4/7/ Disc 2

Exam 2

avg 87
med 92

150	
	14
135	
	5
120	
	4
105	
	2

#4/5

$A = \mathbb{N}$

$R \quad xRy$ if $y = x^2$

$S \quad xRy$ if $x > y$ ←

$5(R \circ S)z$

$5(S \circ R)z$

$\exists w : \underline{5 S w}$ and $\underline{w R z}$

$5 > w$

$\boxed{w = 0, 1, 2, 3, 4}$

$z = w^2$

$z = 0, 1, 4, 9, 16$

\ Fw: $\underline{5RW}$ and $\underline{\underline{w \leq z}}$

$$w = 5^2 = 85$$

$$\underline{25 > z}$$

$$\{0, 1, 2, \ldots \quad 24\}$$

Last time

Method to solve linear homogeneous recurr. relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

c_i const

8.15

Char equation: $a_n \leftarrow x^n$

$\Downarrow$

solve, get $x_1 \ldots x_k$

General solution:

$$a_n = \boxed{b_1 x_1^n + b_2 x_2^n +} \ldots + b_k x_k^n$$

b_i constants

Match $b_1 \ldots b_k$ to initial conditions

How to handle repeats:

$x_1 = x_2$ then general

Gen Solution

$$b_1 x_1^n + b_2 n x_1^n$$

8.16 linear non-homogeneous recurr. relations

$$(*) \quad a_n = \underbrace{c_1 a_{n-1} + \ldots + c_k a_{n-k}}_{H} + \underbrace{g(n)}_{\text{new}}$$

Method:

1. Find general solution to homogeneous part H of equation

a_n^h

② Find <u>one</u> solution to (*)

a_n^p = particular solution

③ Add results

Key

How to find ?? $a_n = a_n^h + a_n^p$

④ Match to initial conditions

<u>Ex 0</u>: a) $a_n = 2a_{n-1} + 3a_{n-2} + 7$

b) $a_n = 2a_{n-1} + 3a_{n-2} + n^2$

c) $a_n = 2a_{n-1} + 3a_{n-3} + 5^n$

H

$\boxed{H}$ $a_n = 2a_{n-1} + 3a_{n-2}$

$x^n = 2x^{n-1} + 3x^{n-2}$

$x^2 = 2x + 3$

$x^2 - 2x - 3 = 0$

$(x-3)(x+1) = 0$

$x = 3, -1$

<u>Gen'l solution</u>: $a_n^h = A3^n + B(-1)^n$

How to find a_n^p ??

(e) <u>Guess</u> 7 const

$\boxed{\text{Guess } a_n^p = c}$

$c = 2c + 3c + 7 \Rightarrow$

$-4c = 7 \quad c = -7/4$

So $a_n = a_n^h + a_n^p =$

$A \cdot 3^n + B(-1)^n - 7/4$

(c) $a_n = 2a_{n-1} + 3a_{n-2} + 5^n$

<u>Guess</u>: $a_n^p = \boxed{c \cdot 5^n}$

$c \cdot 5^n = 2c \cdot 5^{n-1} + 3c5^{n-2} + 5^n$

$\div 5^n$ $\quad \frac{5^{n-1}}{5^n}$

$$c = 2c\frac{1}{5} + 3c\frac{1}{25} + 1 \qquad \cdot 25$$

$$25c = \underline{10c} + \underline{3c} + 25$$

$$12c = 25 \qquad c = \frac{25}{12}$$

so $a_n^p = \frac{25}{12} \cdot 5^n$

and

$$a_n = A \cdot 3^n + B(-1)^n + \frac{25}{12} 5^n$$

(b) $a_n = \underline{\underline{2a_{n-1}}} + \underline{\underline{3a_{n-2}}} + n^2$

Might guess $\underline{a_n^p = cn^2}$

Solve for c:

$$cn^2 = \underline{2c(n-1)^2 + 3c(n-2)^2 + n^2}$$

$$cn^2 = 2c(n^2 - \underline{\underline{2n}} + \underline{\underline{1}}) + 3c(n^2 - \underline{\underline{4n}} + \underline{\underline{4}})^2 + n^2$$

<u>Fails</u>!

Instead guess:

$$a_n^p = a + bn + cn^2$$

Guess: $a_n = a + bn + cn^2$

substitute:

$$a + bn + cn^2 = 2\left(a + b(n-1) + c\underbrace{(n-1)^2}_{n^2-2n+1}\right) + 3\left(a + b(n-2) + c\underbrace{(n-2)^2}_{n^2-4n+4}\right) + n^2$$

Extract n^2 / n / n^0 = const

n^2: $cn^2 = 2cn^2 + 3cn^2 + n^2$

$$c = 2c + 3c + 1$$

$$-4c = 1 \qquad c = -1/4$$

n: $b = 2b - 4c + 3b - 12c$

$$-4b = -16c = -16\left(-\tfrac{1}{4}\right) = 4$$

$$-4b = 4 \Rightarrow b = -1,$$

$\underline{\underline{1}}$:

$$a = \underline{2a} - 2b + 2c$$
$$+ \underline{3a} - 6b + 12c$$

$$-4a = -\underline{2b} + 2c - \underline{6b} + 12c$$
$$= -8b + 14c$$

$$-4a = -8(-1) + 14\left(-\frac{1}{4}\right)$$

$$a = \frac{8 - \frac{7}{2}}{-4} = \frac{\left(\frac{9}{2}\right)}{-4} = -\frac{9}{8}$$

So $a_n^p = -\frac{9}{8} - n - \frac{1}{4}n^2$

and

$$a_n = A \cdot 3^n + B(-1)^n - \frac{9}{8} - n - \frac{1}{4}n^2$$

f) $\underline{\underline{a_n}} = 2a_{n-1} + 3a_{n-2} + 3^n$

Guess: $a_n^p = c \cdot 3^n$

This fails:

$$c3^n = 2c3^{n-1} + 3c3^{n-2} + 3^n$$

$$c = 2c\frac{1}{3} + 3c\frac{1}{9} + 1$$

$$c = \underbrace{\frac{2}{3}c + \frac{1}{3}c}_{} + 1$$

$$c = c + 1 \Rightarrow \boxed{0 = 1} \quad \times$$

Altered guess!

$$a_n = c \cdot n \cdot 3^n$$

(add n)

$$cn \cdot 3^n = 2c(n-1)3^{n-1} + 3c(n-2)3^{n-2} + 3^n$$

divide by 3^n

$c n = 2c(n-1)\left(\frac{1}{3}\right) + 3c(n-2)\frac{1}{9} + 1$

$9cn = 6c(n-1) + 3c(n-2) + 9$

$n: \; 9c = 6c + 3c$ ✓

$1: \; 0 = -6c - 6c + 9$

$12c = 9 \quad c = \frac{9}{12} = 3/4$

so $a_n^p = 3/4 \, n 3^n$

$a_n = A \cdot 3^n + B(-1)^n + 3/4 \, n \cdot 3$

How ~~many~~ to make the guess

<u>Thm</u>: If ~~F_n~~ =

$f_n = c_1 f_{n-1} + \dots + c_k f_{n-k} + \underline{\underline{F_n}}$

where $F_n = p(n) \cdot y^n$??

poly of degree t

Then

(1) If γ is not a root to char equation, then solution

$$f_n^p = (\underbrace{d_0 + d_1 n + \ldots + d_t n^{t-1}}_{\text{poly of degree } t})\gamma^n$$

(2) If γ is a root char ~~eq~~ equation of multiplicity m, then

$$f_n^p = n^m (d_0 + \ldots + d_t n^t)\gamma^n$$

Ex) Guess form of particular solution:

(a) $f_n = 7 f_{n-1} + \underline{3}$ $\gamma = 1$, $p(n) = 3$

guess: $f^p = d_0 = \text{const.}$