10/30/ Discrete

Quiz 13

Defn: $x = \frac{m}{n}$, $m, n \in \mathbb{Z}$, $n \neq 0$

(*) If $\underbrace{x \text{ is not rational}}_{A}$, then $\underbrace{3x-1 \text{ is not rational}}_{B}$

1. contrapositive $\quad \neg B \Rightarrow \neg A$

If $3x-1$ is rational, then
x is rational.

2.

Assume
x is not rational
and
$3x-1$ is rational

$A \Rightarrow B$

$\neg B \Rightarrow \neg A$

$\underline{A \wedge \neg B} \Rightarrow F$

3. <u>Contrapositive proof:</u>

$\underline{3x-1 \text{ rational}} \rightarrow \underline{x \text{ rational}}$

Assume $3x-1$ is rational,

Then $\underline{3x-1 = \frac{m}{n}}$ $\quad m,n \in \mathbb{Z}$
$n \neq 0$

Then $3x = \frac{m}{n} + 1 = \frac{m}{n} + \frac{n}{n} =$

$3x = \frac{m+n}{n}$

so $x = \frac{m+n}{3n}$

Since $m,n \in \mathbb{Z}$, $n \neq 0$,

then $\left.\begin{array}{l} m+n,\ 3n \in \mathbb{Z} \\ 3n \neq 0 \end{array}\right\}$

So x is rational

Last time Functions

$$f : A \to B$$

domain codomain

$$\operatorname{Im} f = \{ f(a) \mid a \in A \}$$

$$\operatorname{Graph} f = \{ (a,b) \mid f(a) = b \}$$

f is 1-1 if $\boxed{f(a_1) = f(a_2) \Rightarrow a_1 = a_2}$

$\left(\boxed{a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)} \right)$

f is onto : $\operatorname{Im} f = B \Leftrightarrow$

$$\forall b \in B \ \exists a \in A \ f(a) = b$$

f is (1-1 correspondence)

if f is 1-1 and onto

Fact: f is 1-1 corresp $\iff$

$\exists\, g : B \to A$: $g(f(a)) = a \ \forall a \in A$

$f(g(b)) = b \ \forall b \in B$

$g = f^{-1}$ inverse

Ex Is $f : A \to B$ 1-1?

Onto? both?

(a) $f : \mathbb{Z} \to \mathbb{Z}$

$f(x) = 2x + 5$

x	-3	-2	-1	0	1	2
$f(x)$	-1	1	3	5	7	9

Claim $\boxed{f \text{ is } 1\text{-}1}$

proof $f(x_1) = f(x_2) \Rightarrow 2x_1 + 5 = 2x_2 + 5$

$\xRightarrow{-5} 2x_1 = 2x_2 \xrightarrow{\div 2} x_1 = x_2$

f onto? NO,

$\text{Im} f = \{\text{odd ints}\} \neq \mathbb{Z}$

(proof: claim $0 \notin \text{Im} f$

suppose not: $\exists x \in \mathbb{Z}$:

$2x+5=0$

$2x=-5$

$x=-5/2 \notin \mathbb{Z}$

$\Rightarrow \Leftarrow$ b/c $x \in \mathbb{Z}$)

(b) $f: \mathbb{Z} \to \mathbb{Z}$ $f(x) = 3-x$

x	-2	-1	0	1	2	3	4	5
$f(x)$	5	4	3	2	1	0	-1	-2

Looks like f 1-1 corr.

Claim f is 1-1 corr:

$3-x = f(x) = y$ same as in Calc I

$3-x=y$ ← switch x & y

$3-y=x$

$\cdot\ 3-y=x$

$y = 3-x$

$\boxed{g(x) = 3-x}$ is inverse:

If $a \in \mathbb{Z}$ $f(g(a)) = a$?

$f(g(a)) = f(3-a)$

$= 3(-(3-a))$

$= 3-3+a = a$ ✓

(c) $f: \mathbb{Z} \to \mathbb{N}$

$f(x) = |x|$

x		-2	-1	0	1	2	
$f(x)$	$-$	2	1	0	1	2	$\ldots$

f is ~~not~~ 1-1 (e.g. $f(1)=f(-1)$
but $1 \neq -1$)

but f is onto:

proof: Let $n \in \mathbb{N}$:

NTS $\exists x \in \mathbb{Z}$, $f(x) = n$

$|x| = n$

but $x = n$ works

$|n| = n$ ✓

Ex 2 Find all functions

$f: A \to B$,

$A = \{1,2\}$, $B = \{a,b,c\}$

To answer, use graph:

$f: A \to B$ $\quad f(1) = a$

$f(2) = a$

graph: $\{(1,a),(2,a)\}$

All of them:

$\{(1,a),(2,a)\}$ $\{(1,a),(2,b)\}$ $\{(1,a),(2,c)\}$

$\{(1,b),(2,a)\}$ $\{(1,b),(2,b)\}$ $\{(1,b),(2,c)\}$

$\{(1,c),(2,a)\}$ $\{(1,c),(2,b)\}$ $\{(1,c),(2,c)\}$

9 total

Which are 1-1?

Those circled in blue

6 of them.

Which are onto? None

$\text{Im} f \neq \{a,b,c\}$

Ex 3 : Count all functions

$f : A \to B,$

$|A| = 3, \quad |B| = 6$

$A = \{a_1, a_2, a_3\}$

tells us what f is $\longrightarrow$ $\{\underset{6}{\underline{f(a_1)}}, \underset{6}{\underline{f(a_2)}}, \underset{6}{\underline{f(a_3)}}\}$

so # $\underline{6^3 = 216}$

<u>How many are 1-1 ?</u>

$$\{f(a_1) \quad f(a_2) \quad f(a_3)\}$$
$$6 \cdot 5 \cdot 4 = 120$$

How many <u>onto</u> ? NONE

Ex 4 Same questions, but $|A| = 6$, $|B| = 3$

$$\begin{pmatrix} f(a_1) & f(a_2) & - & f(a_6) \\ 3 & 3 & & 3 \end{pmatrix}$$

$$\underbrace{3 \cdot 3 - 3}_{6} = 3^6$$

How many are 1-1?, NONE

How many are onto?,

$B = \{b_1, b_2, b_3\}$

How many triples

$(f(a_1) . \quad f(a_6))$

with b_1, b_2 & b_3 appearing?

S_i = set of tuples with b_i **not** appearing

Then **Ans**:

$$3^6 - \underbrace{|S_1 \cup S_2 \cup S_3|}$$

Inclusion - Exclusion:

$$|S_1 \cup S_2 \cup S_3| = |S_1| + |S_2| + |S_3|$$
$$- |S_1 \cap S_2| - |S_1 \cap S_3| - |S_2 \cap S_3|$$
$$+ |S_1 \cap S_2 \cap S_3|$$

$$|S_1| = 2^6 = |S_2| = |S_3|$$

$$|S_1 \cap S_2| = 1 = |S_1 \cap S_3| = 1 = |S_2 \cap S_3| = 1$$

$$|S_1 \cap S_2 \cap S_3| = 0 \ ?$$

So $|S_1 \cup S_2 \cup S_3| = 3 \cdot 2^6 - 3 \cdot 1$

So # onto

$$3^6 - (3 \cdot 2^6 - 3 \cdot 1^6)$$

$$729 - (192 - 3) = 540.$$

Ex 5 $|A| = |B| = 3$

$$\begin{pmatrix} f(a_1) & f(a_2) & f(a_3) \\ 3 & 3 & 3 \end{pmatrix}$$

How many functions?

Ans $3^3 = 27$

How many 1-1?

$3 \cdot 2 \cdot 1 = 3! = 6$

How many onto 6

These suggest a pattern!

Thm 1: Let $f: A \to B$ be a function, A, B finite

(1) $|A| > |B| \Rightarrow f$ is not 1-1

(2) $|A| < |B| \Rightarrow f$ is not onto

Cor If f is 1-1 and onto (f 1-1 corresp) then $|A| = |B|$

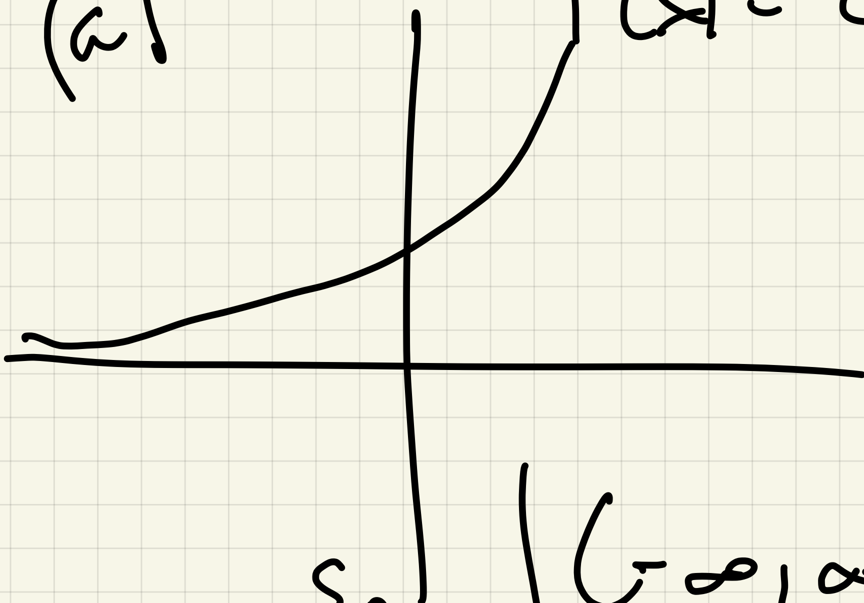
This motivates a definition:

A, B have the same cardinality if there's a 1-1 correspondence $f: A \to B$

Write $|A| = |B|$

Ex 1 $f:(-\infty,\infty)\to(0,\infty)$

(a) $f(x)=e^x$



so $|(-\infty,\infty)| = |(0,\infty)|$

(b) $g:[0,1]\to[0,2]$

$g(x)=2x$

so $|[0,1]| = |[0,2]|$

(c) $|\mathbb{N}| = |\mathbb{Z}|$

0 1 2 3 4 5 6 7 8 9 10

$-4 \quad -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5$

$$f(n) = \begin{cases} n/2 & n \text{ even} \\ \frac{-n-1}{2} & n \text{ odd} \end{cases}$$

(d) $|\mathbb{Z}| = |\mathbb{Q}|$

(e) $|\mathbb{Z}| < |\mathbb{R}|$