10/7/ Discrete:

1. $\exists x \in \mathbb{N}\ \exists y \in \mathbb{N}\ y = x^2$ (T)

$\neg(\exists x \in \mathbb{N}\ \exists y \in \mathbb{N}\ y = x^2)$

$\forall x \in \mathbb{N}\ \neg\exists y \in \mathbb{N}\ y = x^2$

$\forall x \in \mathbb{N}\ \forall y \in \mathbb{N}\ \neg(y = x^2)$

$\rightarrow\ \forall x \in \mathbb{N}\ \forall y \in \mathbb{N}\ y \neq x^2$

2. $\exists x \in \mathbb{N}\ \forall y \in \mathbb{N}\ x = y^2$ (F)

negate

$\forall x \in \mathbb{N}\ \exists y \in \mathbb{N}\ x \neq y^2$

3. $\forall x \in \mathbb{N}\ \exists y \in \mathbb{N}\ x = y^2$ (F)

negate:

$\exists x \in \mathbb{N}\ \forall y \in \mathbb{N}\ x \neq y^2$

$x = 2 \quad y^2 = 2$

4. $\forall x \in \mathbb{N}\ \exists y \in \mathbb{N}\ y = x^3$ T

negate:

$\exists x \in \mathbb{N}\ \forall y \in \mathbb{N}\ y \neq x^3$

Last time

Counting Principles

- # ways to make a list of length k from set of size n is n^k

$$\underbrace{(\_\ \_\ \_\ \_\ \_)}_{k}$$

- # lists of length k with no repetitions is

$$n(n-1)\ldots(n-k+1) = \frac{n!}{(n-k)!} = n_k =$$

$P(n,k)$ = # Permutations

subsets of size k from set of size n is

$$\binom{n}{k} = C(n,k) = \frac{P(n,k)}{k!} = \frac{n!}{(n-k)!\,k!}$$

"n choose k" "Combination"

<u>Consequences:</u>

$$1+2+\dots+n = \binom{n+1}{2} = \frac{n(n+1)}{2}$$

$$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\dots+\binom{n}{n} = 2^n$$

<u>Ex)</u>: A 6 card hand is dealt from deck of cards

(a) How many hands are possible

$$\boxed{\binom{52}{6}} = \frac{52!}{46!\,6!}$$

(b) How many are all hearts

$$\binom{13}{6}$$

(c) How many all one suit?

$$\binom{4}{1}\cdot\binom{13}{6}$$

$$\frac{4!}{3!\,1!} = \frac{4\cdot\cancel{3\cdot2\cdot1}}{\cancel{3\cdot2\cdot1}\cdot1} = 4$$

suit

(d) How many have

3 of one "number"

2 of another number

1 of another number

Eq. (2♢, 2♡, 2♣, 7♡, 7♠, J♢)

choices:

1st number 13 $\binom{13}{1}$

suits 1st number $\binom{4}{1} = 4$

2nd number 12

suits 2nd number $\binom{4}{2}$

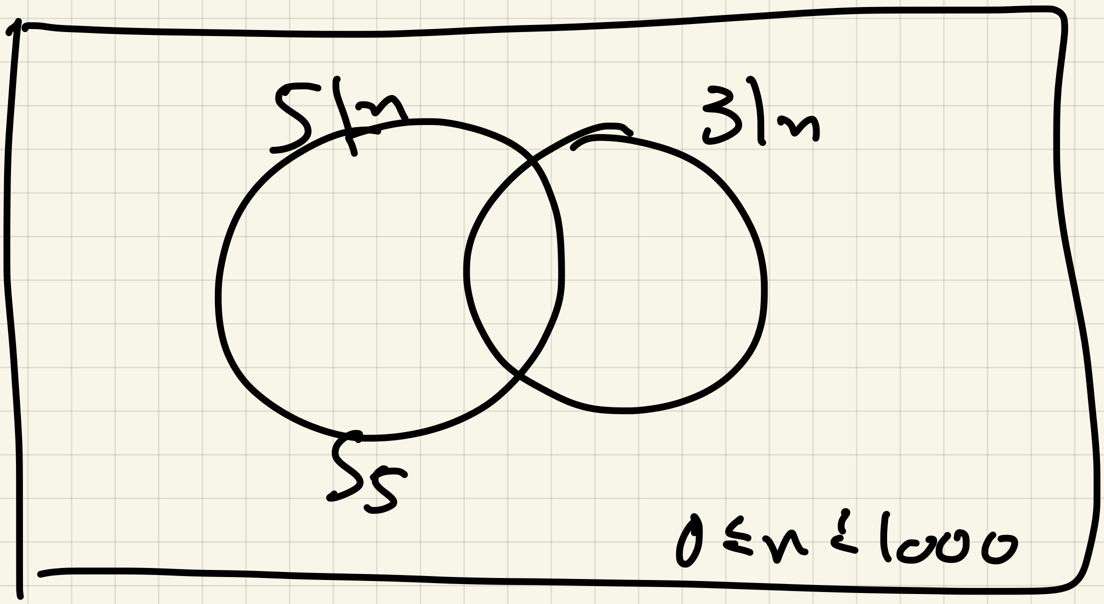
last card 44

$\binom{11}{1}\binom{4}{1} = 11 \cdot 4 = 44$

$\underline{13 \cdot 4} \cdot 12 \cdot 6 \cdot 11 \cdot 4 =$

$\binom{7}{2}$

Ex2 $S = \{n \in \mathbb{N} : n \leq 1000,$
$5 | n \text{ or } 3 | n\}$



$S_5 = \{n \in \mathbb{N} : n \leq 1000, 5 | n\}$

$S_3 = \{n \in \mathbb{N} : n \leq 1000, 3 | n\}$

$S_3 \cap S_5 = \{n \in \mathbb{N} : n \leq 1000,$
$3 | n \text{ and } 5 | n\}$

$S_{15} = \{n : n \leq 1000, 15 | n\}$

$$S = S_3 \cup S_5$$

$$|S| = |S_3 \cup S_5| = |S_3| + |S_5| - |S_{15}|$$

$$|S_5| = \{n \in \mathbb{N} : n \leq 1000, 5 \mid n\}$$

$$\quad\quad\quad = \{5k : k \in \mathbb{N}, \underline{\underline{0 \leq k \leq 200}}\}$$

$$|S_5| = 201$$

$$|S_3| = \{n \in \mathbb{N} : n \leq 1000, 3 \mid n\}$$

$$= \{3k \mid k \in \mathbb{N}, \underline{0 \leq k \leq 333}\}$$

$$|S_3| = 334$$

$$|S_{15}| = \{15k \mid 0 \leq k \leq \frac{1000}{15}\}$$

$$\uparrow$$

$$66\tfrac{2}{3}$$

$$0 \leq k \leq 66$$

$$67$$

$$|S_{15}| = 67$$

so $|S| = |S_3| + |S_5| - |S_{15}|$
$= 334 + 201 - 67$
$= 468$

<u>Aside</u>: <u>Useful notation</u>;

If r is a real number

$\lceil r \rceil$ = least integer n so that $r \leq n$

$\lfloor r \rfloor$ = greatest integer n so that $n \leq r$

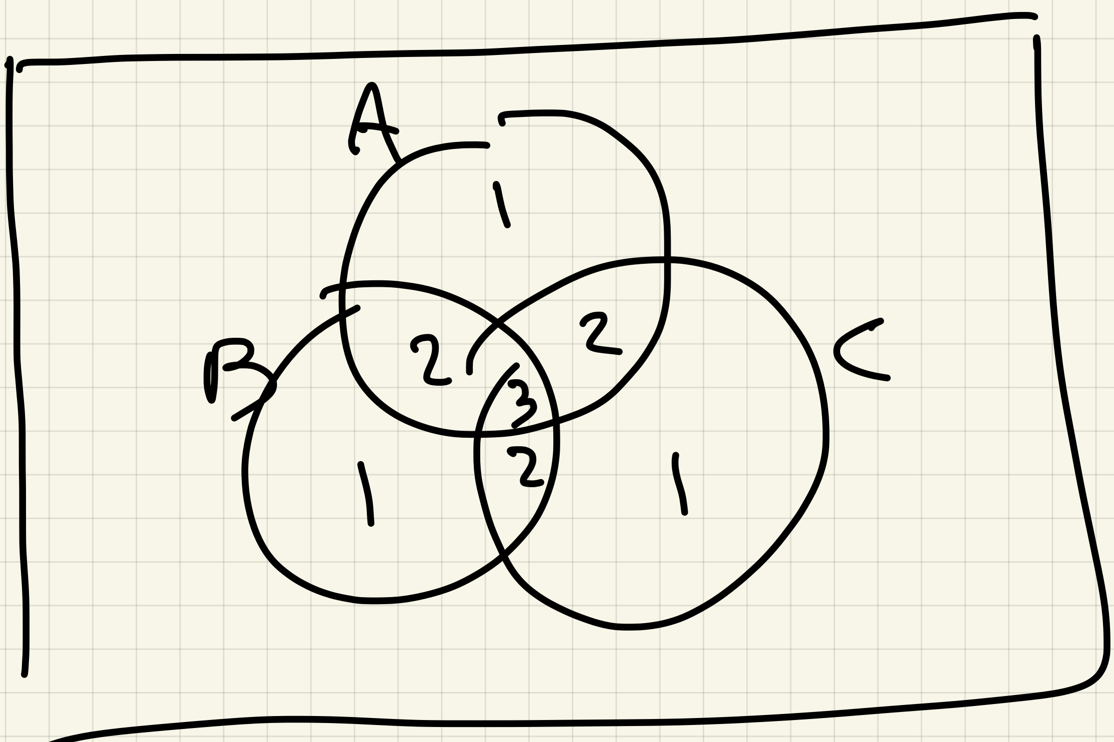
$\lceil 3 \rceil = \lfloor 3 \rfloor = 3$

$\lceil \pi \rceil = 4$ $\qquad$ $\lceil -\pi \rceil = -3$

$\lfloor \pi \rfloor = 3$ $\qquad$ $\lfloor -\pi \rfloor = -4$

Ex 3 $S = \{ n \in \mathbb{N} : n \le 1000,\ 3 \mid n \text{ or } 5 \mid n \text{ or } 11 \mid n \}$

$S = S_3 \cup S_5 \cup S_{11}$



$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

So in context of Ex 3

$$|S_3 \cup S_5 \cup S_{11}| = |S_3| + |S_5| + |S_{11}|$$

$$- |S_3 \cap S_5| - |S_3 \cap S_{11}| - |S_5 \cap S_{11}|$$

$$+ |S_3 \cap S_5 \cap S_{11}|$$

<u>Know</u> $|S_3| = 334$

$|S_5| = 201$

$|S_3 \cap S_5| = |S_{15}| = 67$

$$|S_{11}| = \left\lfloor \frac{1000}{11} \right\rfloor + 1$$

$$\lfloor 90.9 \rfloor + 1 = 90 + 1 = 91$$

$$|S_3 \cap S_{11}| = |S_{33}| = \left\lfloor \frac{1000}{33} \right\rfloor + 1$$

$$30 + 1 = 31$$

$$|S_5 \cap S_{11}| = |S_{55}| = \left\lfloor \frac{1000}{55} \right\rfloor + 1 =$$

$18+1 = 19$

$$|S_3 \cap S_5 \cap S_{11}| = |S_{165}| =$$

$$\left\lfloor \frac{1000}{165} \right\rfloor + 1$$

$$6 + 1 = 7$$

So final answer

$$334 + 201 + 91 - 67 - 31 - 19 + 7$$

$$= 516$$

<u>General case</u> :

If $A_1 \ldots A_n$ are sets,

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = ??$$

$$\left|\bigcup_{i=1}^{n} A_i\right| =$$

union notation

$$\sum_{i=1}^{n} |A_i| - \sum_{i \neq j} |A_i \cap A_j| +$$

$$+ \sum_{\substack{i,j,k \\ \neq}} |A_i \cap A_j \cap A_k| \ldots$$

$$(-1)^n |A_1 \cap A_2 \ldots \cap A_n|$$

(Note there are $2^n - 1$ sets involved!)

Special case $n = 4$

A, B, C, D

$$|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D|$$

$$- |A \cap B| - |A \cap C| - |A \cap D|$$

$$- |B \cap C| - |B \cap D| - |C \cap D|$$

$$+ |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D|$$

$|B \cap C \cap D|$

$- |A \cap B \cap C \cap D|$

<u>Ex 4</u> A Tom Brady fan wants to put 12 into his 5-digit PIN consisting of digits 0–9

$\underline{1}\ \underline{2}\ \_\ \_\ \_ \leftarrow A_1$

$\_\ \underline{1}\ \underline{2}\ \_\ \_ \leftarrow A_2$

$\_\ \_\ \underline{1}\ \underline{2}\ \_ \leftarrow A_3$

$\_\ \_\ \_\ \underline{1}\ \underline{2} \leftarrow A_4$

<u>Want</u>: $|A_1 \cup A_2 \cup A_3 \cup A_4|$

$$|A_1| = |A_2| = |A_3| = |A_4| = 10^3$$
$$= 1000$$

$$|A_1 \cap A_2| = 0$$