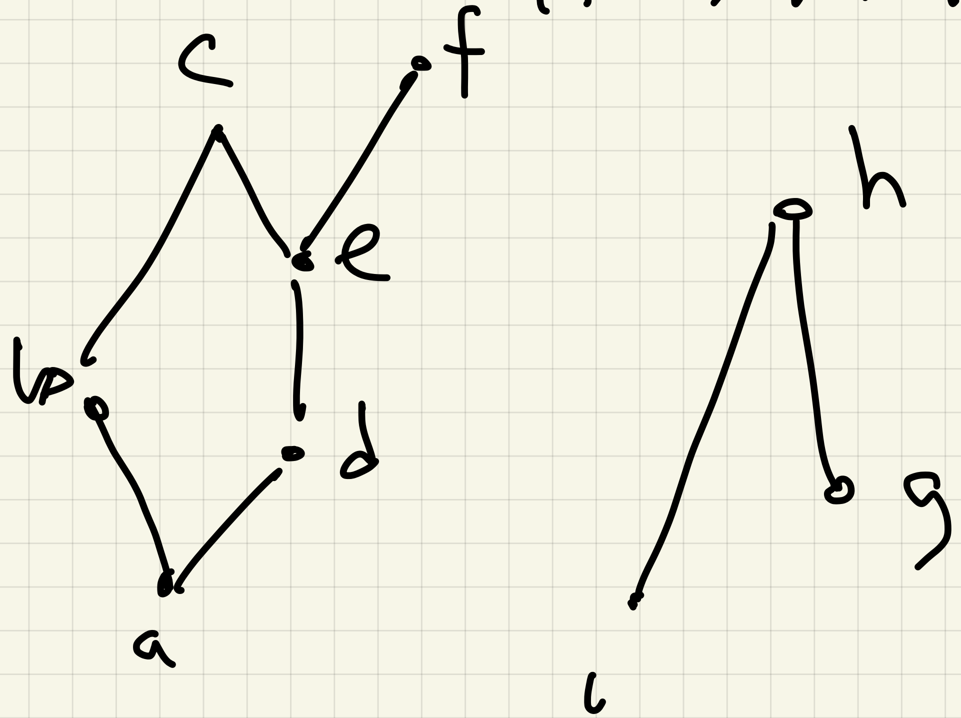
Hasse diagram




min element

## Ex 5 Hasse diagram

$A = \{a, b, c, d, e, f, g, h, i\}$




(a) Not total order:

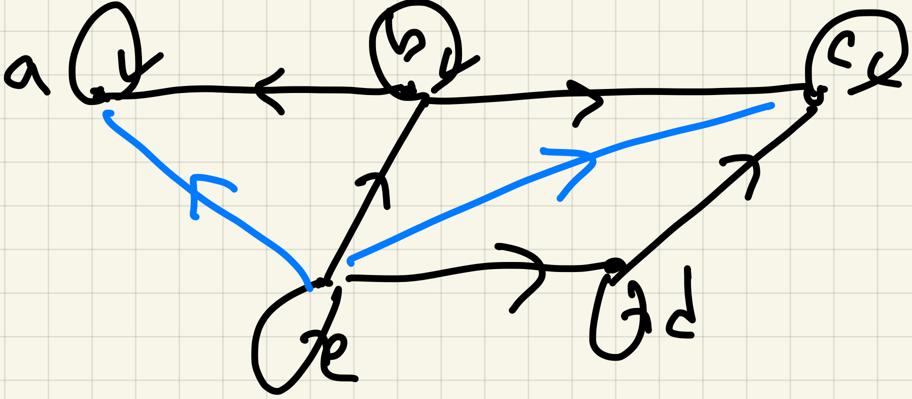
(b) min elts : $a, i, g$

(c) max elts : $c, f, h$

(d) $d, g$ not comparable

(e) but $a \leq c$

## Ex 6 :




reflexive ✓

anti-symm ✓

<u>not</u> <u>transitive</u>

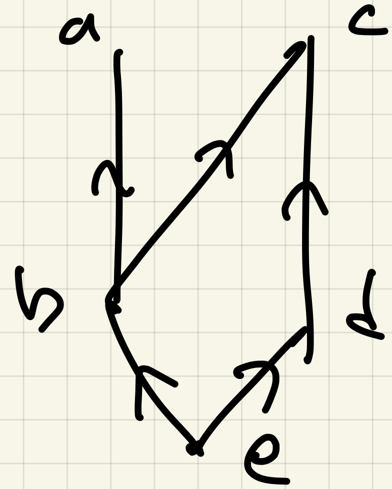
but adding blue edges, it becomes transitive, ∴ partial order,

$$\begin{array}{c} a \\ b \\ c \\ d \\ e \\ \\ \end{array}\begin{array}{c}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \\ \begin{matrix} a & b & c & d & e \end{matrix}\end{array}$$

max elts

← min elt

# Hasse diagram:




Ex 7 $A = P(\{a,b,c\})$

$x \leq y$ if $x \subseteq y$

refl: $x \subseteq x$ ✓

antisymm $x \subseteq y \wedge y \subseteq x \Rightarrow y = x$

transitivity $x \subseteq y \wedge y \subseteq z$

$\Downarrow$

$x \subseteq z$

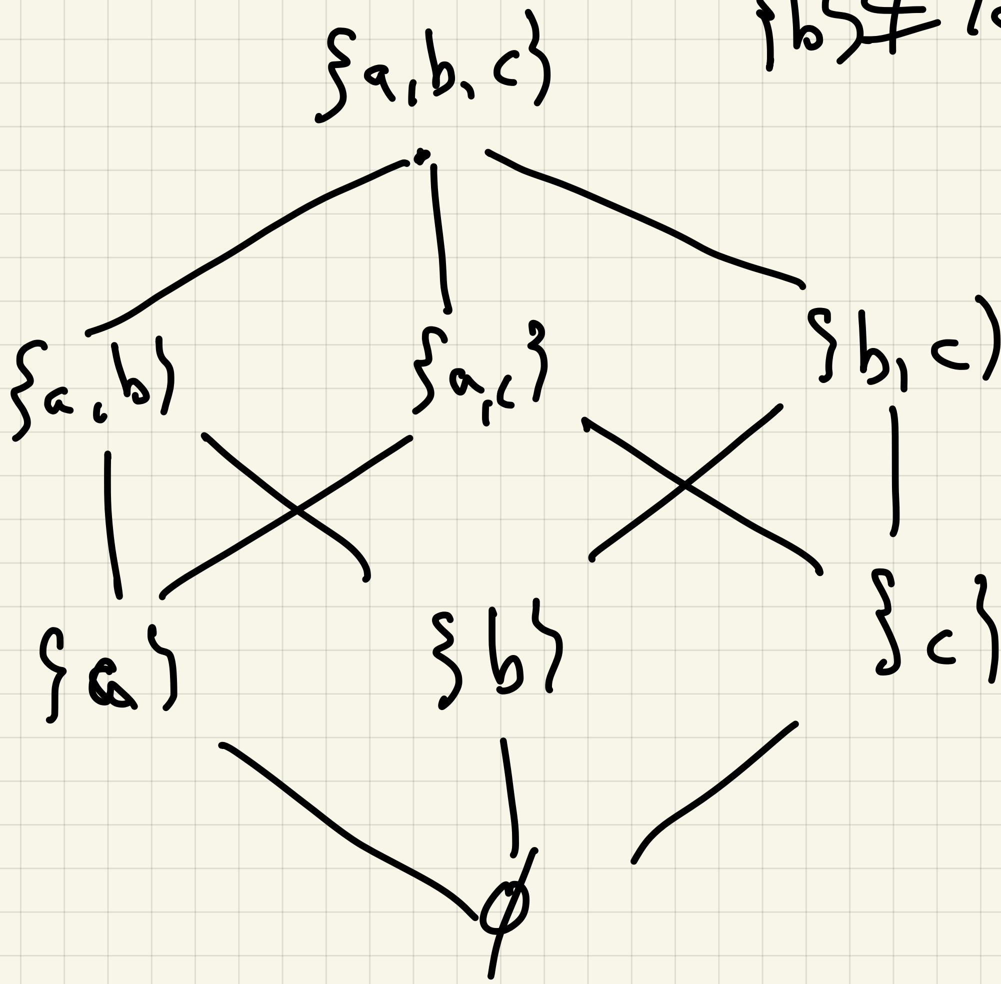
min elt: $\emptyset$

max elt $\{a,b,c\}$

total order?, no

$\{a\} \not\leq \{b\}$

$\{b\} \not\leq \{a\}$




Ex 8

$A = \{$ bit strings of length $\leq 3$ start with 0 $\}$

$x \leq y \qquad x = yw$

001 001

$001 \le 00$

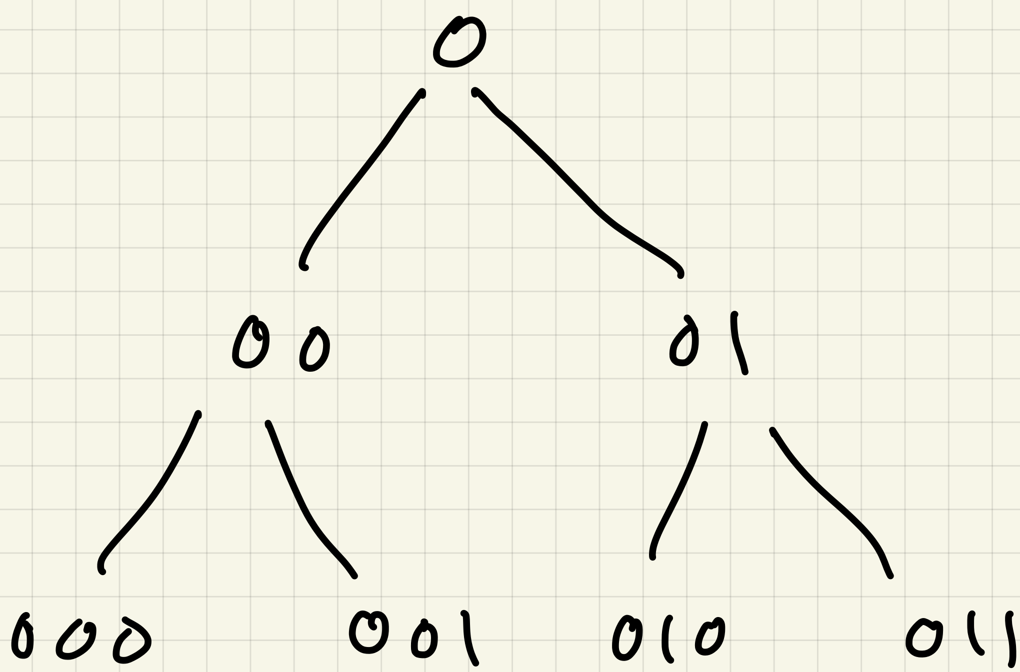
$01 \le 0 \qquad 00 \le 0$

Total order No $\qquad 001$, $010$

max $\quad 0$

min $\quad 000 \quad 001 \quad 010 \quad 011$

Hasse diagram:




§6.9 Equivalence relation:

Defn: A relation R on A is **equivalence relation** if

reflexive, symmetric, transitive

Notation $xRy$ $\quad$ book: $x \sim y$

Ex) (a) $A = \mathbb{R}$

$xRy$ $(x \sim y)$ if $x = y$

(b) $A =$ people

$x \sim y$ if $\text{age}(x) = \text{age}(y)$
(in years)

(c) $A = \mathbb{Z}$

$x \sim y$ if $5 \mid (y - x)$

refl: $5 \mid (x-x)$ $\quad 5 \mid 0$

sym $5 \mid (y-x)$ $\quad 5 \mid (x-y)$

trans $5 \mid (y-x)$ $\quad 5 \mid (z-y)$

$$5 \mid z-x = (z-y) + (y-x)$$

(d) $A = \mathbb{R}^2$

$(a,b) \sim (c,d)$ if $b = d$