# Quiz 7

avg 91%
med 100%

1.

(a) $A = \{x \in \mathbb{N} : x \leq 27 \text{ and } 5 | x\}$

$= \{0, 5, 10, 15, 20, 25\}$

(b) $B = \{x \in \mathbb{Z} : x^2 < 7 \wedge x \neq 0\}$

$\{-1, -2, 1, 2\}$

2. $C = \{x \in \mathbb{Z} : |x| \leq 8\}$

(a) $-8 \ldots -1, 0, 1, 2 \ldots 8$

$|C| = 17$

(b) $|A| = 6$

(c) $|P(A)| = 2^{|A|} = 2^6 = 64.$

Last Time: Set operations

Cartesian product

$$|A \times B| = |A| \times |B|$$

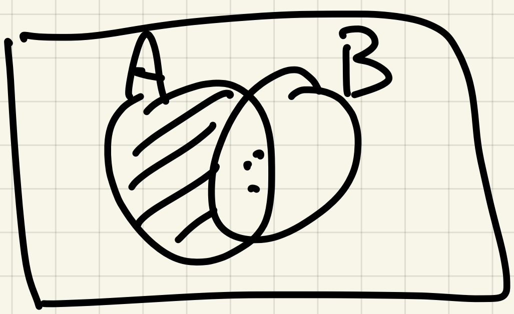
Inclusion - Exclusion:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

(a) $A, B$ disjoint: $A \cap B = \phi$

then $|A \cup B| = |A| + |B|$

(b) $|A - B| = |A| - |A \cap B|$




§17 & §19

More counting:

Know $|P(A)| = 2^{|A|}$;

But how many subsets of A have size ~~size~~ $k$?

<u>Ex)</u> If $A = \{a,b,c,d,e\}$
How many subsets of size 3?

$\boxed{\{a,b,c\},}$ $\{a,b,d\}$, $\{a,b,e\}$
$\{a,c,d\}$, $\{a,c,e\}$, $\{a,d,e\}$
$\{b,c,d\}$, $\{b,c,e\}$, $\{b,d,e\}$
$\{c,d,e\}$

<u>Ans: 10</u>

<u>General principle</u>:

Know how count all lists of 3 from A

without repetition:

$$5 \cdot 4 \cdot 3 = 60$$

$\underline{\ \ 5\ \ }\ \underline{\ \ 4\ \ }\ \underline{\ \ 3\ \ }$

$$\left.\begin{matrix}(a,b,c)\\(a,c,b)\\(b,a,c)\\(b,c,a)\\(c,a,b)\\(c,b,a)\end{matrix}\right\}$$

overcounts the # subsets by factor of $6 = 3!$,

i.e. $\frac{60}{6} = 10$ ✓

__Thm__ # subsets of size $k$ taken from $A$ of size $n$ is

$$\frac{n!}{(n-k)!\,k!}$$

__Why?__ # lists of length $k$ without repetition

$\frac{n!}{(n-k)!}$ but this overcounts

the # subsets by factor of $k!$.

so # subsets $\frac{\frac{n!}{(n-k)!}}{k!}$

$= \frac{n!}{(n-k)!\,k!}$

**Notation:** $n_k = n(n-1)(n-2)\dots(n-k+1)$

$\quad\quad\quad\quad\; \parallel$

**Permutations** $\boxed{P(n,k)}$

$\binom{n}{k}$

**Combinations** $\boxed{C(n,k)} = \frac{n!}{(n-k)!\,k!}$

$\boxed{\text{Ex}}$ Check formula for

$A = \{a, b, c, d, e\}$

| $k$ | subsets of size $k$ | # sets | $\binom{5}{k}$ |
|---|---|---|---|
| 0 | $\emptyset$ | 1 | 1 ✓ |
| 1 | $\{a\}, \{b\}, \{c\}, \{d\}, \{e\}$ | 5 | 5 ✓ |
| 2 | $\{a,b\}$ $\{a,c\}$, $\{a,d\}$ $\{a,e\}$ $\{b,c\}$ $\{b,d\}$ $\{b,e\}$ $\{c,d\}$ $\{c,e\}$ $\{d,e\}$ | 10 | |
| 3 | ten earlier | 10 | |

$$\binom{5}{0} = \frac{5!}{5!\,0!} = 1$$

$$\binom{5}{1} = \frac{5!}{4!\,1!} = \frac{5 \cdot \cancel{4 \cdot 3 \cdot 2 \cdot 1}}{\cancel{4 \cdot 3 \cdot 2 \cdot 1} \cdot 1} = 5$$

$$\binom{5}{2} = \frac{5!}{3!\cdot 2!} = \frac{5\cdot 4\cdot \cancel{3\cdot 2\cdot 1}}{\cancel{3\cdot 2\cdot 1}\; 2\cdot 1} \quad \frac{20}{2} \Downarrow 10 \checkmark$$

$$\binom{5}{3} = \frac{5!}{2!\cdot 3!} = 10$$

| k | | |
|---|---|---|
| 4 | $\{b,c,d,e\}$ $\{a,b,d,e\}$ $\{a,c,d,e\}$ $\{a,b,c,e\}$ $\{a,b,c,d\}$ | 5 |

$$\binom{5}{4} = 5 \checkmark$$

| | | |
|---|---|---|
| 5 | $\{a,b,c,d,e\}$ | 1 |

$$\binom{5}{5} = 1 \checkmark$$

## Two observations:

As a check: $|P(A)| = 2^n$

$n = |A|$

(1) $|P(A)| = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n}$

$\qquad\qquad\;\; \text{//} \qquad 0 \qquad 1 \qquad 2 \qquad\qquad n$

$2^n$

$\underline{n=5}$: $32 = 1+5+10+10+5+1$

i.e. $\sum_{k=0}^{n} \binom{n}{k} = 2^n$

(2) When we counted $k=2$ case, we found that

$$\binom{5}{2} = 4+3+2+1$$

In general,

$$\boxed{\binom{n+1}{2} = 1+2+3+\dots+n}$$

$$= \frac{(n+1)!}{(n-1)!\,2!} = \frac{n(n+1)}{2} \qquad n=100$$

Ex $1+2+3+\dots+100 = \binom{101}{2}$

$$= \frac{101\cdot 100}{2} = \frac{10100}{2} = 5050.$$

Ex) How many ways to list 3 different donuts out of 6

(a) donuts in order!

$$6_3 = P(6,3) = 6\cdot 5\cdot 4 = 120$$

(b) How many ways to select

3 donuts of the six

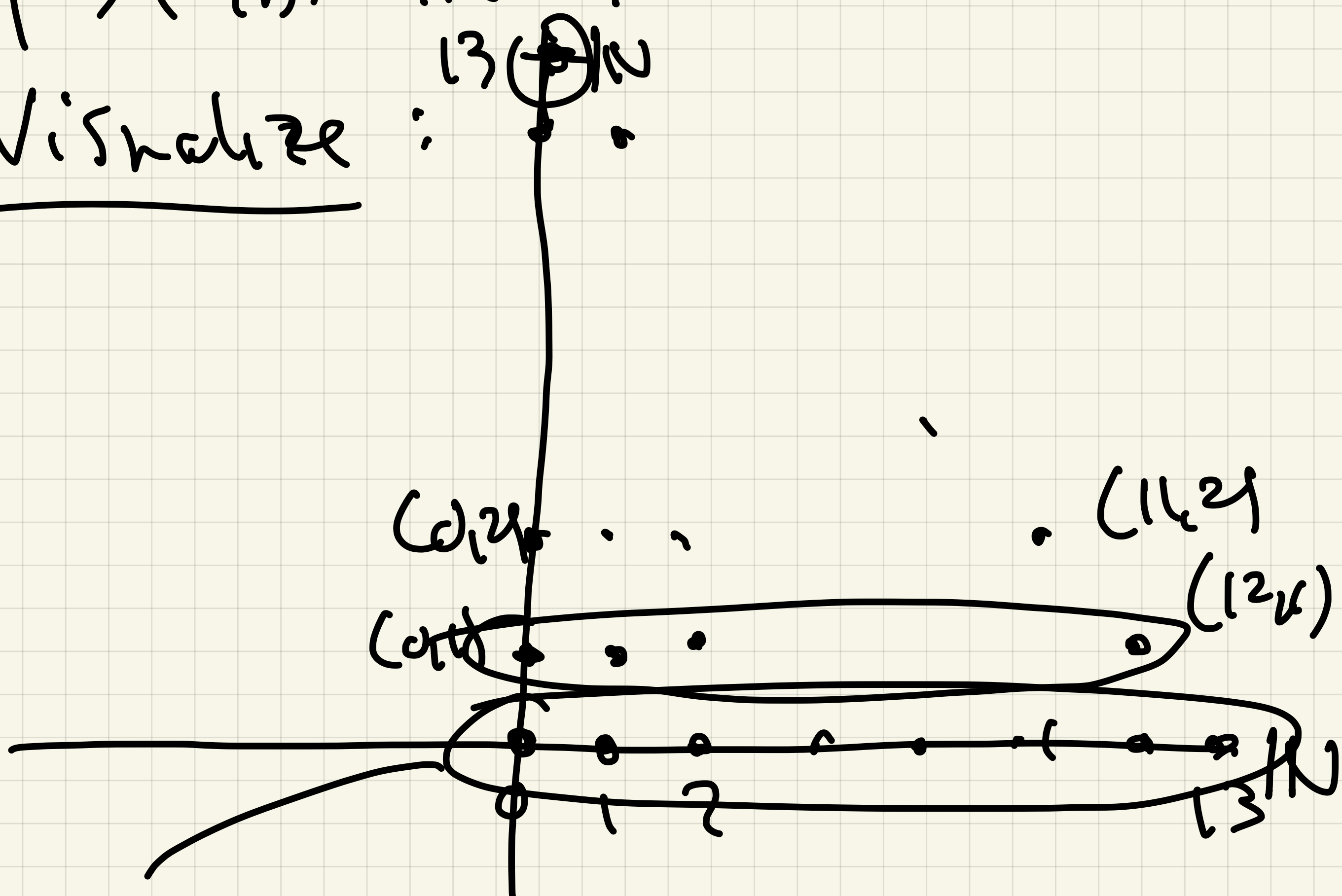
$$\binom{6}{3} = C(6,3) = \frac{6!}{3!\,3!} =$$

$$\frac{720}{6 \cdot 6} = \frac{720}{36} = 20$$

<u>Ex 2</u> Find $\left|\{(a,b) \in \mathbb{N}\times\mathbb{N} : a+b \le 13\}\right|$

$$\left|\{(a,b) : \mathbb{N}\times\mathbb{N} : a+b \le 13\}\right|$$

<u>Visualize</u>:



$$14+13+12+11+\ldots+2+1=$$

$$1+2+3+\ldots+14=\binom{15}{2}=$$

$$\frac{15\cdot 14}{2}=\frac{210}{2}=105$$

<u>Ex3</u> Drama club has 25 members.

(a) How many ways to select pres / VP / treas. / secr

at most one person holds each office.

<u>Ans</u> $_{25}P_4 = 25\cdot 24\cdot 23\cdot 22$

(b) How many ways to choose a committee of size 5 }

Ans $\binom{25}{5} = \frac{25!}{20!\,5!}$

Ex 4 An ice cream shop has 10 ice cream flavors and 20 toppings, and 3 dish sizes

(a) How many ways to place an order with 1 flavor and one topping?

$$\underbrace{3}_{\text{size}} \cdot \underbrace{10}_{\text{flavor}} \cdot \underbrace{20}_{\text{topping}} = 60$$

(b) How many ways with 2 flavors and 4 toppings?

$$\underbrace{3}_{\text{size}} \cdot \underbrace{\binom{10}{2}}_{\text{flavor}} \cdot \underbrace{\binom{20}{4}}_{\text{toppings}}$$

$$3 \cdot 45 \cdot \binom{20}{4} = 654{,}075$$

Ex 5 ~~~~ Consider ~~~~ length 10 strings (codes) from $\{a, b, c, \ldots, z\}$

(a) How many? $26^{10}$

(b) How many begin TCU? $26^7$

(c) How many with no consecutive repeats

$\underset{26}{\_}\ \underset{25}{\_}\ \underset{25}{\_}\ \_\ \ \_\ \_\ \_$

$26 \cdot 25^9$

(d) How many with exactly one T?

$10 \cdot 25^9$

place T — place rest

(e) How many with 3 Ts?

_ _ _ _ _ _ _ _ _ _

place the 3 Ts: $\binom{10}{3}$

place the rest $25^7$

so $\binom{10}{3} \cdot 25^7$

(f) Exactly 3 Ts

3 Cs

3 Us

3TS $\binom{10}{3} \cdot \binom{7}{3} \binom{4}{3} \cdot 23$

T C U