2/12/ Disc 2

Last time # ways to place n identical objects

10.9 into m distinct boxes

" # $n+(m-1)$ bit strings with $(m-1)$ 1s "

dividers

$$\binom{n+(m-1)}{m-1}$$

10.10
ways place n objects into m boxes is

(Zybooks figure 10.10.1)

	no restrictions	at most 1 per box $n \le m$	same # in each box $m \mid n$
objects same	$\binom{n+m-1}{m-1}$	$\binom{m}{n}$	1
objects distinct	m^n	$P(m,n)$	$\frac{n!}{\left(\frac{n}{m}\right)!^m}$

Ex 1 Sally has 50 pieces of taffy, How many ways to give them to 10 friends ....

(a) If they are all different?

$$10^{50}$$

(b) If they're the same?

$$\binom{59}{9}$$

$$x_1 + x_2 + \dots + x_{10} = 50$$

$x_i \geq 0$ x_i ints

(c) If they're different but distributed evenly

$$\frac{50!}{(5!)^{10}} = \binom{50}{5}\binom{45}{5}\binom{40}{5}\cdots$$

(d) If they're same, distributed evenly,

$$1$$

(e) Taffies different, but 100 friends, each gets at most one.

$$P(100, 50) = \frac{100!}{50!}$$

(f) Taffies same, 100 friends

$$\binom{100}{50}$$

10.11 Saw in Disc 1 that

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Ex1: Consider 5 card hands from deck of 52

$\binom{52}{5}$ such hands

(a) How many hands have

A $\diamondsuit$ or A $\heartsuit$

A $\diamondsuit$: $\binom{51}{4}$

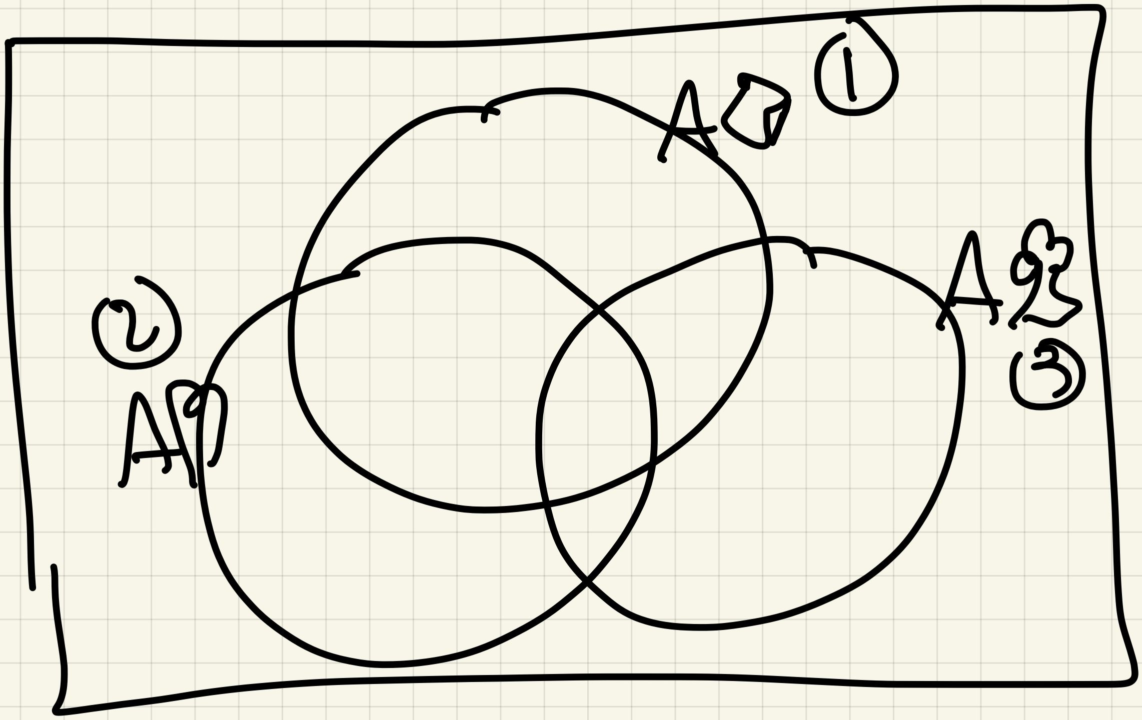
A $\heartsuit$: $\binom{51}{4}$

A $\diamondsuit$ and A $\heartsuit$ $\binom{50}{3}$

so $\binom{51}{4} + \binom{51}{4} - \binom{50}{3}$

(b) How many hands have

A♦, A♠ or A♣ ??



$$|1 \cup 2 \cup 3| = |1| + |2| + |3|$$

$$- |1 \cap 2| - |1 \cap 3| - |2 \cap 3|$$

$$+ |1 \cap 2 \cap 3|$$

$$3 \cdot \binom{51}{4} - 3\binom{50}{3} + 1\binom{49}{2}$$

In general:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Ex 3 $D = \{ n \in \mathbb{Z} \mid 1 \leq n \leq 1000,\ 5 \mid n,\ 7 \mid n \text{ or } 8 \mid n \}$

$A = \{ n \in \mathbb{Z} \mid 1 \leq n \leq 1000,\ 5 \mid n \}$

$B = \{ n \in \mathbb{Z} \mid \text{''},\ 7 \mid n \}$

$C = \{ n \in \mathbb{Z} \mid \text{''},\ 8 \mid n \}$

$|A| = |\{ 5k : 1 \leq k \leq 200 \}|$

200

$$|B| = \left\lfloor \frac{1000}{7} \right\rfloor = 142$$

$$|C| = \left\lfloor \frac{1000}{8} \right\rfloor = 125$$

$$|A \cap B| = \{n : 5 \mid n \text{ and } 7 \mid n\}$$

$$= \{n : 35 \mid n\}$$

$$= \left\lfloor \frac{1000}{35} \right\rfloor = 28$$

$$|A \cap C| = |\{n : 40 \mid n\}| =$$

$$\left\lfloor \frac{1000}{40} \right\rfloor = 25$$

$$|B \cap C| = |\{n : 56 \mid n\}| =$$

$$\left\lfloor \frac{1000}{56} \right\rfloor = 17$$

$$|A \cap B \cap C| = |\{n > 280 \mid n\}| = 3$$

so

$$\# = 200 + 142 + 125$$
$$- 28 - 25 - 17 + 3 = 400$$

<u>In general</u>

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum |A_i| - \sum |A_i \cap A_j|$$
$$+ \sum |A_i \cap A_j \cap A_k| - \cdots$$

<u>Ex 4</u> How many length 5 strings can formed from $\{a, b, c, d\}$

(a) with no repetition?

<u>NONE!</u>

(b) How many with at least <u>one letter missing</u>?

bbcdc

abbab

S_a = strings without a

S_b =

S_c =

S_d =

PIE

<u>Want</u>: $|S_a \cup S_b \cup S_c \cup S_d| =$

$(|S_a| + |S_b| + |S_c| + |S_d|$

$+ |S_a \cap S_b| - |S_a \cap S_c| - |S_a \cap S_d|$

$- |S_b \cap S_c| - |S_b \cap S_d| - |S_c \cap S_d|$

$+ |S_a \cap S_b \cap S_c| + |S_a \cap S_b \cap S_d| +$

$$|S_a \cap S_c \cap S_d| + |S_b \cap S_c \cap S_d|$$

$$- |S_a \cap S_b \cap S_c \cap S_d|$$

$$|S_a| = 3^5 = |S_b| = |S_c| = |S_d|$$

$$|S_a \cap S_b| = 2^5 \ldots$$

$$|S_a \cap S_b \cap S_c| = 1$$

$$|S_a \cap S_b \cap S_c \cap S_d| = 0$$

Ans: $4 \cdot 3^5 - 6 \cdot 2^5 + 4 \cdot 1 - \underline{1 \cdot 0}$

§11.3 Pigeon hole principle

Ex10 I place 14 marbles into 4 boxes

(a) Can a box be empty?

YES

(b) Can all boxes be empty?

NO

(c) Must some box have at least 2 marbles

yes

PHP

(d) Must some box have at least 3? yes

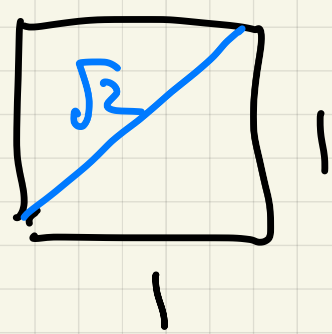
(e) 4 yes

(f) 5 no

<u>PHP</u>: If p pigeons are placed into h ~~boxes~~ boxes and $p > h$, then some box has at least 2 pigeons.

[<u>Alternative</u>: If $f: A \to B$ is a function, and $|A| > |B|$ then f is not 1-1]

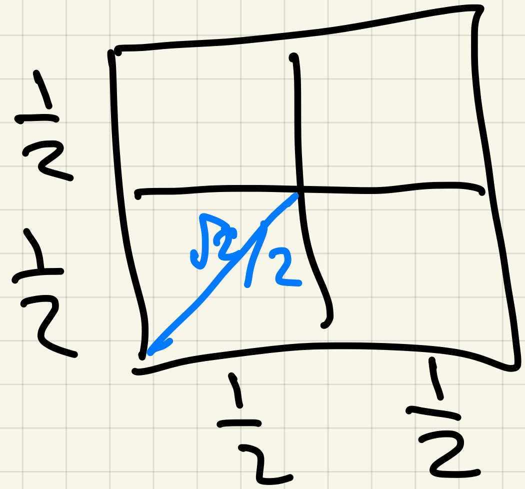
<u>Ex)</u> Among any 5 points in the unit square, there are 2 points a distance of at most



$\sqrt{2}/2$ apart.

Why?

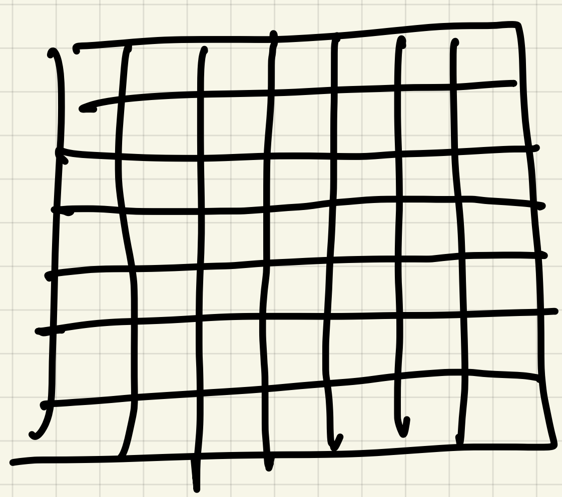
At least 2 of 5 points lie in one of the smaller boxes



(b) Among 50 points in unit square, there are at least two within $\sqrt{2}/7$ of each other.

Same idea:

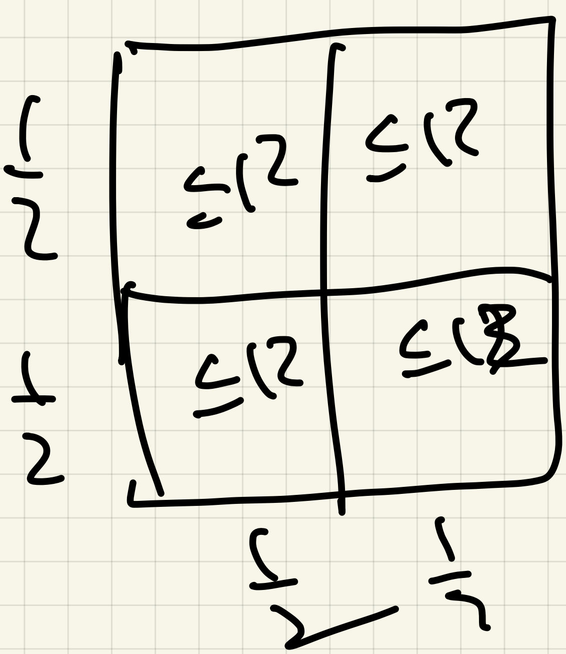
break 49 squares



$\frac{1}{7} \times \frac{1}{7}$ in size

$50 > 49 \Rightarrow 2$ in one box ✓

Ex) How many points are guaranteed to be within $\sqrt{2}/2$ of each other if we put 50 points into unit square?



50 points

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