1/29/ Disc 2

Generalized product rule

Last time

X = set of size n

r-permutations from X

# of these is $n(n-1)\ldots(n-r+1)$

$$= \frac{n!}{(n-r)!}$$

Permutations of X

# of these is

$n(n-1)(n-2)\ldots(1) = n!$

$$\frac{n!}{0!} = n!$$

$0! = 1$

Ex ① drawing 5 cards in order from deck of 52

$$\frac{52!}{47!}$$

But usually you're concerned about actual cards in hand, not order

10.5 Combinations

Ex1 How many ways to select 3 different cookies from 5 varieties

$C = \{c_1, c_2, c_3, c_4, c_5\}$

choc. chip raisin etc.

Can count the 3-permutations

$5 \cdot 4 \cdot 3 = P(5,3) = \frac{5!}{2!}$

But this is wrong:
We have overcounted

To count sets of size 3,

We can form a function

$$f : \left\{ \begin{array}{c} 3 \text{ permutations} \\ \text{from } C \end{array} \right\}_{P_3} \longrightarrow \left\{ \begin{array}{c} \text{subsets of} \\ \text{size } 3 \\ \text{from } C \end{array} \right\}_{S_3}$$

$$f(a_1 a_2 a_3) = \{a_1, a_2, a_3\}$$

f is <u>onto</u>:
But not 1-1:

$a_1\ a_2\ a_3$
$a_1\ a_3\ a_2$
$a_2\ a_1\ a_3$
$a_2\ a_3\ a_1$
$a_3\ a_1\ a_2$
$a_3\ a_2\ a_1$

There are 6 3-permutations that map to $\{a_1, a_2, a_3\}$

By k-1 rule

$$|S| = \frac{|P|}{6} = 10 = \frac{5!}{2!\,3!}$$

Defn: an r-combination of S is a subset of size r. (order not important)

If $|S| = n$, then the number of r-combinations is

$$\frac{P(n,r)}{\#\text{ permutations of set of size } r} = \frac{\frac{n!}{(n-r)!}}{r!} = \boxed{\frac{n!}{(n-r)!\,r!}}$$

Notation:

$$\binom{n}{r} = \text{"n choose r"} = C(n,r)$$

Ex: $X = \{a,b,c,d,e\}$

The 2-combinations from X

are $\{a,b\}$ $\{a,c\}$, $\{a,d\}$ $\{a,e\}$
$\{b,c\}$ $\{b,d\}$ $\{b,e\}$
$\{c,d\}$ $\{c,e\}$
$\{d,e\}$

10 of them!

$n=5, r=2$

$$\frac{5!}{3!2!} = \frac{5\cdot 4\cdot \cancel{3\cdot 2\cdot 1}}{2\cdot 1\cdot \cancel{3\cdot 2\cdot 1}}$$

$$\frac{20}{2} = 10$$

$$\binom{n}{2} = 1+2+3+\dots(n-1)$$

Ex1 Pizza from Buffalo Bros.

small/large

(a) 21 choices of toppings

How many 4-topping pizzas?

$$2 \cdot \binom{21}{4} =$$

size

$$2 \cdot \frac{21!}{17!\,4!}$$

(b) How many if one double topping is allowed?

pp olives/onions

$$2 \cdot \left[\binom{21}{4} + \binom{21}{1}\binom{20}{2}\right]$$

all different

choose topping to double

choose other 2

(c) Allow one or two double toppings

Easy:

$$2\left[\binom{21}{4}+\binom{21}{1}\binom{20}{2}+\binom{21}{2}\right]$$

size

choose 2 toppings

Ex2: (a) How many 5-card hands from deck of 52?

$$\binom{52}{5}$$

(b) How many flushes (all same suit)

$4 \to \binom{4}{1}\binom{13}{5}$

suit

(c) How many full houses?

$(\underbrace{3H, 3S, 3C}_{3}, \underbrace{JD, JH}_{2})$

Choose number for 3: $\binom{13}{1}$

choose suits for 3: $\binom{4}{3}$

Choose number for 2 $\binom{12}{1}$

choose suits $\binom{4}{2}$

$$\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}$$

$$\underline{13 \cdot 4} \cdot 12 \cdot 6 = 52 \cdot 72$$

(d) 4 of a kind

(4H, 4S, 4C, 4D, QS)

$\binom{13}{1}\binom{48}{1}$

last card

(e) At least one ace??

How many with no aces?

$\binom{52}{5} - \binom{48}{5}$

Ex3 Length 7 strings from

$\{a, b, c, \ldots\ z\}$

(a) How many strings are there?

26^7

(b) How many with distinct

letters?

$$P(26,7) = \frac{26!}{19!}$$

(c) Exactly 3 ms ??

_ _ _ _ _ _ _

Place the 3 ms $\binom{7}{3} \cdot 25^4$

(d) Exactly 3 "xs" and one "b"

$$\binom{7}{3}\binom{4}{1} 24^3$$

x b

(e) Exactly 3 xs, one b, only duplicates are x s

$$\binom{7}{3}\binom{4}{1} P(24,3)$$

(f) At least one "a" in string

$$26^7 - 25^7$$

Ex 4 A hiring committee receives 200 applications for a job listing

(a) How many ways to make a short list of 20?

$$\binom{200}{20}$$

(b) 50 applications of the 200 come from out of state.

How many with at least one out of state app?

$$\binom{200}{20} - \binom{150}{20}$$

a) ↑
ur out of
state apps

(c) How many ways to make a ranked short list of 20?

$P(200, 20)$

(d) How many ways to decide which 6 of 20 to bring in for an interview?

$$\binom{20}{6}$$

(e) How many ways to rank the 6 after the interview?

$6! = 720$

S = length 6 strings from
{1, 2, 3 . . 9}

1. How many strings ?

2. How many start
111 or 222 ?

3. How many start 11
and end 44 ?

4. T = set of strings that
triples 1st / 2nd / 3rd
sets of 2 digits'

353535 ∈ T 676747

676767 ∈ T

(a) B = strings of length 2,

Find a bijection from

$f: B \to T$

(b) How many strings in T?