9/16 Discrete

Quiz 4

For x an integer,

$\underbrace{\boxed{x \text{ is odd}}}_{A}$ iff $\underbrace{\boxed{5x \text{ is odd}}}_{B}$.

iff: if and only if

$A \Rightarrow B$ Let x be an odd integer.

Then there is an integer c so that $x = 2c+1$

Therefore $5x = 5(2c+1) =$

$10c + 5 = 2(5c+2) + 1$

since c in $\mathbb{Z}$, so $5c+2$ in $\mathbb{Z}$

so $5x$ is odd.

$B \Rightarrow A$ Let $5x$ be odd.

Then there's an integer d in $\mathbb{Z}$

so that $5x = 2d+1$

Therefore

$$x = \underline{5x} - 4x$$
$$= 2d+1 - 4x$$
$$= 2(d-2x) + 1$$
$$= 2(d-2x) \quad + 1$$

is an integer b/c d, x are integer

$\therefore$ x is odd

<u>Last time</u> Counting lists
multiplication principle

If you make lists $(a_1, \ldots a_n)$ where a_i are chosen from

n possibilities, then

\# lists is n^r if allow repetition

is $n(n-1)(n-2)\cdots(n-r+1)$ if no repetition

with factorials, $\dfrac{n!}{(n-r)!}$

product notation:

$$\prod_{k=1}^{n} a_k = a_1 a_2 \cdots a_n$$

§ 10 Sets

Defn: A set is an unordered repetition-free collection of objects.

Ex1 $S = \{A, B, C\}$

$\{B, A, C\} = \{B, B, C, A\}$

(not equal as lists)

The <u>cardinality</u> of a set S is the number of objects in it. <u>Notation</u> $|S|$

<u>Ex</u> $|\{A, B, C\}| = 3$

$|\{B, B, C, A\}| = 3$

$|\mathbb{Q}| = \infty$ $\quad |\{\mathbb{Q}\}| = 1$

$|\{2, \mathbb{Q}, E, 2, cc\}| = 4$

counted as one object

$\mathbb{Q} = \{ \frac{m}{n} \mid m, n \text{ in } \mathbb{Z}, n \neq 0 \}$

Notation: If x is an element of set A, write $x \in A$

Also $x \notin A$ means $\neg(x \in A)$

Ex 3 $\frac{2}{3} \in \mathbb{Q}$, $\frac{2}{3} \in \mathbb{R}$

$\frac{2}{3} \notin \mathbb{N}$

Usually, sets come naturally from bigger sets via some condition

Set-builder notation: $\quad$ / condition

$$E = \{x \in \mathbb{Z} : 2 \mid x\}$$

$$= \{x \in \mathbb{Z} : x \text{ is even}\}$$

$$= \{2y : y \in \mathbb{Z}\}$$

$$P = \{p \in \mathbb{N} : p \text{ is prime}\}$$

$$= \{2, 3, 5, 7, 11, 13, 17, 19 \ldots\}$$

$$S = \{x^2 : x \in \mathbb{Z}\} = \{0, 1, 4, 9, 16, \ldots\}$$

Squares

<u>Containment and equality of sets</u>

If A, B are sets, then A is a <u>subset</u> of B if every element of A is also an element of B,

i.e., $\boxed{x \in A \Rightarrow x \in B}$

__Notation__ $A \subseteq B$

$A \subsetneq B$ if $A \subseteq B$ but $A \neq B$

(__Many books__ $A \subset B$ subset means $A \subsetneq B$)

The __empty set__ has no __elements__:

__Notation__: $\emptyset = \{\}$

$$|\emptyset| = 0$$

Two sets A and B are __equal__ if $A \subseteq B$ and $B \subseteq A$.

Ex) $A = \{x \in \mathbb{Z} \mid 3 \mid x\}$

$B = \{x \in \mathbb{Z} \mid 15 \mid x\}$

Is $A \subseteq B$?, Is $B \subseteq A$?,

Is $A = B$?,

Claim $B \subseteq A$

Proof Need to show

that $x \in B \Rightarrow x \in A$

Let $x \in B$.

Then $15 \mid x$, so there's an

integer c such that

$x = 15c$

therefore $x = 15c = 3(5c)$

since $5c \in \mathbb{Z}$ (b/c $c \in \mathbb{Z}$),

$\therefore 3 \mid x$ so $\cancel{x} \in A$.

Claim $A \not\subseteq B$

Example $x = 3 \in A$
but $3 \notin B$.

Ex 2 $E = \{ x \in \mathbb{Z} : 2 | x \}$

$C = \{ 10a + 14b : a, b \in \mathbb{Z} \}$

Is $E \subseteq C$?, $\boxed{C \subseteq E?}$ ✓ $C = E$?

Claim: $C \subseteq E$

proof Let $c \in C$.

Then there are $a, b \in \mathbb{Z}$ so that $c = 10a + 14b =$

$2(5a + 7b)$

$a, b \in \mathbb{Z} \Rightarrow 5a + 7b \in \mathbb{Z}$

so $2 | c$ $\therefore c \in E$

So $C \subseteq E$,

In fact, $C = E$

Claim $E \subseteq C$

[Note $2 = 10 \cdot 3 + 14(-2)$]

Let $e \in E$. Then there is an $q \in \mathbb{Z}$ such that

$$e = 2q =$$

$$= (10 \cdot 3 + 14(-2))q$$

$$= 10(3q) + 14(-2q)$$

$3q, -2q \in \mathbb{Z}$ (b/c $q \in \mathbb{Z}$)

so $e \in C$ ✓

Ex 3 Find the cardinality of set S

(a) $S = \{x \in \mathbb{Z} : -3 \leq x \leq 3\}$

$|S| = 7$

(b) $S = \{x \in \mathbb{R} : -3 \leq x \leq 3\}$

$|S| = \infty$

(c) $S = \{x \in \mathbb{Z} : -3 \leq x^2 \leq 3\}$

$\Downarrow$ $\quad |S| = 3$

$\{-1, 0, 1\}$

(d) $S = \{z \in \mathbb{Z} \mid 5 | z \text{ and } |z| < 101\}$

$\Downarrow$

$\{0, \pm 5, \pm 10, \pm 15 \, \_ \, \_ \, \_ \, \pm 100\}$

$|S| = 41$

(e) $S = \{n \in \mathbb{N} : n! < 5000\}$

$4! = 4 \cdot 3 \cdot 2 = 24$

$5! = 120$

$$6! = 720$$

$$7! = 5040$$

$$\underbrace{0, 1, 2, \ldots, 6}_{7} \qquad |S| = 7$$

<u>Ex 4</u> $A = \{a, b, c, d\}$

List the subsets of A,

$\{a\}$ $\{b\}$ $\{c\}$ $\{d\}$

$\{b, c, d\}$ $\{a, c, d\}$ $\{a, b, d\}$ $\{a, b, c\}$

$\{a, b, c, d\}$,

$\{a, b\}$ $\{a, c\}$ $\{a, d\}$,

$\{b, c\}$ $\{b, d\}$, $\{c, d\}$

$\emptyset$

So there are $16 = 2^4$ subsets