10/9 | Discrete Quiz 9

$A = \{n \in \mathbb{Z} : |n| \leq 9 \wedge 3 | n\}$

(a) $= \{0, \pm 3, \pm 6, \pm 9\}$

$|A| = 7$

$B = \{n \in \mathbb{Z} \mid |n| \leq 9, n \text{ odd}\}$

(b) $= \{\pm 1, \pm 3, \pm 5, \pm 7, \pm 9\}$

$|B| = 10$

(c) $A \cap B = \{\pm 3, \pm 9\}$

(d) $A \cup B = \{0, \pm 1, \pm 3, \pm 5, \pm 6, \pm 7, \pm 9\}$

(e) $A - B = \{0, \pm 6\}$

(f) $B - A = \{\pm 1, \pm 5, \pm 7\}$

Last time

$$\left.\begin{aligned} P(n,k) &= n_k \\ C(n,k) &= \binom{n}{k} \end{aligned}\right\}$$

Principle of Inclusion Exclusion

(PIE)

$$|A \cup B| = \underline{|A| + |B|} - |A \cap B| \qquad 3$$

$$|A \cup B \cup C| = |A| + |B| + |C|$$
$$- |A \cap B| - |A \cap C| - |B \cap C|$$
$$+ |A \cap B \cap C| \qquad 7$$

$$|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D|$$
$$- |A \cap B| - |A \cap C| - |A \cap D| -$$
$$|B \cap C| - |B \cap D| - |C \cap D|$$
$$+ |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D|$$
$$+ |B \cap C \cap D| \qquad 15$$
$$- |A \cap B \cap C \cap D|$$

Ex | A Tom Brady fan wants the number 12 to in his

5 digit PIN, from
$\{0, 1 \ldots 9\}$

$\underline{1}\ \underline{2}\ \_\ \_\ \_$: A_1 12 1/2

$\_\ \underline{1}\ \underline{2}\ \_\ \_$: A_2 12 2/3

$\_\ \_\ \underline{1}\ \underline{2}\ \_$: A_3 12 3/4

$\_\ \_\ \_\ \underline{1}\ \underline{2}$ A_4 12 4/5

<u>Want:</u> $|A_1 \cup A_2 \cup A_3 \cup A_4|$

$|A_1| = |A_2| = |A_3| = |A_4| = 10^3$

$|A_1 \cap A_2| = |A_2 \cap A_3| = |A_3 \cap A_4| = 0$

$|A_1 \cap A_3| = 10$ $|A_2 \cap A_4| = 10$

$|A_1 \cap A_4| = 10$

$|A_i \cap A_j \cap A_k| = 0$ $|A_1 \cap A_2 \cap A_3 \cap A_4| = 0$

$$10^3 + 10^3 + 10^3 + 10^3$$
$$- 10 - 10 - 10 =$$
$$4000 - 30 = 3970$$

Ex2 How many ways to form a 6 letter string from letters a, b, c, d if each letter must appear at least once?

Know: all strings 4^6

We want remove those with

no a $\quad B_a$
no b $\quad B_b$

no c $\qquad B_c$

no d $\qquad B_d$

<u>Answer</u> : $4^6 - |B_a \cup B_b \cup B_c \cup B_d|$

$|B_a| = 3^6 = |B_b| = |B_c| = |B_d|$

$|B_a \cap B_b| = 2^6 = |B_a \cap B_c| =$
$|B_a \cap B_d| = |B_b \cap B_c| =$
$|B_b \cap B_d| = |B_c \cap B_d|$.

$|B_a \cap B_b \cap B_c| = 1$ $\qquad$ dddddd

$= |B_a \cap B_b \cap B_d| = |B_a \cap B_c \cap B_d| =$

$|B_b \cap B_c \cap B_d|$

$|B_a \cap B_b \cap B_c \cap B_d| = 0$

$|B_a \cup B_b \cup B_c \cup B_d| = 4 \cdot 3^6 -$
$6 \cdot 2^6 + 4 \cdot 1^6 - 0$

So final answer

$$4^6 - 4\cdot 3^6 + 6\cdot 2^6 - 4\cdot 1^6 + 0$$

$$= 1{,}560$$

§20 More proof technique

In §5, learned how to prove

If A then B

$\boxed{A \Rightarrow B,}$ $\qquad A \Leftrightarrow B$

Observe

$$x \to y = \neg x \vee y = y \vee \neg x$$

$$= \neg(\neg y) \vee \neg x$$

$$= \neg y \to \neg x$$

Logically equivalent

Defn The contrapositive of

$A \Rightarrow B$ is $\neg B \Rightarrow \neg A$

Ex1 If $\overbrace{\text{it rains}}^{A}$, then Jane

(a) takes her umbrella

(b) If Jane does not take her umbrella, then it does not rain

Notice: this not

(c) If Jane takes her umbrella then it rains

NOT same

Ex2 (a) If $x^2+1=0$, then

$\boxed{x \text{ is not a real number}}$

Contrapositive:

(b) If x is a real number,

then $x^2+1 \neq 0$

Proof: Let x be a real number,

then $x^2 \geq 0$, therefore

$x^2+1 \geq 1 > 0$, $x^2+1 > 0$

$\Rightarrow x^2+1 \neq 0$,

Ex3 Let $x \in \mathbb{Z}$. If x^3 is not

even, then neither is x.

Contrapositive:

Let $x \in \mathbb{Z}$, If x is even, then x^3 is even.

<u>proof</u>: Let x be even, then $\exists c \in \mathbb{Z} \therefore x = 2c$

$$x^3 = 8c^3 = 2(2c^3)$$

$$c \in \mathbb{Z} \Rightarrow 2c^3 \in \mathbb{Z} \therefore$$

$2 | x^3$ by definition,

so x^3 is even

<u>Ex 4</u> If $A \subseteq B$ then $A - B = \emptyset$

<u>proof</u>: We'll prove contrapositive;

$A - B \neq \emptyset$, then $A \not\subseteq B$.

Assume $A - B \neq \emptyset$.

then $\exists x \in A - B$.

i.e. $\exists x;\ x \in A$ but $x \notin B$.

so $A \not\subseteq B$ by definition

($\underline{\text{defn}}$: $x \in A \Rightarrow x \in B$
$A \subseteq B$)

<u>Proof by contradiction</u>:

<u>Observe</u>:

$$(x \wedge \neg y) \to F \equiv \neg(x \wedge \neg y) \vee F \equiv$$

$$\neg(x \wedge \neg y) \overset{\text{DeMorgan}}{=} \neg x \vee \neg(\neg y) \equiv$$

$$\neg x \vee y \equiv x \to y$$

<u>Conclusion</u>: to prove $A \Rightarrow B$

it's equivalent to prove

If A true and B is false

leads to a <u>contradiction</u>.

Ex 5 If x^3 is not even, then x is not even.

Proof (BWOC = By way of contradiction)

Assume x^3 is not even and x is even

Since x is even, $\exists c \in \mathbb{Z}$: $x = 2c$, therefore $x^3 = 8c^3 = 2(4c^3)$ even,

$\therefore$ $\boxed{x^3 \text{ is not even and } x^3 \text{ is even}}$ $\Leftarrow$

contradiction

$\Rightarrow \Leftarrow$

Ex 6 No integer is both even and odd.

Proof: BWOC Let $x \in \mathbb{Z}$:

x is even and x is odd

Since x is even,

$\exists c \in \mathbb{Z} : \quad x = 2c$

Since x is odd,

$\exists d \in \mathbb{Z} : \quad x = 2d + 1$

It follows that

$$2c = x = 2d+1 \implies$$

$$2c = 2d + 1 \implies$$

$$1 = 2c - 2d = 2(c-d)$$

and $\frac{1}{2} = (c-d)$, $c, d \in \mathbb{Z}$

This ~~is~~ is a contradiction

b/c $\frac{1}{2} \notin \mathbb{Z}$,