2/10/Disc2 Quiz 3

1. 26^9

$\underset{26}{\_}\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \underset{26}{\_}$

2. $P(26,9) = \frac{26!}{17!}$

3. $P(21,9) = \frac{21!}{12!}$

4. $\binom{9}{4} 25^5$

$\binom{9}{4}$ ← place Ns; 25^5 ← place rest

5. $\binom{9}{4} P(25,5)$

$\binom{9}{4}$ ← Ns; $P(25,5)$ ← Now rest are distinct

6. $\binom{9}{3}\binom{6}{2} 24^4$

$\binom{9}{3}$: N, $\binom{6}{2}$: P, 24^4 ← the rest.

Last time Ex1 Select 3 cookies from 5 types A B C D E

3 different: $\binom{5}{3}$ $=$ 10

2 different $\binom{5}{1}\binom{4}{1} = 20$

all same $\binom{5}{1} = 5$

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35 total.

With larger numbers, this method is horrible!

BETTER WAY: encode possibilities:

2B1D → A | $\overset{oo}{B}$ | C | $\overset{o}{D}$ | E ⟶ 1001101

place dividers and cookies

This gives bijection between

Cookie selections ⟷ length 7, bit strings w 4 1s

$1A|1B|1C \longleftrightarrow 0|0|0|011$

$3A \longleftrightarrow 0001111$

so # selections = # length 7 bit string with 4 1s

$$= \binom{7}{4} = 35 \checkmark$$

<u>In general</u>: # ways to choose n items from m types

= # length n+(m-1) bit strings with m-1 1s

$$= \binom{n+(m-1)}{m-1}.$$

<u>Ex2</u> (a) How many ways to place 52 cards in 10 boxes?

10^{52} $\underbrace{\underline{10}\,\underline{10}\,\underline{10} \cdots \underline{\ }\ \underline{\ \ }}_{52}$

(b) # ways to place 52 <u>identical</u> marbles into 10 boxes?

Think of boxes as 10 types
want to select 52

$$\Rightarrow \binom{52+10-1}{10-1} = \binom{61}{9}$$

<u>Another viewpoint:</u>

Ex 2(b) equivalent to finding all integer solutions to

$$\begin{cases} x_1 + x_2 + \cdots + x_{10} = 52 \\ x_i \geq 0 \end{cases}$$

(x_i = # marbles in box i)

Ex3 (a) # integer solutions to

$$\begin{cases} x_1 + \dots + x_{10} = 52 \\ x_i \geq 0 \end{cases}$$

is $\binom{61}{9}$

Variations:

(b) How many if $x_1 = 10$?

i.e. # Int solutions to

$$\begin{cases} \underbrace{x_2 + \dots + x_{10}}_{\text{9 variables}} = 42 \\ x_i \geq 0 \end{cases}$$

i.e. $\binom{42+8}{8} = \binom{50}{8}$

(c) How many with $x_1 \geq 10$?

If $y_1 = x_1 - 10$, then want

int solutions to

$$y_1 + x_2 + \dots + x_{10} = 42$$

$$x_i \geq 0, \; y_1 \geq 0$$

$\Longrightarrow$

$$\binom{42+9}{9} = \binom{51}{9}$$

(d) How many with $x_1 = x_2 = 10$, $x_3 \geq 5$?

let $y_3 = x_3 - 5$, then

$$\underbrace{y_3 + x_4 + \dots + x_{10}}_{8 \text{ vars}} = 27$$

$$\Downarrow \quad \binom{27+7}{7} = \binom{34}{7}$$

(e) How many with

$$x_1 + x_2 + \dots + x_{10} \leq 52$$

Idea: let $x_{11} = 52 - \sum_{i=1}^{10} x_i$

solve $\begin{cases} x_1 + \dots + x_{10} + x_{11} = 52 \\ x_i \geq 0 \end{cases}$

is $\binom{52+10}{10} = \binom{62}{10}$

(f) How many if at most 5 go into box 10?

Idea: complement:

with at least 6 in box 10 is

solutions to $\begin{cases} x_1 + \dots + x_{10} = 46 \\ x_i \geq 0 \end{cases}$

is $\binom{55}{9}$, so

Answer is $\binom{61}{9} - \binom{55}{9}$

(g) How many if

at least 10 in box 1

at most 5 in box 10

i.e. $x_1 + \dots + x_{10} = 52$

$x_1 \geq 10, \; x_{10} \leq 5$

Combine parts (c) + (f):

$y_1 = x_1 - 10$

$x_1 + x_2 + \dots + x_{10} = 42 \Rightarrow \binom{51}{9}$

\# with $x_{10} \geq 6$: $\binom{36+9}{9} = \binom{45}{9}$

$\therefore$ Ans is $\binom{51}{9} - \binom{45}{9}$

Ex4

(a) \# Ways to place 100 identical cookies in 50 jars is $\binom{149}{49}$ $n=100$, $m=50$

(b) \# Ways if Jar 1 gets 10?

→ 9 jars, 90 cookies $\Rightarrow \binom{98}{8}$

(c) \# ways if Jar 1 gets ≥ 10?

10 jars, 90 cookies $\Rightarrow \binom{99}{9}$

(d) \# ways if jar 1 gets at most 10?

Complement: jar1 at least 11, so

$n = 89$
$m = 50$ $\rightarrow$ $\binom{138}{49}$

so Ans $\binom{149}{49} - \binom{138}{49}$

(e) How many if I keep some cookies?

Let x_{51} = # I keep, then

$n = 51$
$m = 100$ $\rightarrow$ $\binom{150}{50}$

(51st jar for me)

(f) # ways with equal number in each jar?

1! Exactly 2 cookies go in each jar! No choices!

Summary:

ways to place n marbles in m boxes

	No restriction	At most 1 per box, $m \geq n$	same # in each box, $m \mid n$
marbles same	$\binom{n+m-1}{m-1}$	$\binom{m}{n}$	1
marbles different	m^n	$P(m,n)$	$\frac{n!}{((n/m)!)^m}$