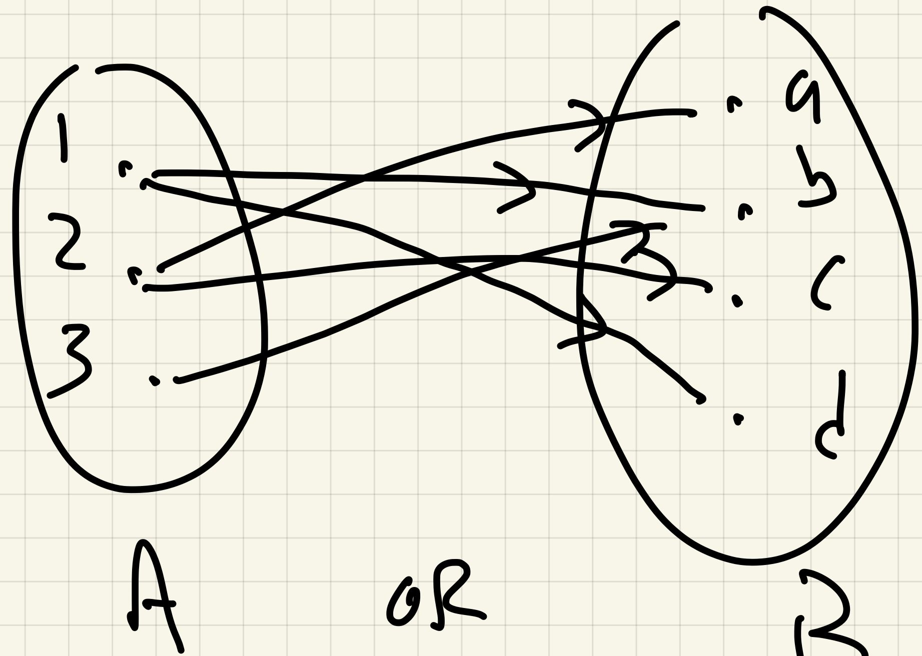
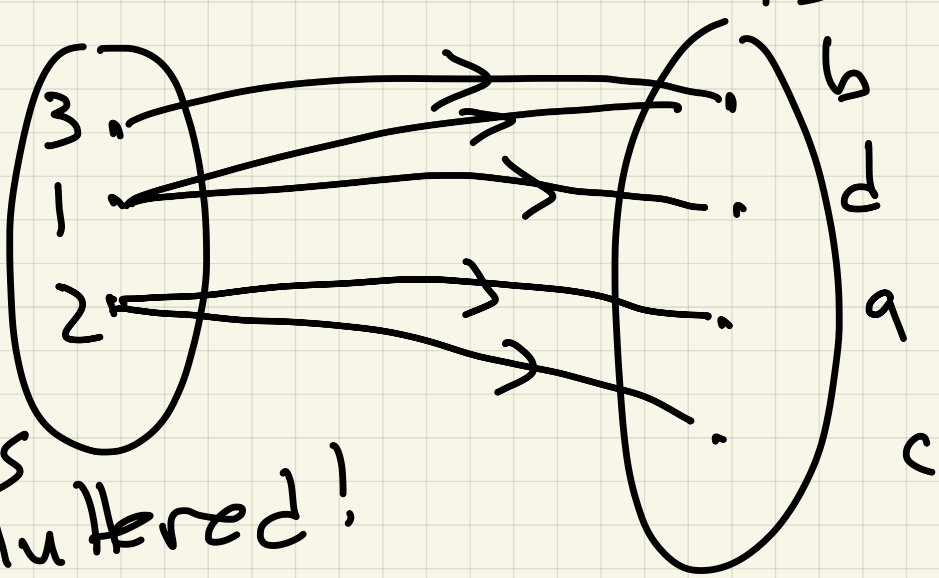
3/16 | Disc 2   Quiz 6

1.

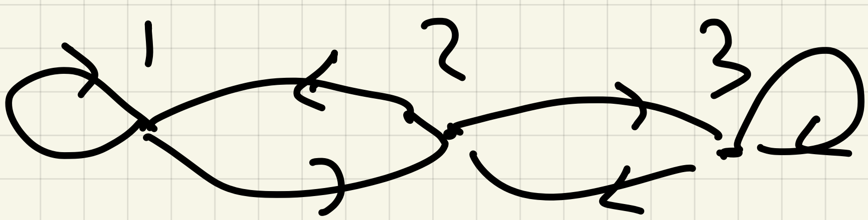


OR



less cluttered!

2. S:



(a)

$$M = \begin{array}{c} 1 \\ 2 \\ 3 \\ \end{array} \begin{array}{c} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \\ \begin{matrix} 1 & 2 & 3 \end{matrix} \end{array}$$

(b) Not reflexive b/c $\neg 2S2$

(c) Not antireflexive b/c $1S1$

(d) Symmetric yes

(e) Anti-symmetric no b/c

$2S1 \wedge 1S2$ but $\neg 2S2$

(f) transitive NO $1S2 \wedge 2S3$ but

$\neg 1S3$

Last time composition of relations:

R, S relations on $A \Rightarrow$

$S \circ R$ is relation:

$a(S \circ R)c$ if $\exists b \in A: aRb \wedge bSc$

Special Case: $R = S$:

$R, \; R \circ R = R^2, \; R \circ R \circ R = R^3 \ldots$

<u>Interpretation</u>: $a(R\circ R)b \iff$

$\exists c \in A: aRc \wedge cRb \iff$

(a,c,b) is length two walk from a to b (in R)

<u>Similarly</u>: $(a,b) \in \underbrace{R \circ \ldots \circ R}_{n} = R^n$

if $\exists$ length n walk from a to b.

<u>Rmk</u> R transitive $\iff$

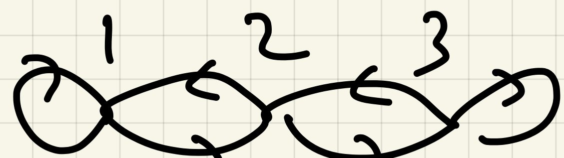
$\underbrace{aRb \wedge bRc}$ $\Rightarrow aRc$

$\Downarrow$

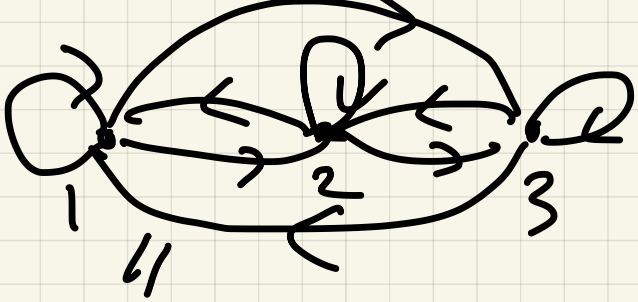
$a(R\circ R)c$, so

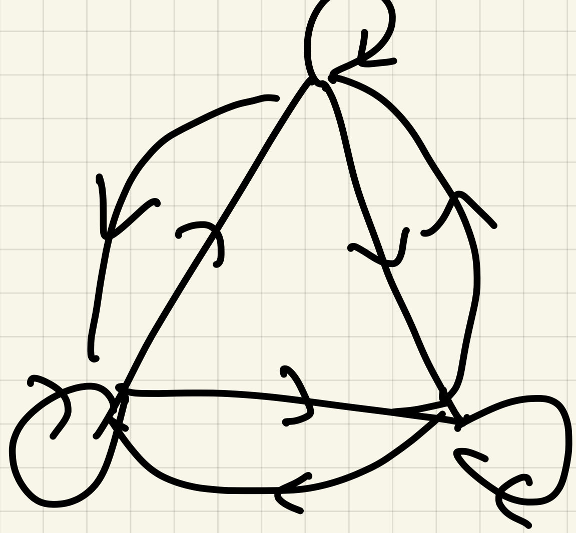
R <u>transitive</u> $\iff R^2 \subset R$.

<u>Ex1</u>: Quiz 6 R:



R^2





$R^2 \not\subseteq R \Rightarrow R$ not transitive, as we saw.

<u>Defn</u> The <u>transitive closure of R</u> is $R^+ = R \cup R^2 \cup R^3 \cup R^4 \cup \ldots$

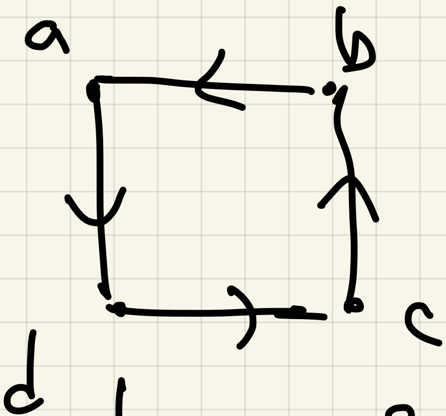
$$= \bigcup_{n=1}^{\infty} R^n$$

<u>In practice</u>, enough to take $n \leq |A|$.

<u>Note</u> $(a,b) \in R^+ \Leftrightarrow \exists$ walk from a to b so R^+ is transitive.

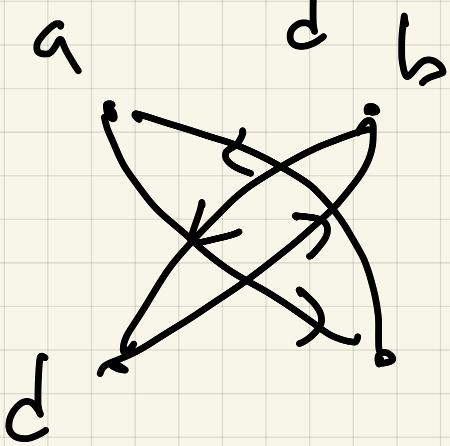
<u>Ex2</u> In Ex1, $R^2 = A \times A$, so $R^+ = A \times A$.

Ex3 R :

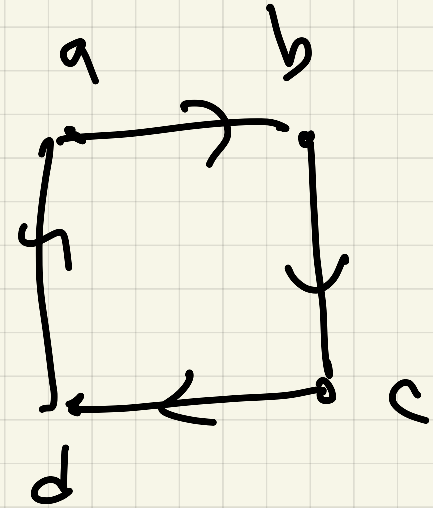


on $A=\{a,b,c,d\}$

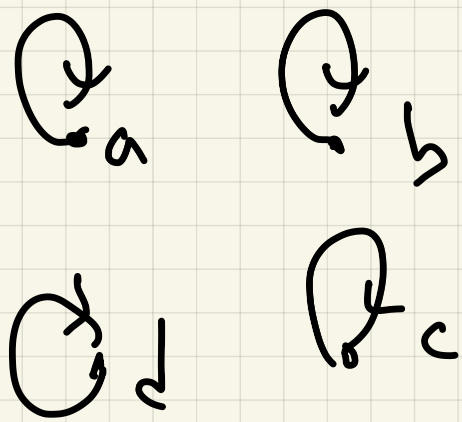
R^2



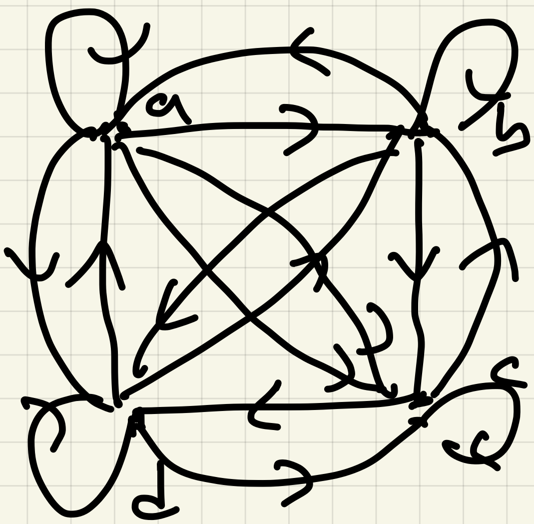
R^3



$R^4 =$



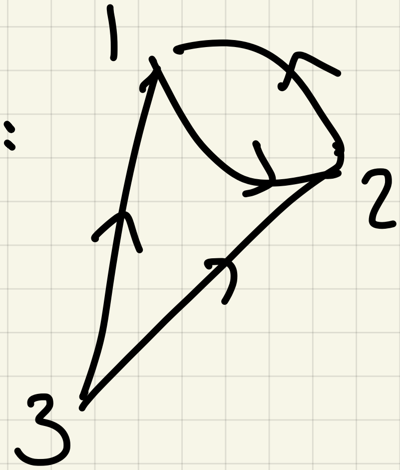
so $R^+ = R\cup R^2\cup R^3\cup R^4 =$

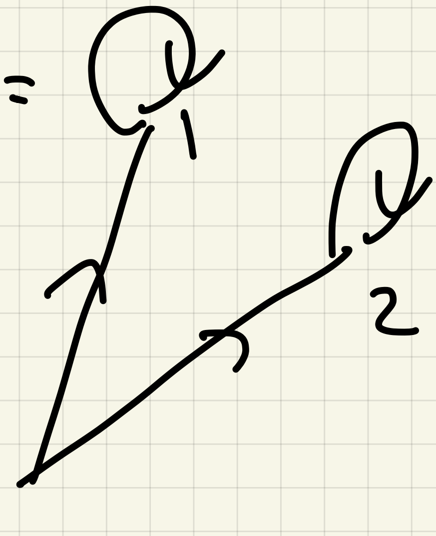


$= A\times A$

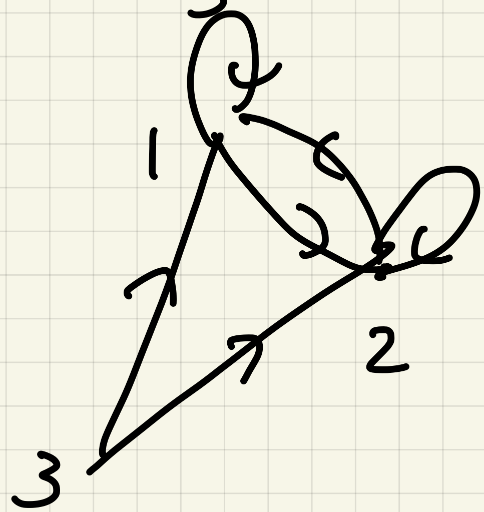
Ex4 Find transitive Closure

(a)

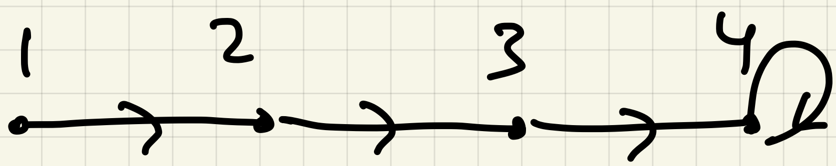
$R:$ 

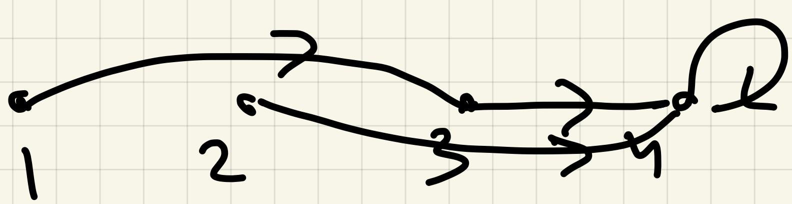
$R^2 =$ 

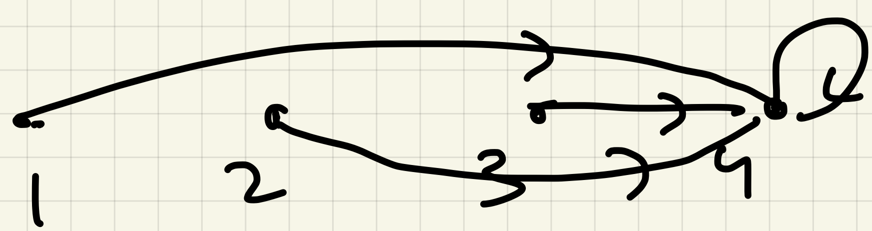
$R^3 = R$!

so $R^+ = R \cup R^2 =$ 

$$M = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

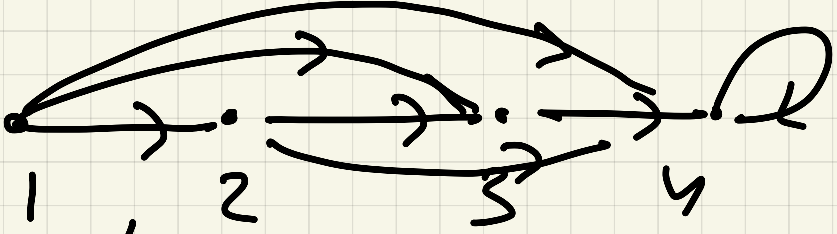
(b) R 

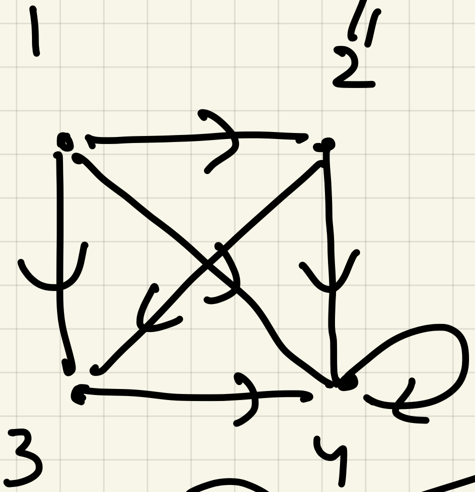
R^2 

R^3 

$R^4 = R^3$

so R^+





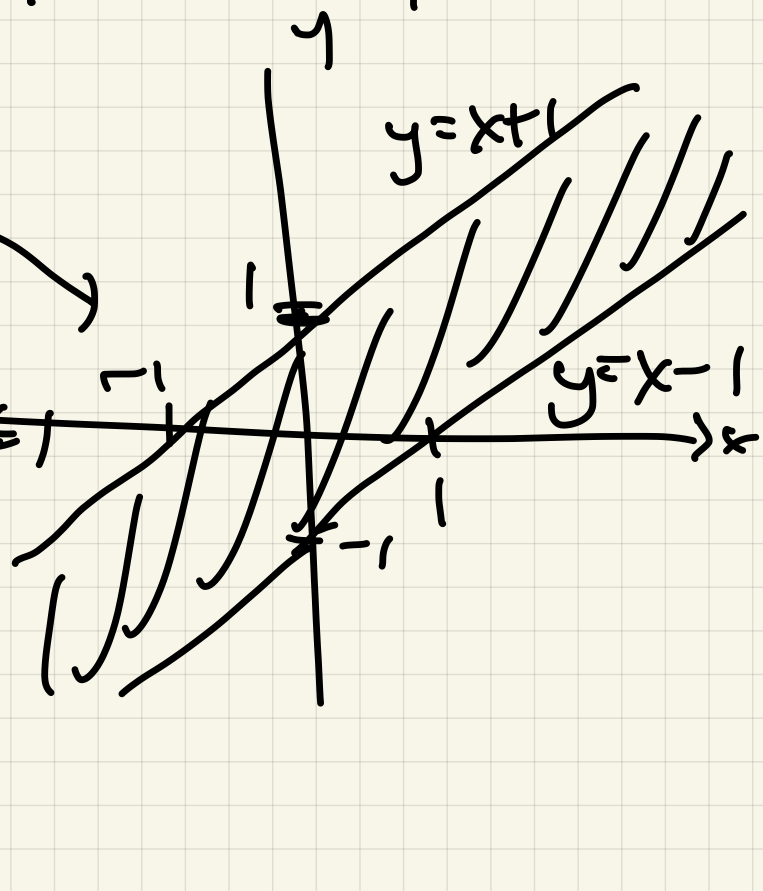
I used $\mathbb{Z}$ in class, but $\mathbb{R}$ easier.

(c) $A = \mathbb{R}$,

xRy if $|x-y| \le 1$

R is reflexive, symmetric, transitive, not antireflexive, not antisymmetric.

R: $|x-y| \le 1$ given by



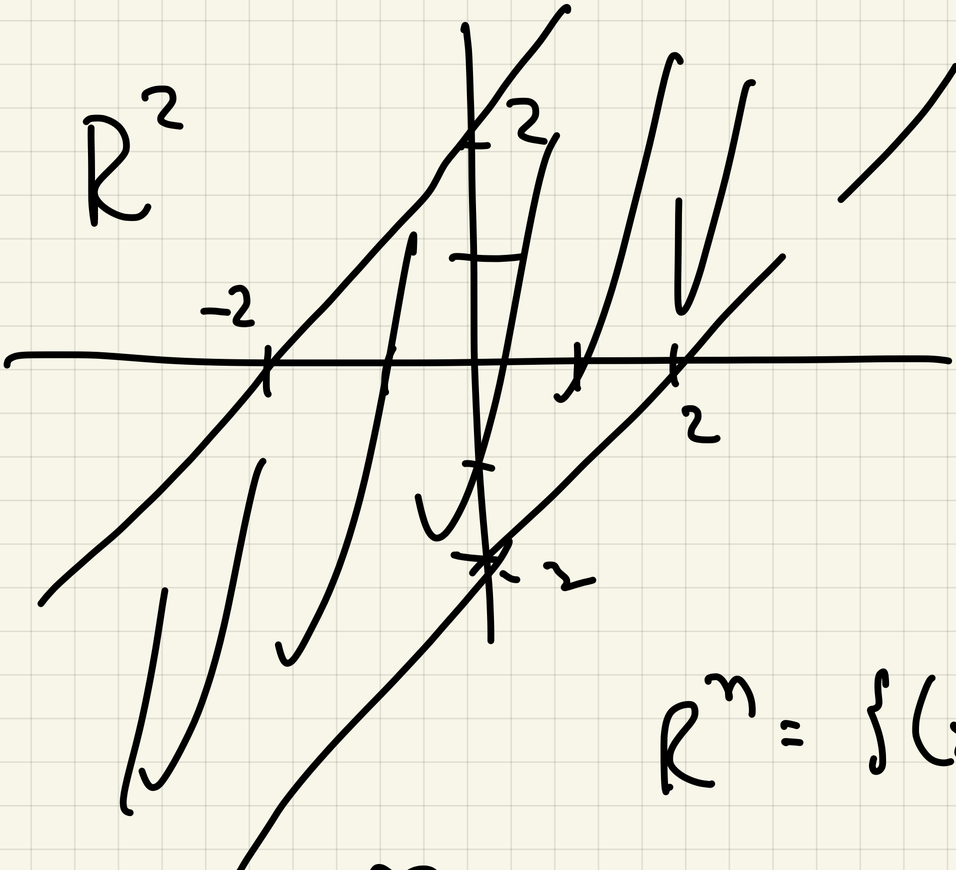
R^2: $|x-y| \le 1$ → $|x-z| \le 1$

$|y-z| \le 1$,

Δ inequality

so

R^2



$$R^n = \{(x,y) \mid |x-y| \le n\}$$

$$R^+ = \bigcup_{n=1}^{\infty} R^n = \mathbb{R}^2$$

§6.7 Partially ordered sets = Posets

A relation R on a set A is a partial order if R is reflexive, transitive, anti-symmetric,

Ex1 The quintessential example is

$A = \mathbb{R}$, xRy if $x \le y$.

Note: For this reason, partial orders R usually use symbol $\leq$ instead.

Ex2 $A = \mathbb{R}$, xRy if $x<y$.

R not reflexive, so not partial order.

Ex3 $A = \{1, 2, 3, \ldots 12\}$

aRb if $a \mid b$

reflexive: $a \mid a$ ✓

transitive $a \mid b \wedge b \mid c \Rightarrow a \mid c$ ✓

anti-symm $a \mid b \wedge b \mid a \Rightarrow a = b$

OK b/c numbers are > 0:

$$\left.\begin{array}{l} a \mid b \Rightarrow a \leq b \\ b \mid a \Rightarrow b \geq a \end{array}\right\} \Rightarrow a = b.$$

Defn A **poset** is $(A, \leq)$,

A = set, $\leq$ = partial order

Defn For Poset $(A, \leq)$,

1. $x, y \in A$ comparable if $x \leq y$ or $y \leq x$
2. $x \in A$ is minimal if $\nexists y \in A$: $y \leq x$ and $y \neq x$
3. $x \in A$ is maximal if $\nexists y \in A$: $y \geq x$ and $y \neq x$.
4. $(A, \leq)$ is a total order if $\forall x, y \in A$, x, y comparable.

Exy $(\mathbb{R}, \leq)$ is a total order.
It has no minimal or maximal elements.