4/9 | Disc 2

Last time: Solving linear $\boxed{\text{non}}$-homogeneous recurr. relations

$(*)$ $\underbrace{a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}}_{H} + F_n$

① a_n^h = General solution to homogeneous equation

② a_n^p = particular solution

③ $a_n = a_n^h + a_n^p$

④ match initial conditions

How to find a_n^p ??

Thm 1: If $F_n = \boxed{p(n)\, y^n}$

polynomial of degree t

y = constant

Then

① If γ is not a solution to char equation, then guess

$$a_n^p = (d_0 + d_1 n + \dots + d_t n^t)\gamma^n$$

② If γ is a root of char equation of multiplicity m, then

$$a_n^p = n^m(d_0 + \dots + d_t n^t)\gamma^n$$

Solve for $d_0 \dots d_t$

<u>Ex1</u>: Find the <u>form</u> of guess for a_n^p for recurrence relations:

(a) $\underbrace{a_n = 7a_{n-1}}_{H} + \underbrace{3}$

$$X^n = 7X^{n-1} \Rightarrow \boxed{X = 7}$$

$$a_n^h = A \cdot 7^n$$

$$3 = 3 \cdot 1^n$$

poly?
degree 0 $\quad r=1$

Guess $\quad \underline{\underline{d_0}} \cdot 1^n = \underline{\underline{d_0}}$

(b) $\quad a_n = 7a_{n-1} + \underline{3n^{2}} \cdot \underline{\underline{1}}^n$

poly degree 2 $\quad r = 1$.

$\underline{\text{Guess}}$: $\quad a_n^p = (\underline{a + bn + cn^2})1^n$

$= a + bt + ct^2$

(c) $\quad a_n = 7a_{n-1} + 3n^2 \cdot 5^n$

$\underline{\text{Guess}}$: $\quad a_n^p = (a + bn + cn^2) \cdot 5^n$

(d) $\quad a_n = 7a_{n-1} + \underline{\underline{3n}}^3 (-1)^n$

$\underline{\text{Guess}}$: $a_n^p = (a + bn + cn^2 + dn^3)(-1)^n$

(e) $\quad a_n = 7a_{n-1} + 3n^3 \cdot 7^n$

$\underline{\text{Guess}}$: $a_n = n^1 (a + bn + cn^2 + dn^3) \cdot \underline{7}^n$

(f) $a_n = 14a_{n-1} - 49a_{n-2} + 3n^3 \cdot 7^n$

$$x^n = 14x^{n-1} - 49x^{n-2}$$

$$x^2 = 14x - 49$$

$$x^2 - 14x + 49 = 0$$

$$(x-7)^2 = 0$$

$7, 7$

$m = 2$

Guess: $n^2(a + bn + cn^2 + dn^3) \cdot 7^n$

(b): Solve for a, b, c

$$a_n = 7a_{n-1} + 3n^2$$

Guess: $a_n^p = a + bn + cn^2$

$$a + bn + cn^2 = 7(a + b(n-1) + c(n-1)^2) + 3n^2$$

$n^2 - 2n + 1$

?

n^2: $c = 7c + 3$ (1)

$\underline{n}$: $b = 7b - 14c$ ②

$\underline{1}$: $a = 7a - 7b + 7c$ ③

① $\rightarrow$ $-6c = 3$ $\quad c = -\frac{1}{2}$

② $b = 7b - 14c$

$b = 7b + 7$

$-6b = 7 \rightarrow b = -\frac{7}{6}$

③ $a = 7a - 7\left(\frac{-7}{6}\right) + 7\left(-\frac{1}{2}\right)$

$-6a = \frac{49}{6} - \frac{7 \cdot 3}{2 \cdot 3} = \frac{28}{6}$

$a = \frac{28}{6(-6)} = \frac{28}{36}$

so $a_n^p = \frac{28}{36} - \frac{7}{6}n - \frac{1}{2}n^2$

Ex) Solve

$$\begin{cases} \underbrace{A_n = 1.08A_{n-1}}_{} - 10{,}000 \\ A_0 = 100{,}000 \end{cases}$$

$$X^n = 1.08X^{n-1} \to$$

$$X = 1.08$$

General solution to H is

$$\boxed{A_n^h = A \cdot (1.08)^n}$$

<u>Particular solution</u>:

<u>Guess</u>: $A_n^p = c = \text{const}$

<u>find</u>: $\underline{c = 1.08c} - 10000$

$$-.08c = -10000$$

$$c = \frac{10000}{.08} = \frac{1000000}{8} =$$

$$\boxed{A_n^p = 125000}$$

So $A_n = A \cdot (1.08)^n + 125000$

find A:

$$A_0 = 100,000 = A(1.08)^0 + 125000$$

$$100000 = A + 125,000$$

$$A = -25000$$

$$A_n = -25000(1.08)^n + 125000$$

Pay off date:

$$0 = A_n = -25000(1.08)^n + 125000$$

$$\frac{-125000}{-25000} = (1.08)^n$$

$$\parallel$$

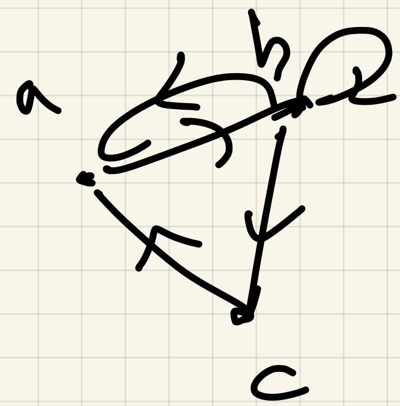
$$5 = (1.08)^n$$

$$n = \log_{1.08} 5 = \frac{\ln 5}{\ln 1.08} = 20.91 \text{ years}$$

§13.1 Graphs

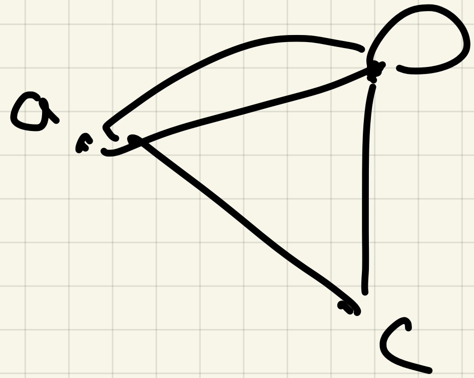
We already saw digraphs:

Ex 0



A graph (undirected) is given by set V of vertices, and E of edges connecting pairs of vertices

↳

Ex) 

5 edges
3 vertices

$V = \{a, b, c\}$

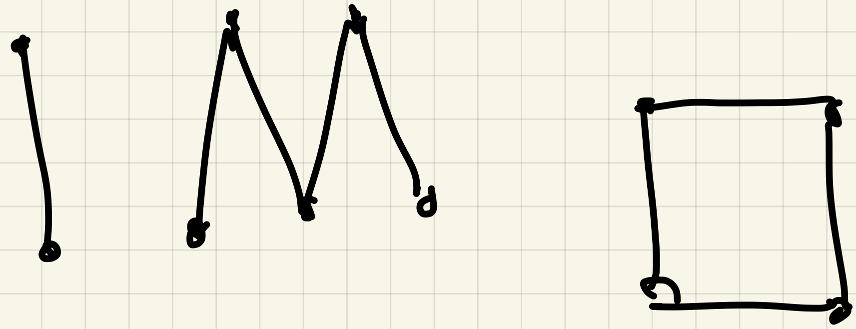
$E = \{(a,b), (a,b), (b,b), (a,c), (b,c)\}$

Note: edge (a,b) is repeated,

Two edges between same vertices are called parallel.

(b,b) is self-loop

Defn: A graph with no parallel edges and no self-loops is a simple graph



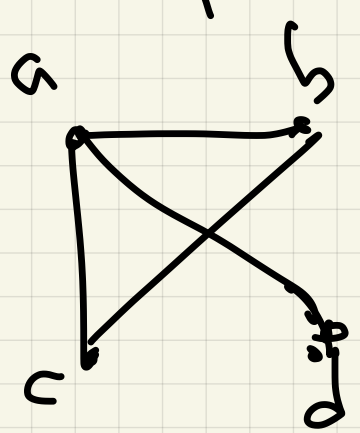
If $e = (v_1, v_2)$ is an edge
v_1, v_2 are adjacent (or neighbors)

The edge e is incident to
v_1 and v_2

Degree of a vertex is the
number of incident edges

Total degree of graph is sum
of degrees of vertices

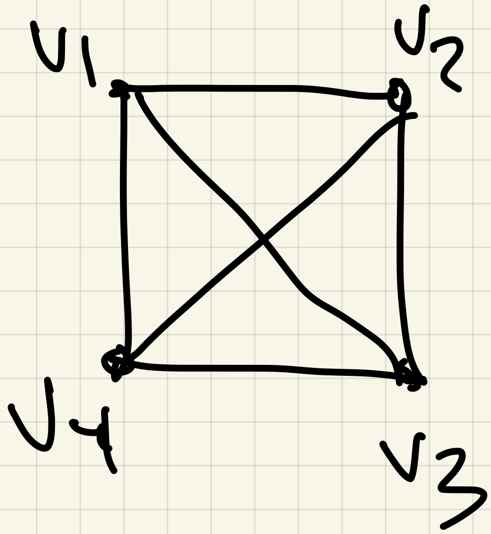
Ex 2



$\deg(a) = 3$
$\deg(b) = 2$
$\deg(c)$
$\deg(d) = 1$

Total degree = $3+2+2+1 = 8$

Ex3



$\deg(v_i) = 3$
all
$1 \leq i \leq 4$

A d-regular graph : all vertices have degree d

Total degree 12

Defn: A graph (V_H, E_H) is a subgraph of $G = (V_G, E_G)$ if $E_H \subset E_G$ and $V_H \subset V_G$

Theorem: If $G = (V, E)$ is a graph
Total degree of G

$$\parallel$$

$$\sum_{v \in V} \deg(v) = 2|E|$$

$|E|$ = # edges

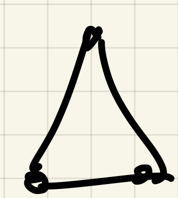
<u>Proof</u>: add edges to vertices one at ~~at~~ a time, $\sum \deg(v)$ increases by 2 with each new edge

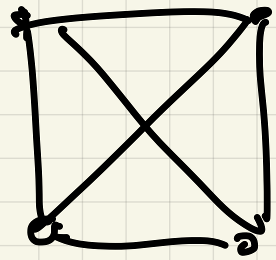
~~Ex~~ <u>Special Graphs</u>:

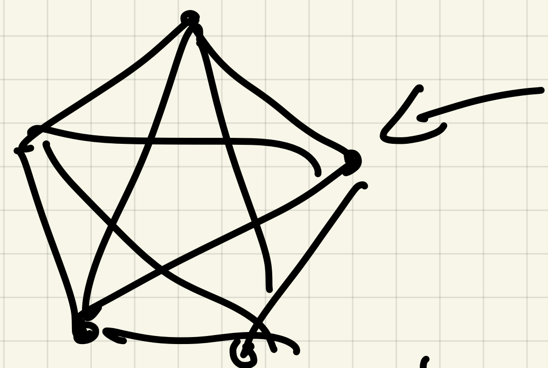
K_n = complete graph of n vertices; has every possible edge between vertices

K_1 

K_2 

K_3 

K_4 

$K_5 =$ 

$\forall$ vertex v in K_n

$\deg v = n-1$, so

$$2|E| = \sum \deg(v) = n(n-1)$$

so $$|E| = \frac{n(n-1)}{2} = \binom{n}{2}$$