10/21 Discrete

Exam 2 → Oct 23

§10-12, 17, 19 (20)

Quiz 12

$|A_1| = |A_2| = 10, \quad |A_3| = 15$

$|A_1 \cap A_2| = 6 \qquad |A_2 \cap A_3| = 4$

$|A_1 \cap A_3| = 5 \qquad |A_1 \cap A_2 \cap A_3| = 3$

(a) $|A_1 \cup A_2| = \boxed{|A_1| + |A_2| - |A_1 \cap A_2|}$

$= 10 + 10 - 6 = 14$

(b) 21

(c) 20

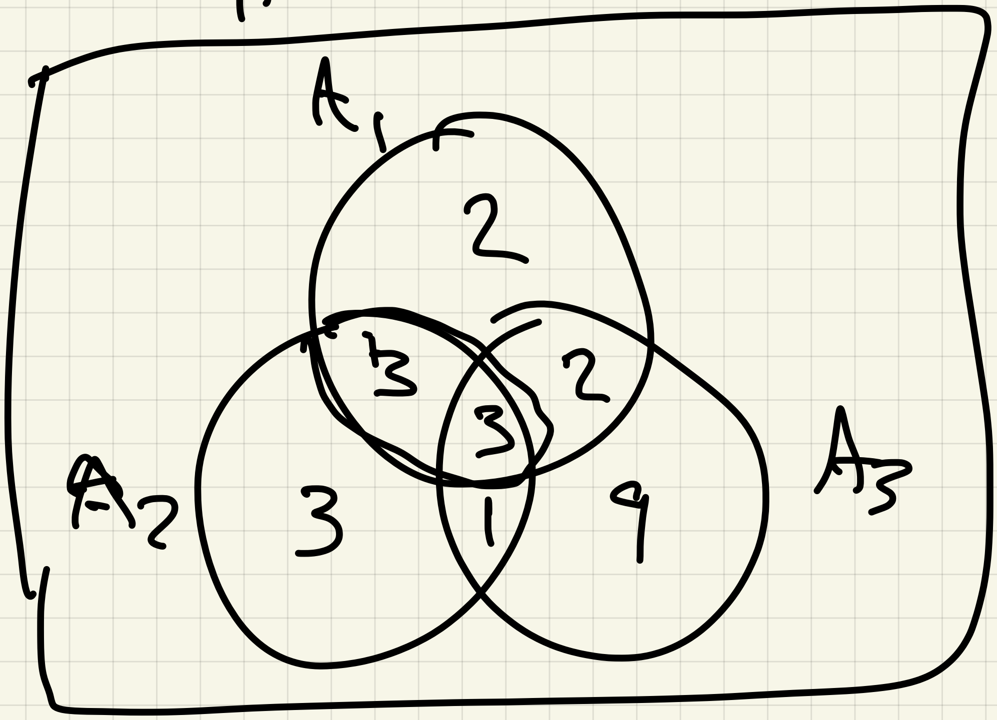
(d) $|A_1 \cup A_2 \cup A_3| =$

$|A_1| + |A_2| + |A_3| -$

$|A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3|$

$+ |A_1 \cap A_2 \cap A_3| = 23$

Alternately,



Last time: Induction Proofs

$P(n)$ = math statements

for $n \geq n_0$

To prove all $P(n)$ at once
enough to prove

1. Base $P(n_0)$
2. Step $P(n) \Rightarrow P(n+1)$, $n \geq n_0$

Ex1 Define sequence b_n

for $n \geq 0$ by

$b_0 = 2$, $b_n = 2b_{n-1} - 1$, $n \geq 1$

recurrence relation

(a) Find 5 terms

$b_0 = 2$, $b_1 = 2b_0 - 1 = 2 \cdot 2 - 1 = 3$

$b_2 = 5$, $b_3 = 9$, $b_4 = 17$

(b) Prove that $b_n = 2^n + 1$, $n \geq 0$

$P(n)$: $b_n = 2^n + 1$, $n \geq 0$

Proof $P(n)$ true by induction

Base $P(0)$: $b_0 = 2$, $2^0 + 1 = 1 + 1 = 2$ ✓

Step $P(n) \rightarrow P(n+1)$

Assume $\hat{P}(n)$: $\overset{\circ}{b_n} = 2^n + 1$

(NTS: $P(n+1)$: $\underline{b_{n+1}} = 2^{n+1} + 1$)

$$b_{n+1} \underset{\text{defn}}{=} 2\underline{b_n} - 1$$

$$\underset{\substack{\text{Ind}\\\text{hyp}}}{=} 2(2^n+1) - 1$$

$$= 2 \cdot 2^n + 2 - 1$$

$$= 2^{n+1} + 1 \checkmark \checkmark$$

Sometimes need <u>strong</u> <u>induction</u>:

To prove $P(n)$ for $n \geq n_0$, enough to prove

<u>Base</u> $P(n_0)$

<u>Step</u> $\boxed{P(n_0) \wedge \ldots \wedge P(n)} \rightarrow P(n+1)$

for $n \geq n_0$

i.e. $P(n_0), P(n_0+1), \ldots P(n)$
true, then so is
$P(n+1)$.

Stronger induction hypothesis

<u>Ex 2</u> Define a_n by
$a_0 = -1$, $a_1 = 2$, and
$\boxed{a_n = 2a_{n-1} + 3a_{n-2}}$ for $n > 1$

Several terms:

$a_0 = -1$, $a_1 = 2$, $a_2 = 1$,
$a_3 = 8$, $a_4 = 19$, $a_5 = 62$

<u>Prove</u>: $P(n)$: $a_n = \frac{1}{4} \cdot 3^n - \frac{5}{4}(-1)^n$
$n \geq 0$

<u>Proof</u> by <u>strong</u> induction:

Base $a_0 = -1$

$$\frac{1}{4}\cdot 3^0 - \frac{5}{4}(-1)^0 =$$

$$\frac{1}{4} - \frac{5}{4} = -\frac{4}{4} = -1 \checkmark$$

ALSO $a_1 = 2$:

$$\frac{1}{4}\cdot 3^1 - \frac{5}{4}(-1)^1 = \frac{3}{4} + \frac{5}{4} = \frac{8}{4} = 2 \checkmark$$

Step: Assume $a_k = \frac{1}{4}\cdot 3^k - \frac{5}{4}(-1)^k$

for $0 \leq k \leq n$

(NTS $a_{n+1} = \frac{1}{4}\cdot 3^{n+1} - \frac{5}{4}(-1)^{n+1}$)

$$\text{ok}, \quad a_{n+1} \overset{\text{DEFN}}{=} 2a_n + 3a_{n-1}$$

$$\underset{\text{Induct hypoth}}{=} 2\left(\frac{1}{4}\cdot 3^n - \frac{5}{4}(-1)^n\right) + 3\left(\frac{1}{4}3^{n-1} - \frac{5}{4}(-1)^{n-1}\right)$$

$$2\cdot\frac{1}{4}\cdot 3^n + 3\cdot\frac{1}{4}\cdot 3^{n-1} =$$

$$\frac{1}{4}(2\cdot 3^n + \underline{3\cdot 3^{n-1}})$$

$$- 5/4(2(-1)^n + \underline{3(-1)^{n-1}})$$

$$\frac{1}{4}(2\cdot \underset{=}{3^n} + \underset{=}{3^n}) = \frac{1}{4}(\underbrace{(2+1)}_{3}\cdot 3^n) =$$

$$\frac{1}{4}(3\cdot 3^n) = \frac{1}{4}\cdot 3^{n+1}$$

$$-5/4(2(-1)^n + \underline{3(-1)^{n-1}}\ (-1)(-1)$$

$$(2(-1)^n - 3(-1)^n)$$

$$= (2-3)(-1)^n = (-1)(-1)^n = (-1)^{n+1}$$

$$\frac{1}{4}3^{n+1} - 5/4(-1)^{n+1} \checkmark$$

Fibonacci Sequence

$$f_0 = f_1 = 1,\ f_n = f_{n-1} + f_{n-2},\ n > 1$$

defines the Fibonacci sequence

n	0	1	2	3	4	5	6	7	8	9
f_n	1	1	2	3	5	8	13	21	34	55

Can show by strong induction that

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$$

Golden ratio

i.e.

$f_0 = 1$

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{1}$$

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2} - \left(\frac{1-\sqrt{5}}{2}\right)\right)$$

$$\frac{1}{\sqrt{5}}\left(\frac{\cancel{1}+\sqrt{5}-\cancel{1}+\sqrt{5}}{2}\right)$$

$$=\frac{1}{\cancel{\sqrt{5}}}\left(\frac{\cancel{2}\cancel{\sqrt{5}}}{\cancel{2}}\right)=1 \checkmark$$

__Ex 3__ Find a formula for

$$q_n=\sum_{k=0}^{n} f_k = f_0+f_1+\dots+f_n$$

Try to find pattern:

n	0	1	2	3	4	5	6	7
f_n	1	1	2	3	5	8	13	21
q_n	1	2	4	7	12	20	33	54

__Pattern:__ $q_n+1=f_{n+2}$

i.e. $q_n=f_{n+2}-1$

__Prove it!__

Strong induction:

base $g_0 = 1 = f_2 - 1$

step Assume $g_n = f_{n+2} - 1$

(NTS : $g_{n+1} = f_{n+3} - 1$)

$$g_{n+1} \underset{\text{defn}}{=} \sum_{k=0}^{n+1} f_k = \underbrace{f_0 + \dots + f_n}_{g_n} + f_{n+1}$$

Ind hypoth

$$= (f_{n+2} - 1) + f_{n+1} =$$

$$= \underbrace{f_{n+2} + f_{n+1}}_{} - 1$$

$$= f_{n+3} - 1$$

defn of Fibonacci numbers

Ex 4 Let $2 \le n \in \mathbb{N}$.

Then n is a product of prime numbers.

<u>Proof</u>: By strong induction;

<u>Base</u> $n=2$ $\quad 2 =$ prime ✓

<u>Step</u>: Assume $P(2) - P(n)$
(i.e. $\underline{2, 3, 4, \ldots n}$ are products of prime numbers)

NTS $(P(n+1))$ i.e.,
$n+1 =$ product of primes,

<u>Consider $n+1$</u>:

If $n+1$ is prime, then OK ✓

If $n+1$ is <u>not</u> <u>prime</u>,

then $n+1 = d \cdot e$,
$1 < \underline{\underline{d}}, \underline{\underline{e}} < n+1$

By induction hypothesis,

d, e are products of primes

($P(e)$ $P(d)$ true)

so $d \cdot e = n+1$ is prod of primes

§23 – rec defs

<u>§24</u> <u>Functions</u>

A <u>function</u> $f: A \to B$ is an assignment of each $a \in A$ to exactly one $b \in B$

<u>Notation</u>: $f(a) = b$

A = domain

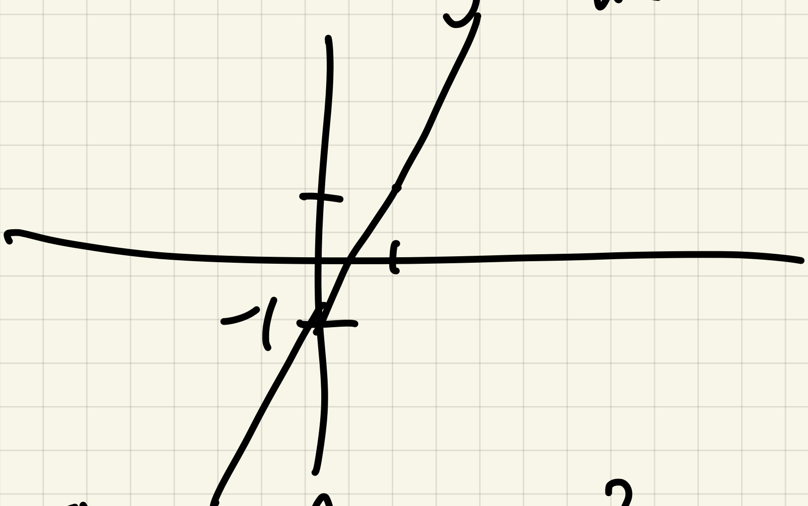
B = codomain = target

$\operatorname{Im} f = \{ f(a) : a \in A \}$ = Range of f

Graph of f : $\{(a,b) \mid f(a)=b\}$
$\qquad\qquad\quad \overset{\wedge}{A \times B}$

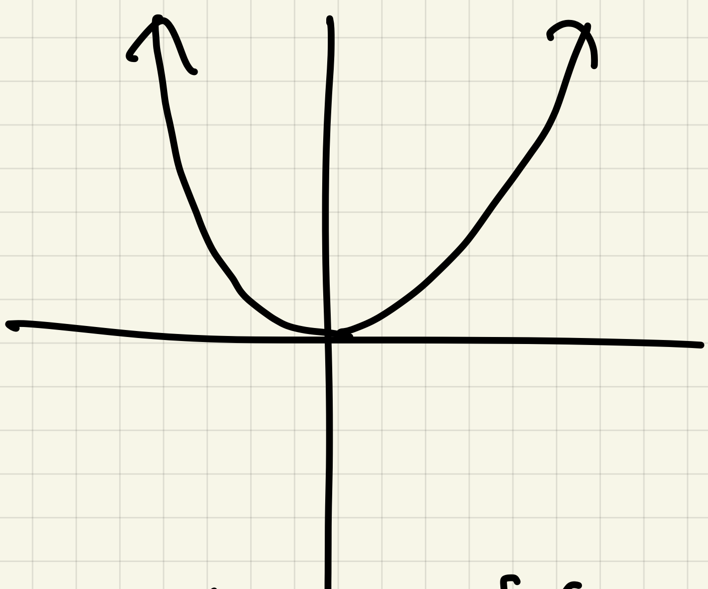
Calculus examples :

Ex (a) $f(x) = 2x-1$
$f : \mathbb{R} \to \mathbb{R}$



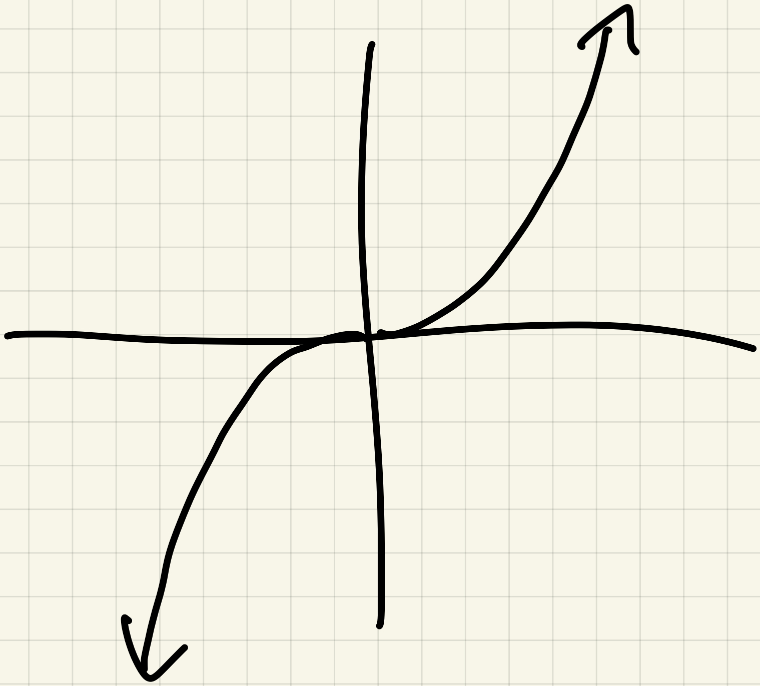
$\mathrm{Im} f = \text{range} = \mathbb{R}$

(b) $f(x) = x^2$



$\mathrm{Im} f = [0, \infty)$

(c) $f(x) = x^3$

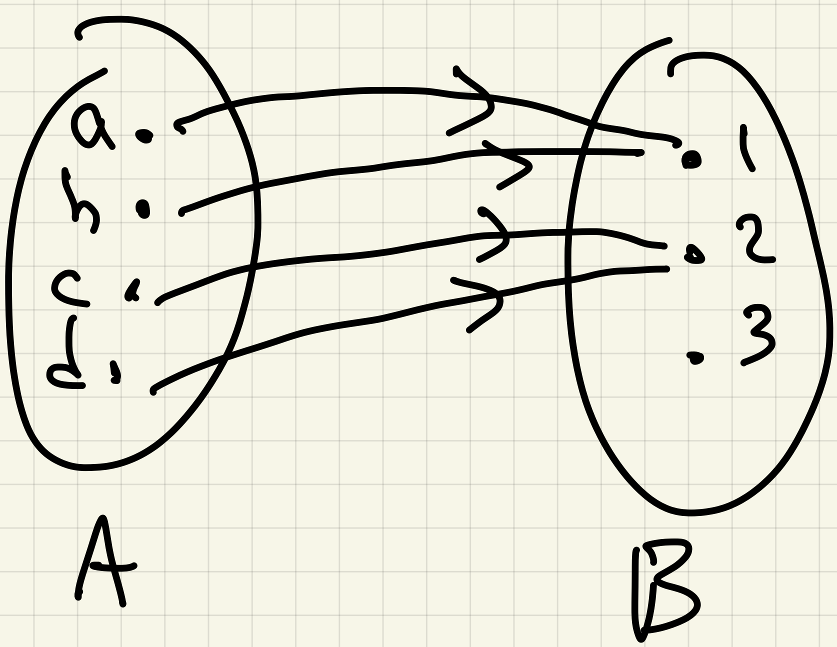


$\mathrm{Im} f = (-\infty, \infty) = \mathbb{R}$

Ex2 discrete:

$\mathrm{Im} f = \{1, 2\}$

(a)

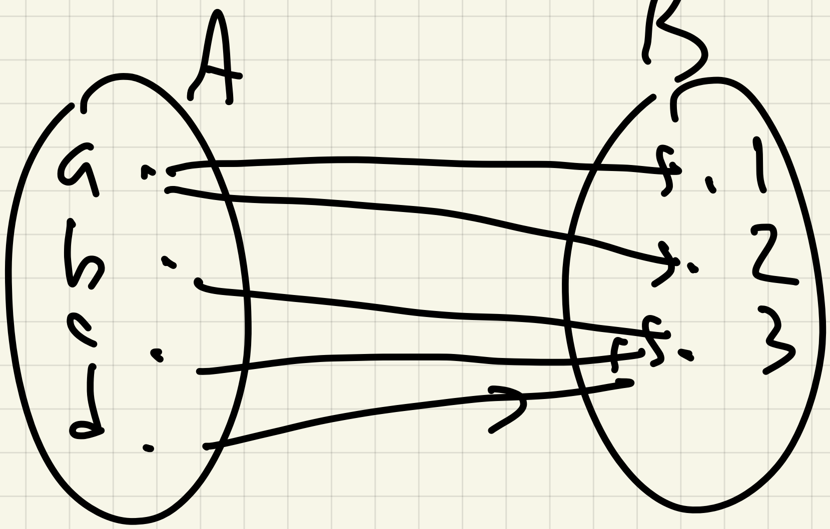


$f(a) = 1$ $\quad f(c) = 2$

$f(b) = 1$ $\quad f(d) = 2$

(b)

f :



Not a function

$f(a) = \{1, 2\}$

one output.