4/23 | Discrete 2 :

Last time Graphs

2+ verts

Trail : (open) walk

with dis tinct edges

Euler circuit for G

uses all edges.

Thm G has Euler circuit

$\Downarrow$

G connected and

each vertex has even degree.

<u>Ex</u> K_4, $K_{5,10}$ do <u>not</u> have

but $\boxed{K_5}$, Q_4 does have Euler circuit.

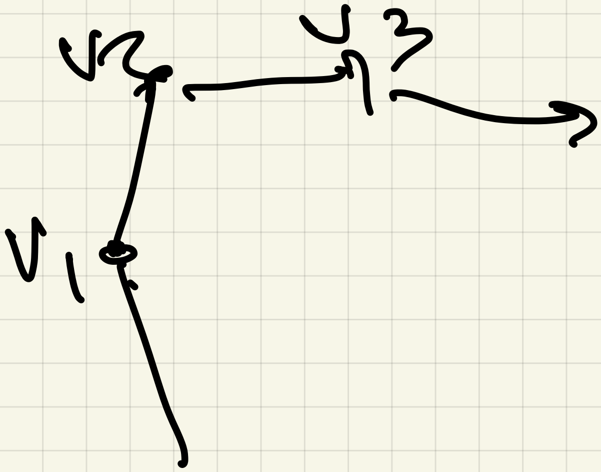
<u>Proof Thm</u> $(\Downarrow)$

observed in class

$(\Uparrow)$

Claim 1 If vertices of G have even degree and $v \in V_G$ has $\deg v > 0$, then there's a circuit C containing v, (G need ~~not~~ not be connected)

Pf: Suppose $v_1 = v$ has $\deg v > 0$

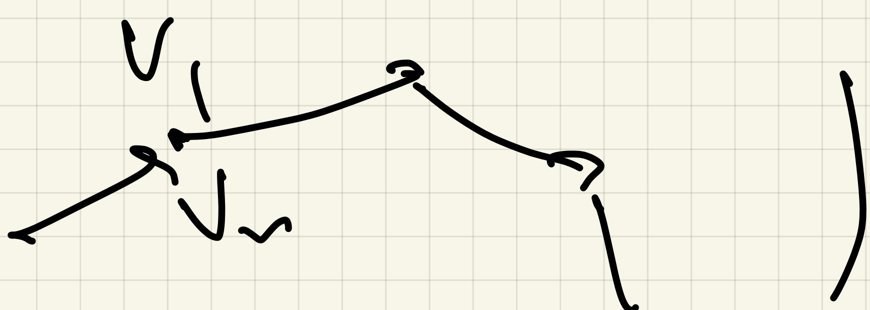


Make trail starting at v_1

$v_1 \, v_2 \, v_3 \, \cdots \, v_n$

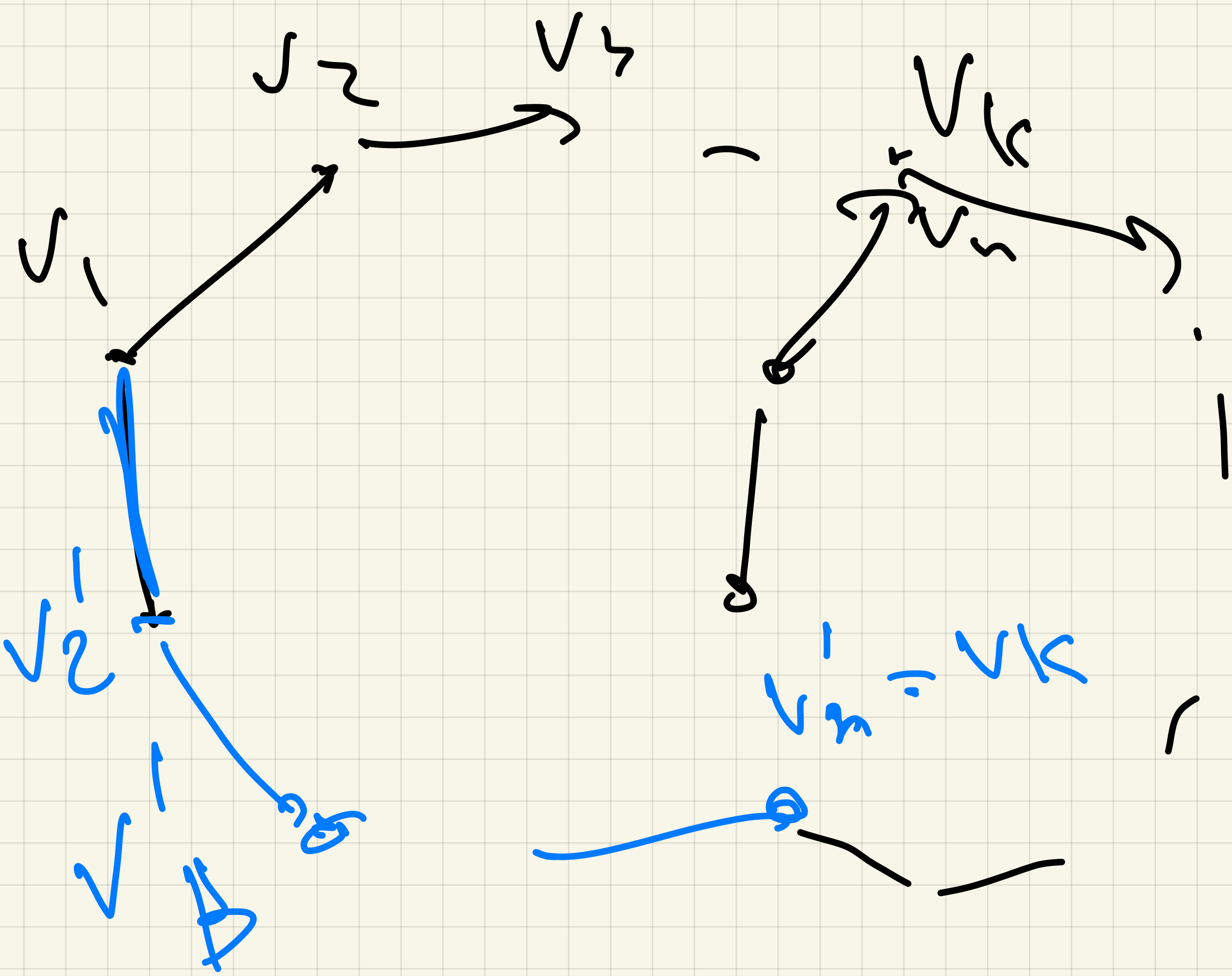
eventually, v_n is among $v_1 \sim v_{n-1}$

Case 1: $v_n = v_1$ Then



$\langle v_1, \ldots, v_n \rangle$ is circuit.

<u>Case 2</u> $v_n = v_k,\ k > 1$



Then

$\langle v_1', v_2', v_3' \ldots v_n' = v_k, v_{k-1}$

$v_{k-2} \ldots v_1 \rangle$

<u>Claim 2</u>: If G connected, $\deg v$ even $\forall v \in V_G$ then G has a Euler circuit.

(B) <u>Algorithm</u>:

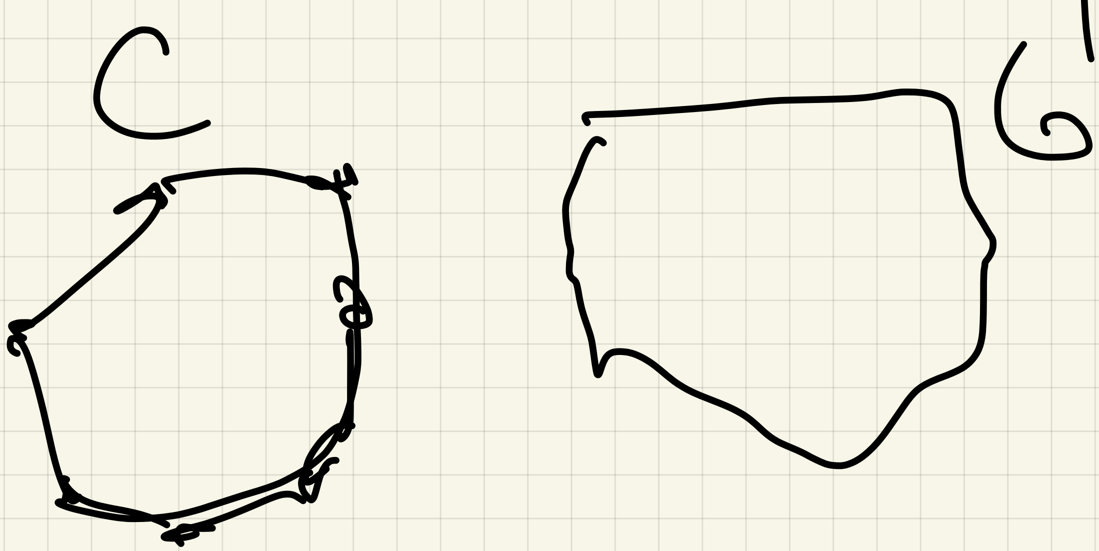
① Pick any circuit C in G

② Remove edges of C from G and isolated vertices to get G'

③ Choose vertex w in $G' \cap C$

Why does w exist?

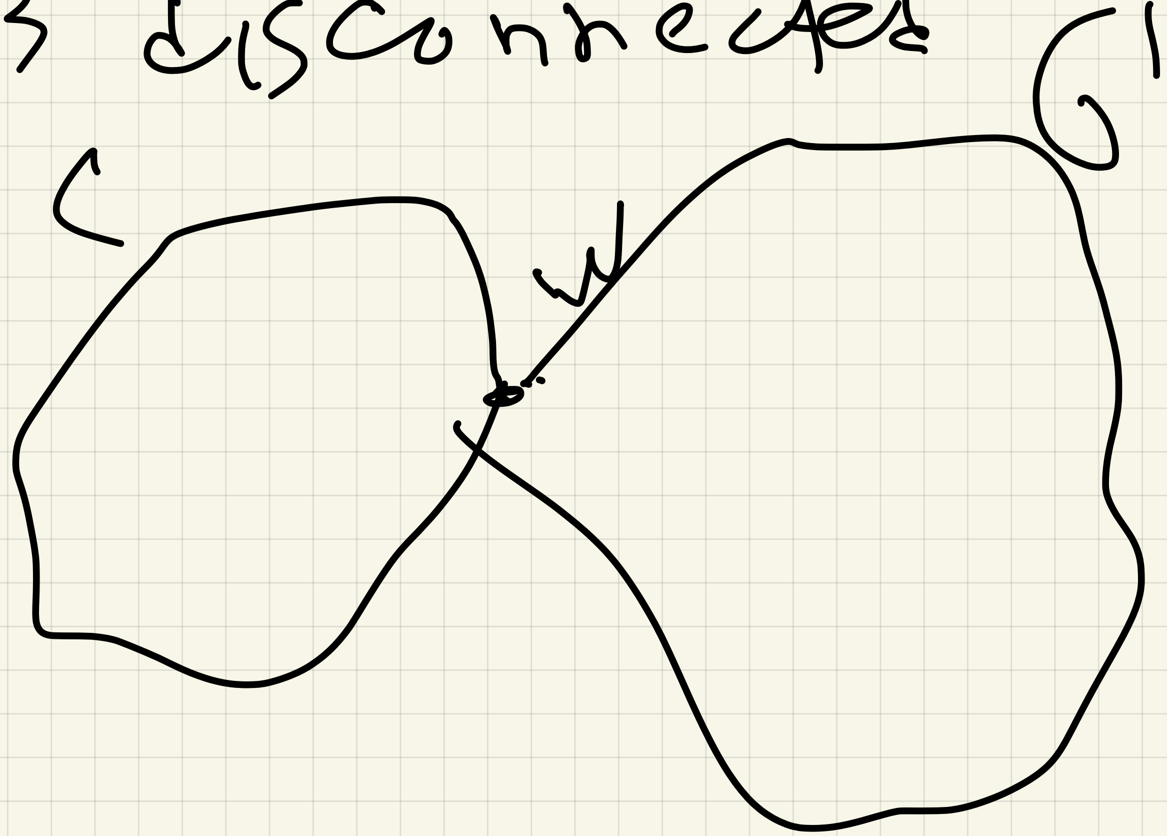
If no such w



Then $G = C \cup G'$

is disconnected

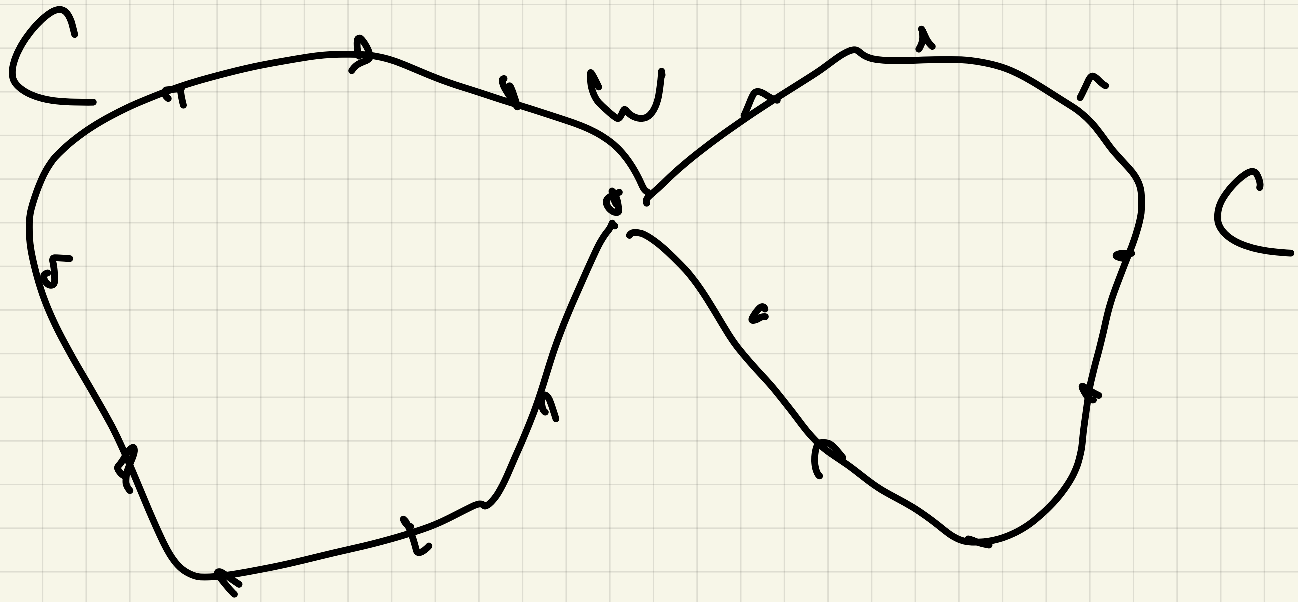
(4)



<u>Claim!</u> $\Rightarrow$ $\exists$ circuit C'

in G' contains w.

(5)



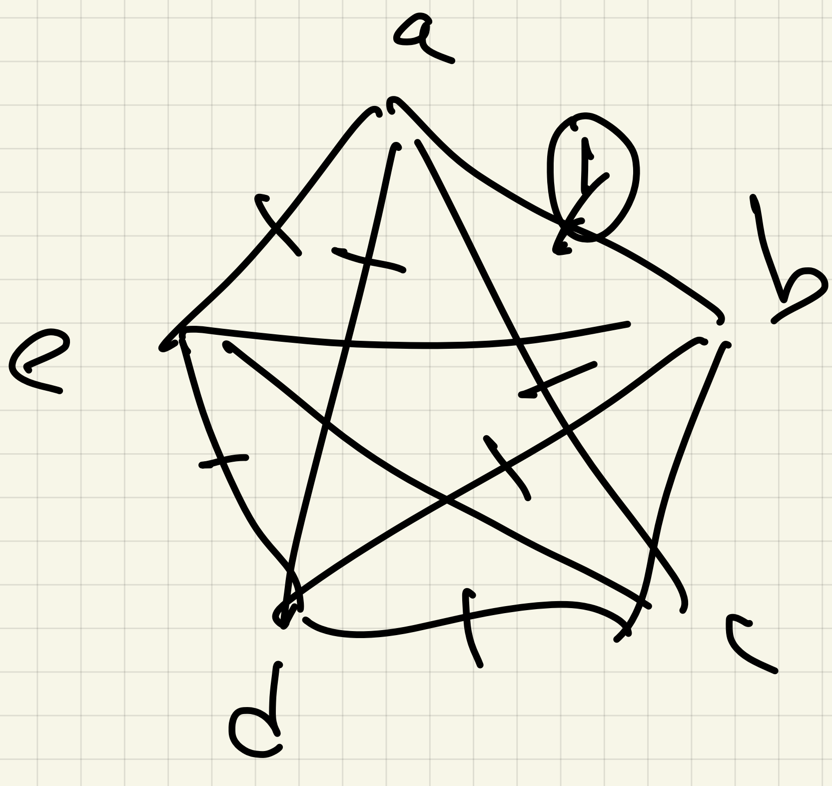
$C \cup C'$ is a circuit.

If $C \cup C' \neq G$,

Then continue.

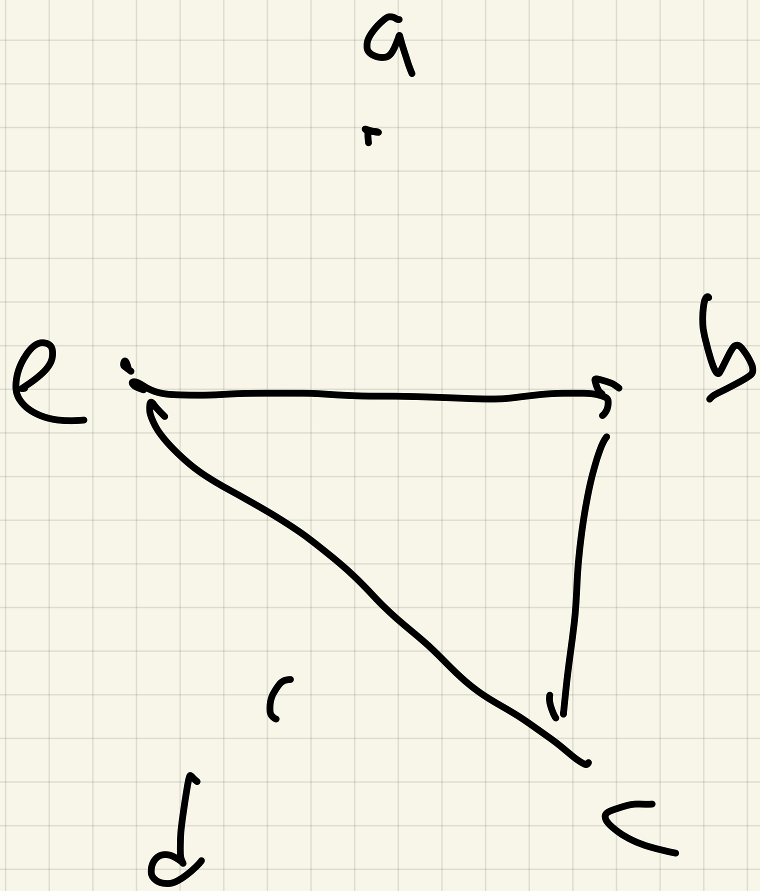
Ex1:

K_5

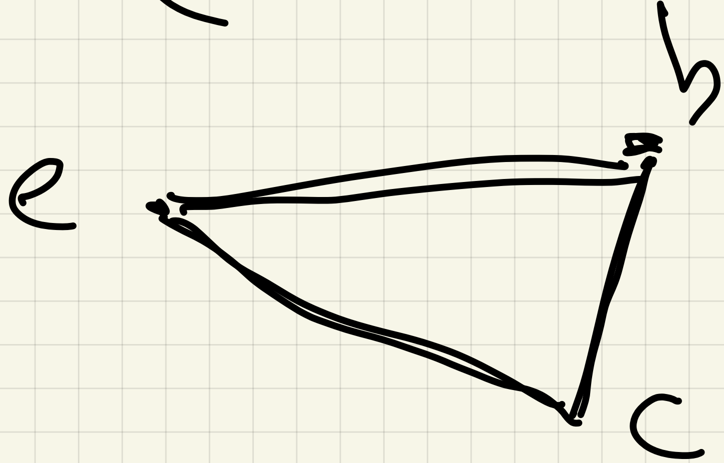


Take: C : (a, b, d, c, a, d, e, a)

$G - C$



G' :



take $w = b$ (in proof)

$C' \quad \langle \underline{b, c, e, b} \rangle$

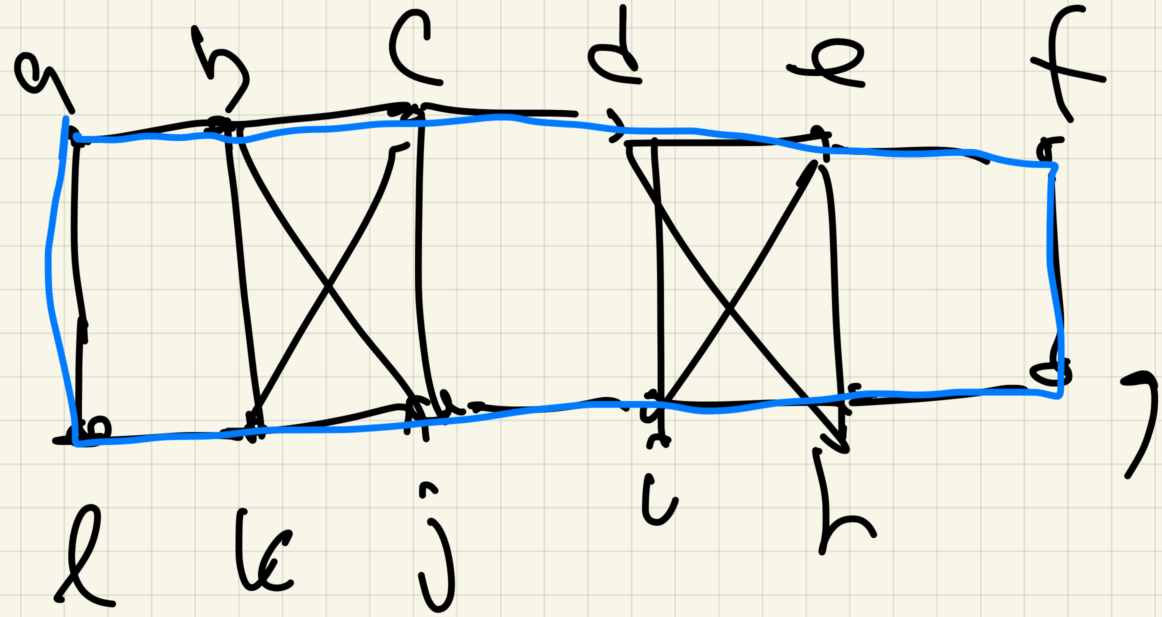
combine C & C' :

$\langle a, b, d, c, a, d, e, a \rangle$

$\langle a, \underbrace{b, c, e, b}_{C'}, d, c, a, d, e, a \rangle$

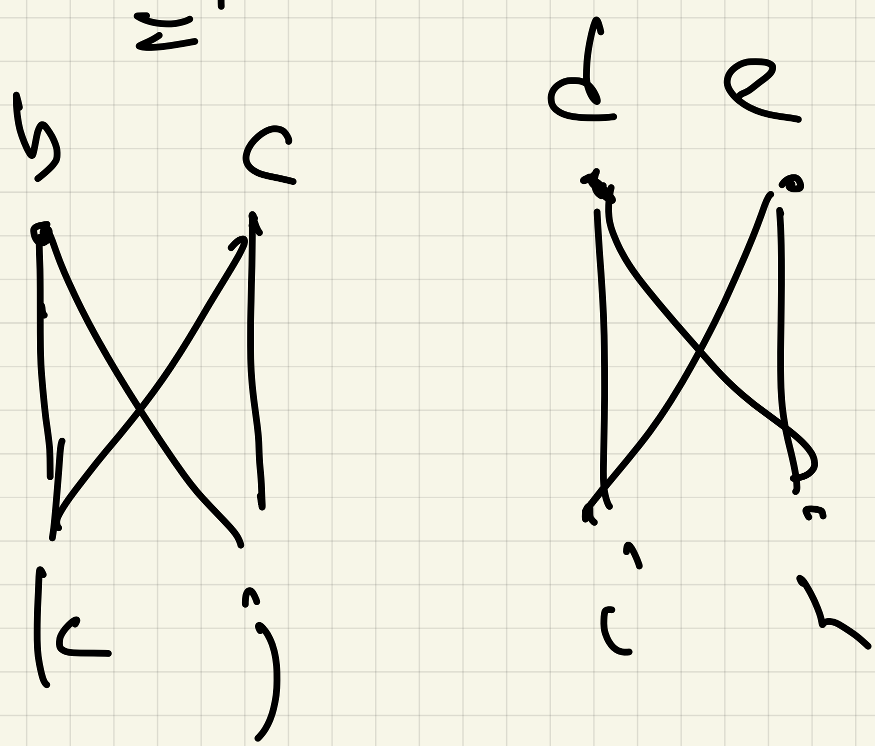
Ex 2

G:



$C : \langle a, \underline{b}, c, d, e, f, g, h, i, j, k, l, a \rangle$

G'



$\underline{w = b}$

$C' = \langle b, j, c, k, b \rangle$

$C'' = \langle \underline{\underline{d}}, h, e, i, d \rangle$

Insert into C:

so

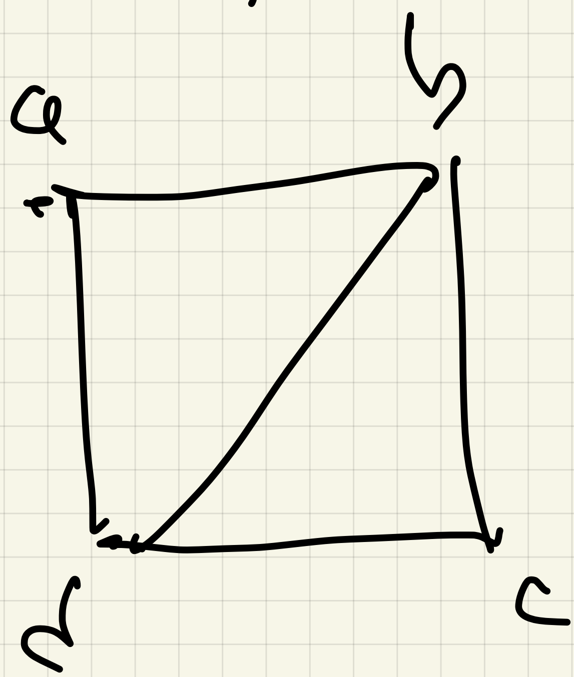
$C \cup C' \cup C'' =$

$$\langle a, \underbrace{b, j, c, k, b}_{C'}, c, \underbrace{d, h, e, i, d}_{C''}, e, f, g, h, i, j, k, l, a \rangle$$

<u>Defn</u>: An <u>Euler trail</u> is an (open) trail that

uses every edge

Ex



(d, a, b, c, d, b)

Theorem 2:

G has an (open) Euler trail $\iff$ 1 G connected and 2 There are exactly 2 vertices with odd degree.

($\Rightarrow$) like before

($\Leftarrow$) How to construct Euler trail:

Let v, w be the two vertices with odd degree

Add edge $\langle v, w \rangle$ to G to G'

all verts have <u>even</u>

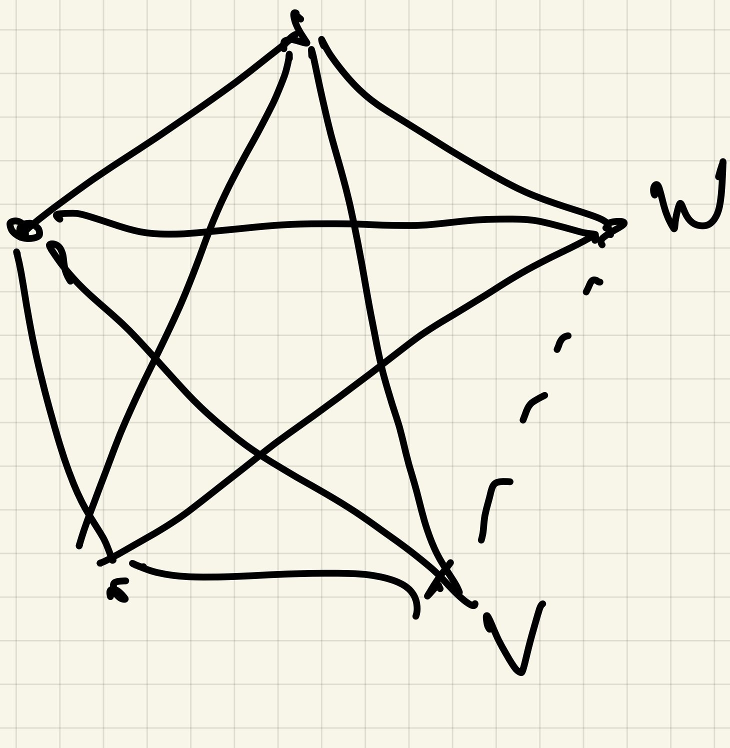
degree, G' has an Euler circuit:

$\langle v_1, w, v_2, v_m \quad v_n, v_1 \rangle$

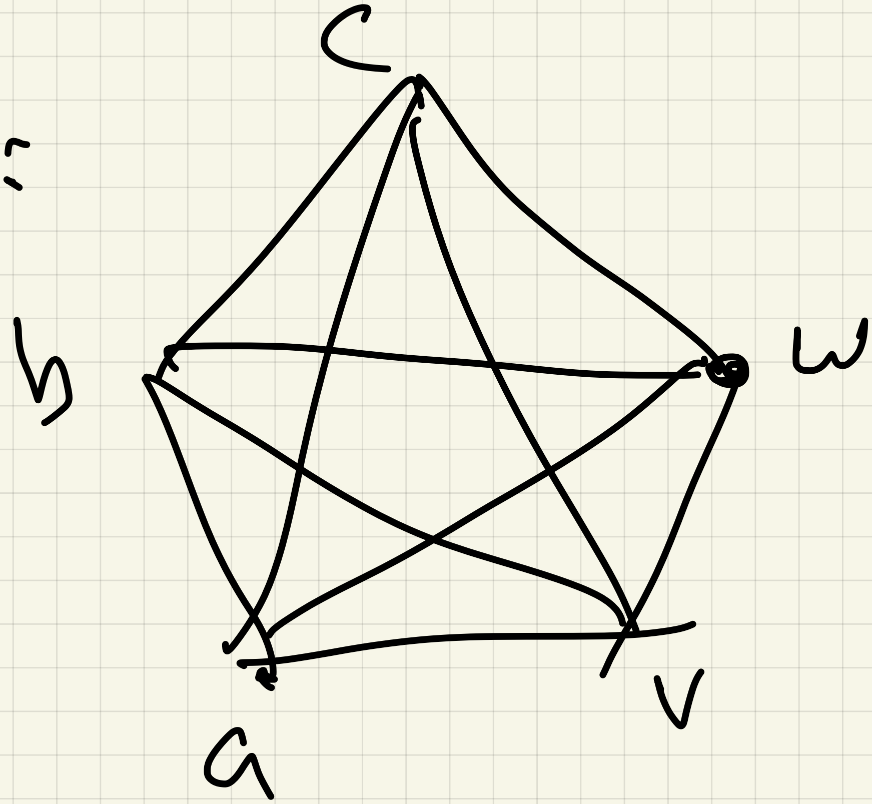
$\langle w, v_2, \ldots \quad v_n, v_1 \rangle$

Euler trail

Ex

$G =$ 

G':

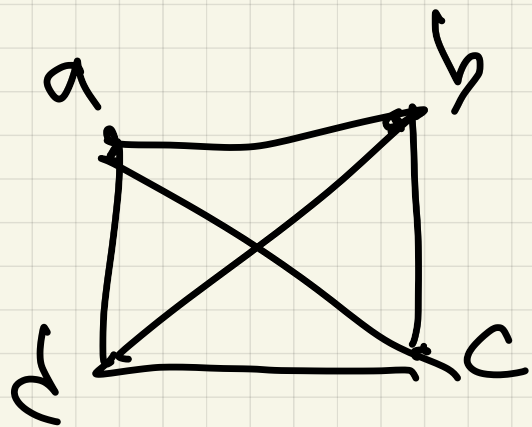


Euler circuit: G':

$\langle \cancel{w}, v, a, b, c, w, a, c, v, b, w \rangle$

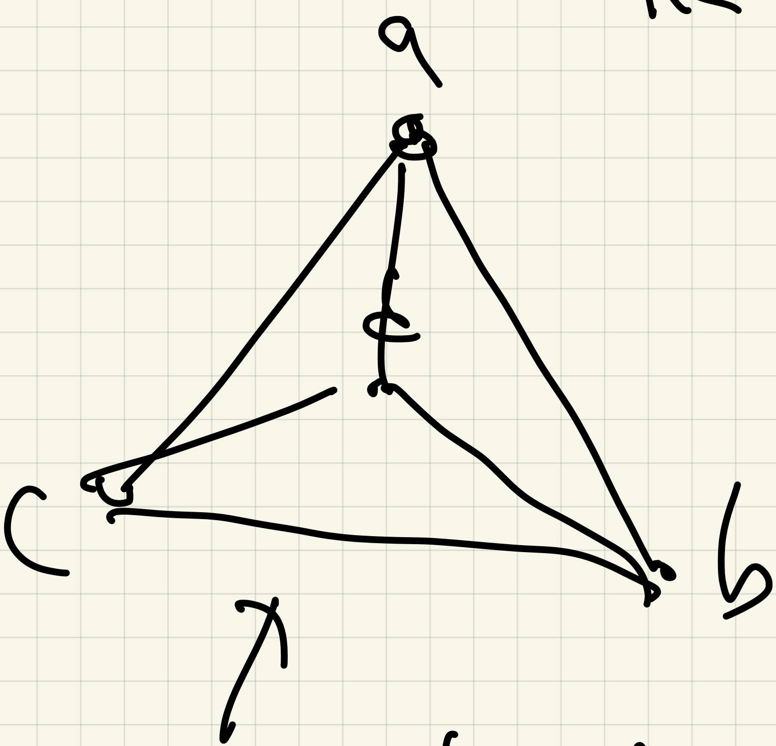
§13.8 <u>Planar Graphs</u>:

<u>Ex 1</u> K_4



12

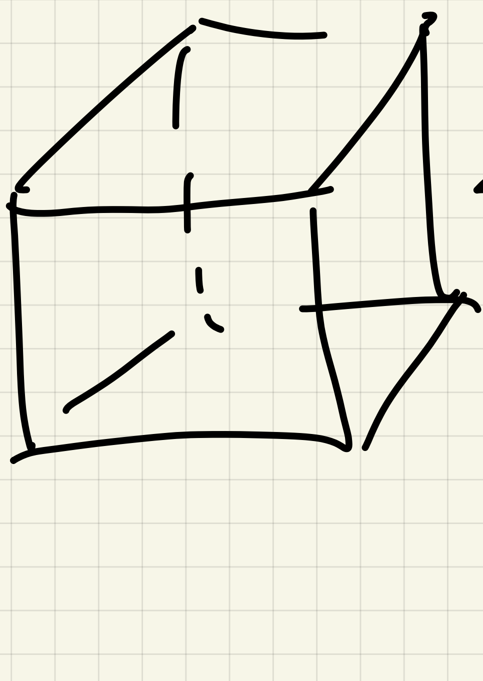
K_4



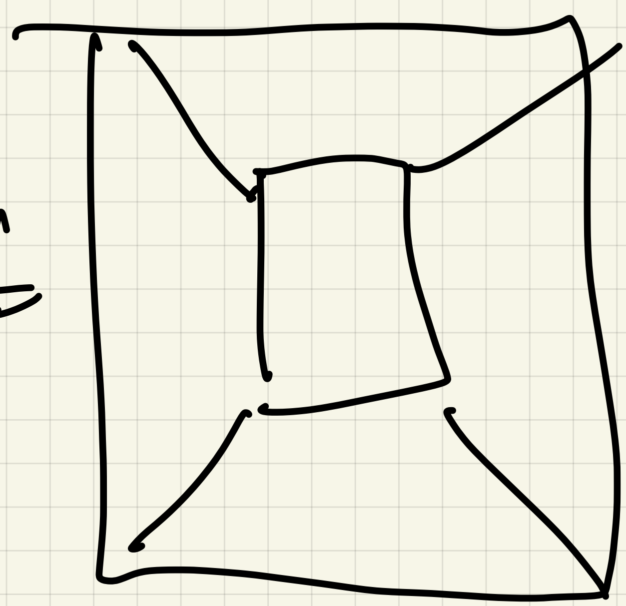
drawn in plane with no crossings.

<u>Ex 2</u> Q_3

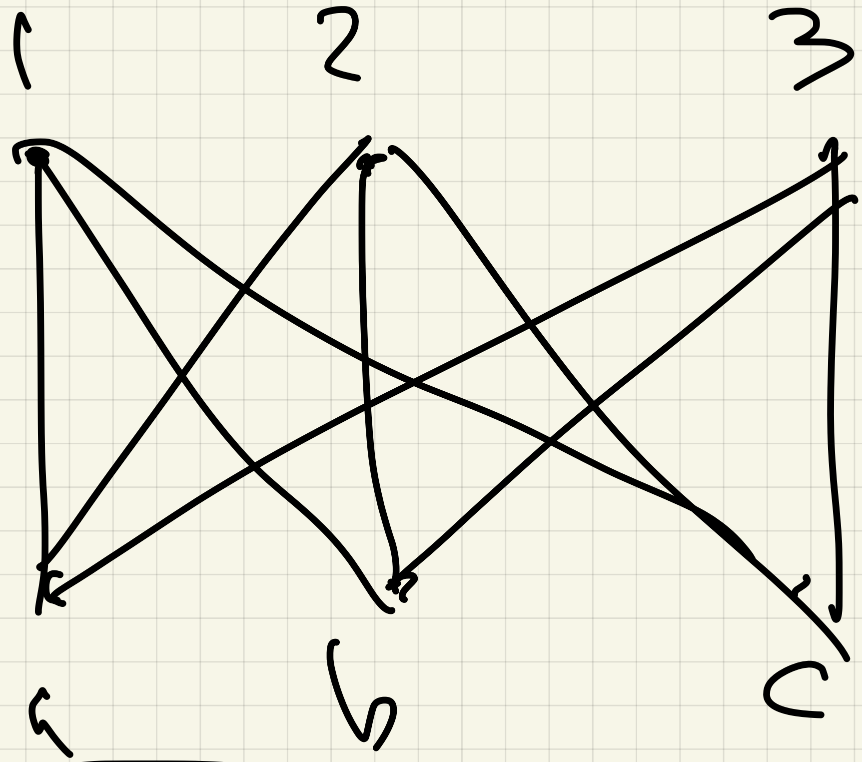
Q_3

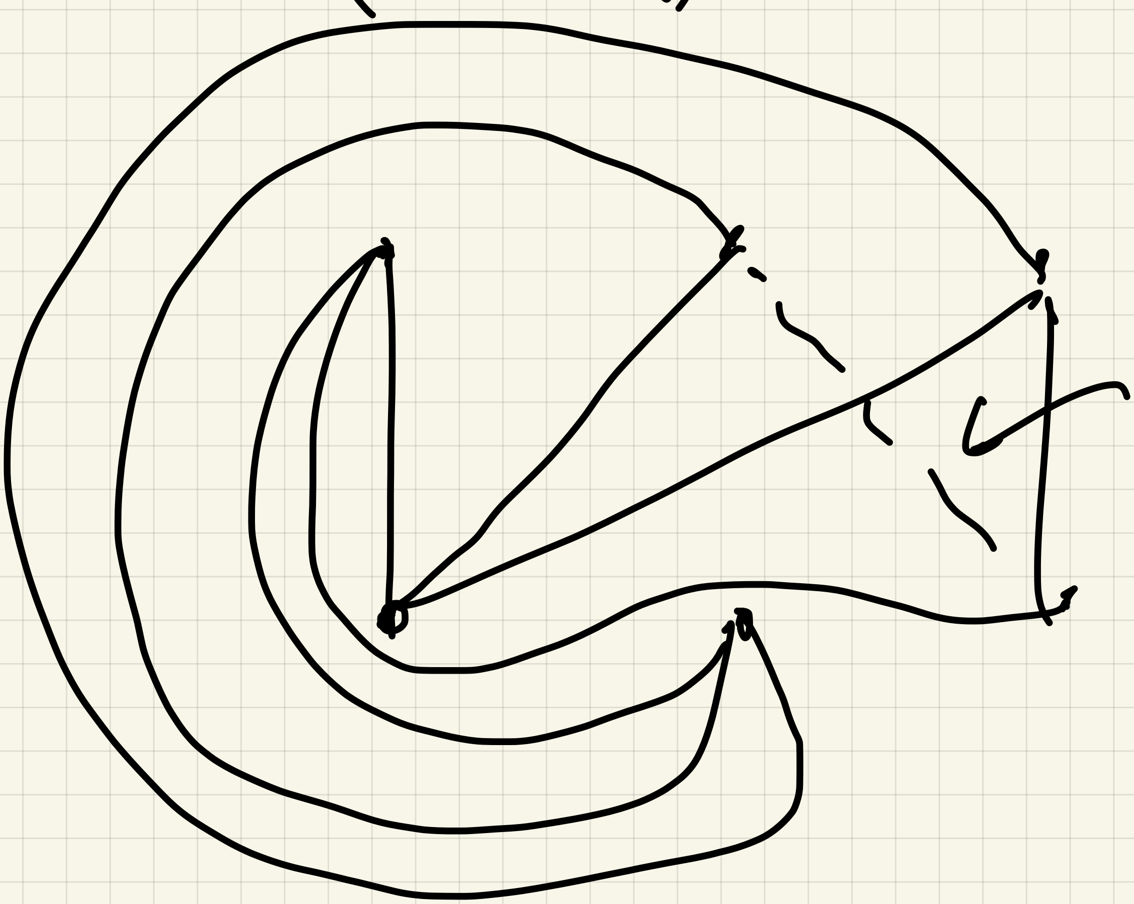


$\cong$



Ex3 $K_{3,3}$

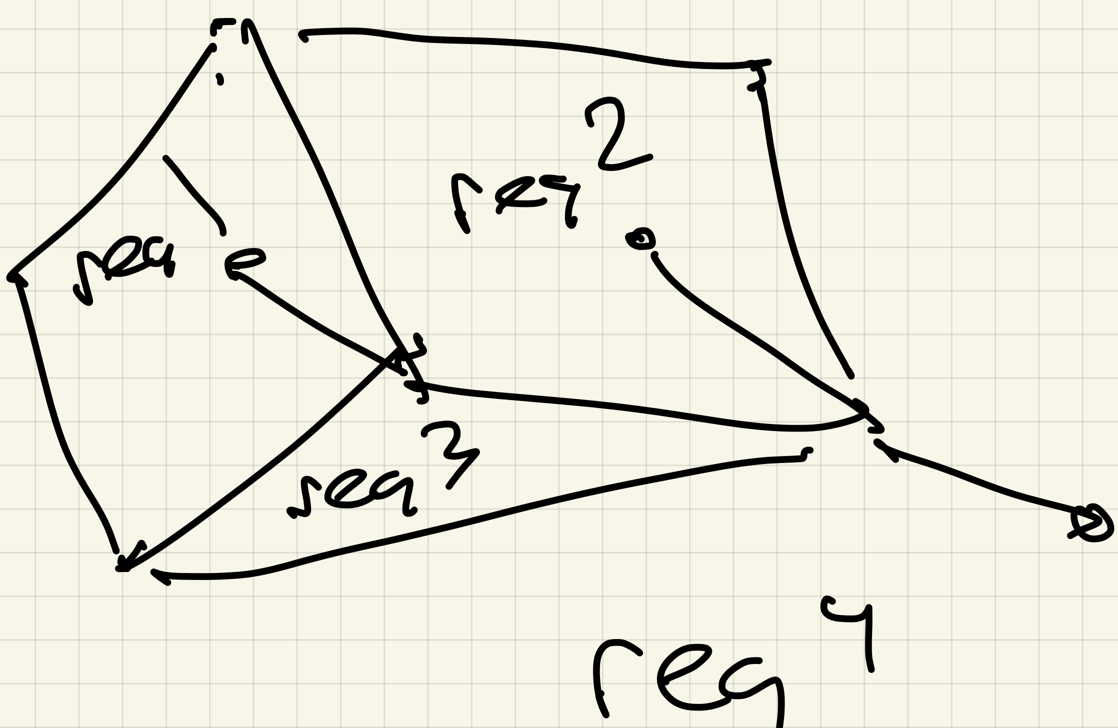




can't add w/out crossings

A plane embedding of a <u>connected</u> graph breaks complement of plane into regions

<u>Ex</u>



<u>Euler Identity</u>:

If G planar,

$n = |V_G|$, $m = |E_G|$

$r = \#$ regions

<u>Then</u>: $\underline{n - m + r = 2}$

<u>InEx)</u>

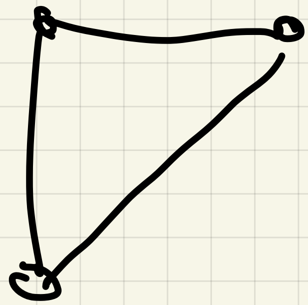
$n = |V_G| = 9$

$m = |E_G| = 11$

$r = 4$

$9 - 11 + 4 = 2$

<u>Idea</u>:



$n=2$
$m=1$
$r=1$
$2-1+1=2$ ✓

$n'=3$
$m'=2$
$r'=1$

Build graph one edge at a time,

$n-m+r$ stays same.

Degree of a region = # edges bounding region.