1/22 Discrete2

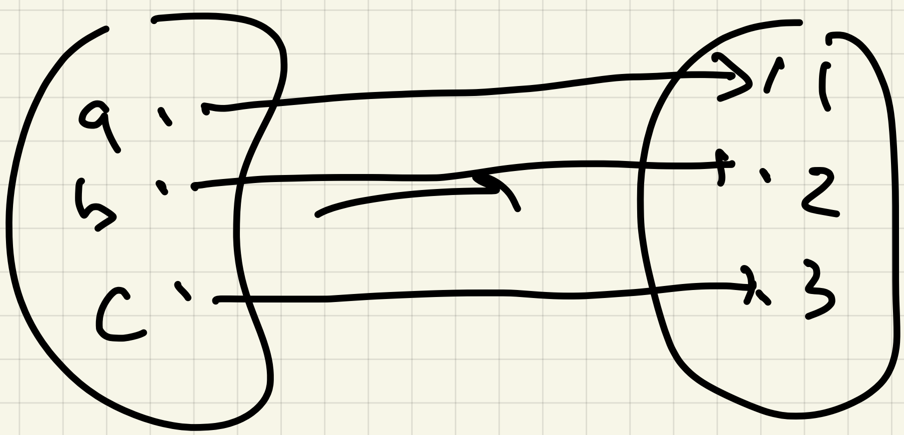
functions $f: A \to B$

Last time f 1-1 $\left(f(a_1) = f(a_2) \Rightarrow a_1 = a_2 \right)$

f onto $\left(\forall b \in B \; \exists a \in A : f(a) = b \right)$

f bijection (1-1 and onto)

Bijection rule: f bijection $\Rightarrow$ $|A| = |B|$



Ex): $A = \{1, 2, \ldots, 10\}$

$P(A)$ = power set

Discrete $|P(A)| = 2^{|A|}$ (know this)

Why?

Idea: Encode subsets of A

$$B = \underbrace{\{0,1\} \times \{0,1\} \times \ldots \times \{0,1\}}_{10} = \{0,1\}^{10}$$

define $f : P(A) \to B$

$$S \mapsto \underline{b_1 b_2 \ldots b_{10}}$$

subset of A

where

$$b_i = \begin{cases} 1 & i \in S \\ 0 & i \notin S \end{cases}$$

Examples

$$S = \{1,2,3\} \xrightarrow{f} \underset{1\,2\,3\,4 \qquad\qquad 10}{1110000000}$$

$$\{3,5,7,8\} \to 0010101100$$

$$\emptyset \to 0000000000$$

$$\{1,2, \ldots 10\} \to 1111111111$$

$\{4,5,6,7\} \leftarrow 0001111000$

f is 1-1 and onto

Bijection rule $\Rightarrow$

$$|P(A)| = |B| = 2^{10}$$

(b) Another idea for counting:

$$g: P(\{1, \dots 10\}) \longrightarrow P(\{1,2,3,\dots 9\})$$

$$S \longmapsto S - \{10\}$$

$$\{1,2,3\} \longmapsto \{1,2,3\}$$

$$\{2,3,7,10\} \longmapsto \{2,3,7\}$$

$$\{2,3,7\} \longmapsto \{2,3,7\}$$

<u>Notice</u>: g is <u>not</u> 1-1

but for each $T \in P(\{1, .. 9\})$,
there are **two** sets A, B
in $P(\{1, .. 10\})$ with

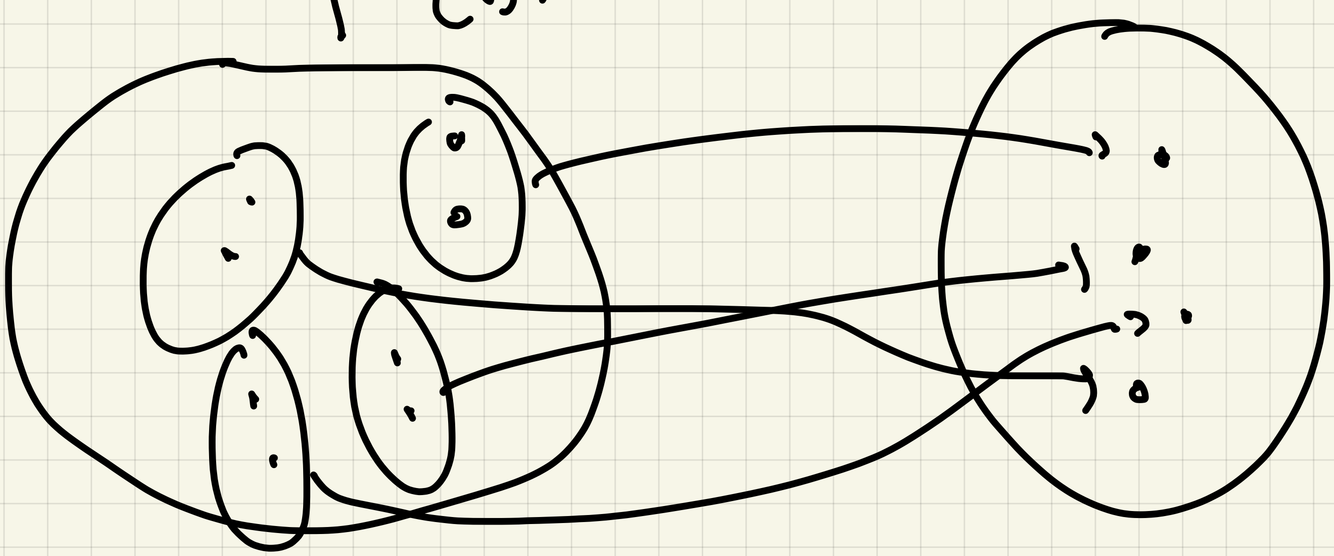
$$g(A) = g(B) = T$$

namely

$$A = T$$
$$B = T \cup \{10\}$$

Definition A function $f : A \to B$
is k-1 if for each $b \in B$,

$$\Big| \underbrace{\{a \in A : f(a) = b\}}_{= \, f^{-1}(b)} \Big| = k$$



$k=2$

$$\boxed{\begin{array}{l}\underline{\text{Connection}}: \text{If } f \text{ is a } k\text{-}1 \\ \text{map, then } |B| = \dfrac{|A|}{k} \\ |A| = k|B|\end{array}}$$

<u>Ex 1 (b)</u> $g: P(\{1,..10\}) \to P(\{1,..9\})$

g is 2-1 map

$$|P(\{1..10\})| = 2|P(\{1,2..9\})|$$

$$= 2\cdot 2\,|P(\{1..8\})|$$

$$\vdots$$

$$\underbrace{2\cdot 2 \cdots 2}_{10} \frac{|P(\phi)|}{1} = 2^{10} \checkmark$$

Ex 1 $A = \{0, 1, 2, \ldots 999\}$

$B = \{0, 1, 2, \ldots 99\}$

find a 10-1 function

$f : A \to B$

(a) $f(a) = a \bmod 100$

$f^{-1}(0) = \{a \in A \mid f(a) = 0\} =$

$\{0, 100, 200, 300, \ldots - 900\}$

(b) $g(a) = \left\lfloor \frac{a}{10} \right\rfloor \leftarrow$

Recall For $x \in \mathbb{R}$,

$\lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \leq x\}$

$\lfloor 3 \rfloor = 3 \qquad \lfloor \pi \rfloor = 3$

$\lfloor -\pi \rfloor = -4$

9 15 10-1

Variation:

$A' = \{1, 2, \ldots \quad 1000\}$

$B' = \{1, 2, \ldots \quad 100\}$

harder to write a nice formula: but

$h(x) = 1 + \left\lfloor \frac{(x-1)}{10} \right\rfloor$ works

$H(x) = 1 + (x-1) \bmod 100$ also OK

10.3 Generalized Product Rule

In a multi-step process,

if there are

n_1 ways to step 1

n_2 ways to step 2

⋮

n_m ways to do step m

Then the number of ways to to do the entire m-stage process is

$$n_1 n_2 \cdots n_m = \prod_{i=1}^{m} n_i$$

<u>Ex1</u>: How many ways can you draw <u>in order</u> five cards from standard deck of 52?

$$\underline{52}\ \underline{51}\ \underline{50}\ \underline{49}\ \underline{48}$$

1st card | || 5th card

2nd card

$$52 \cdot \underset{4}{51} \cdot 50 \cdot 49 \cdot 48$$

$$\frac{52!}{47!}$$

(b) How many ways ~~of~~ with all hearts?

Ans: $13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 = \frac{13!}{8!}$

(c) How about all face cards

$$12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 = \frac{12!}{7!}$$

(d) How many ways to draw 3 of one value followed by 2 of another

$$52 \cdot 3 \cdot 2 \cdot 48 \cdot 3$$

$\uparrow$

1 2 3

<u>Ex 2</u> How many outcomes:

(a) flip a coin 10 times

$2 \; 2 \; 2 \ldots \; 2 = 2^{10}$

$\uparrow$
1st flip

2^{nd}

(b) flip a coin 5 times
roll a die 5 times

$2 \, 2 \, 2 \, 2 \, 2 \; 6 \; 6 \, 6 \; 6 \, 6 = 2^5 \cdot 6^5$

(c) select 5 cards from a deck, but replace each time

$52 \cdot 52 \cdot 52 \cdot 52 \cdot 52 = 52^5$

(d) Same, but do <u>not</u>

replace,

$52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 = \frac{52!}{47!}$

(e) Select 2 cards,

replace

Select 3 more

replace

select 2

$52 \cdot 51 \cdot 2 \cdot 51 \cdot 50 \cdot 49 \cdot 4 \cdot 49 \cdot 48$

Ex3 20 students in grade school class

10 boys, 10 girls

(a) How many ways to line them up?

$\underline{20}\ \underline{19}\ \underline{18}\ \underline{\cdot}\ \underline{\cdots} \qquad \underline{1}$

$$20!$$

(b) How about if all 10 girls are first in line?

$$(10!)(10!)$$

(c) How many are in order

B G B G B G ...

10 10 9 9 8 8 ...

$$(10!)^2.$$

(d) How many ways if Ben and Jerry are not together in line?

__Idea__: How many ways with

Ben and Jerry together
in line ??