10/16/ Discrete :

Exam 2 → Oct 23

Quiz 11 avg 79% med 88%

1 (a)

$$\underset{P}{\underline{\quad}}\ \underset{VP}{\underline{\quad}}\ \underset{T}{\underline{\quad}}$$

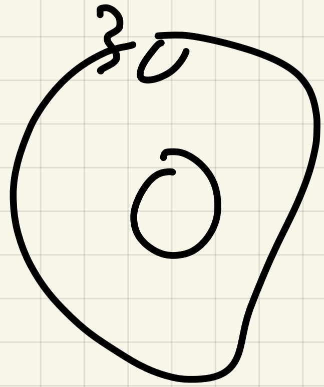
$$\boxed{30\cdot 29\cdot 28} = 30_3 = \frac{30!}{27!}$$

(b)

$$\underset{P}{\underline{20}}\ \underset{VP}{\underline{19}}\ \underset{T}{\underline{10}}$$

$$20\cdot 19\cdot 10 = 20_2\cdot 10_1$$

(c) committee of 8



$$\binom{30}{8} = C(30,8) = \frac{30!}{22!\,8!}$$

(d) $\binom{20}{5} \cdot \binom{10}{3}$

$\boxed{5 \text{ women}}$ 3 men

(e) 2. $\binom{6}{4} = \frac{6!}{(6-4)!\,4!}$

$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$

$$= \frac{6 \cdot 5 \cancel{\cdot 4 \cdot 3 \cdot 2 \cdot 1}}{2 \cdot 1 \cancel{\cdot 4 \cdot 3 \cdot 2 \cdot 1}} = \frac{30}{2} = 15$$

<u>Last time</u> §20 Contradiction

§21 Least counter example

§22 <u>Mathematical Induction</u>

To prove statements

$P(n)$, $n > n_0$

$$\left[P(n_0),\ P(n_0+1),\ P(n_0+2) \ldots \right]$$

It's enough to prove 2 things:

<u>Induction base</u> $\quad P(n_0)$

<u>Induction step</u> $\quad P(n) \Rightarrow P(n+1)$ $\quad n \geq n_0$

<u>Ex 0</u> $\quad$ <u>Claim</u>: $\overbrace{\boxed{n! > 3^n}}^{P(n)}$ for $n \geq 7$

<u>Proof</u>: By induction;

<u>Base</u> $\quad P(7)$: $\quad 7! > 3^7$

$$\underset{5040}{\overset{7!}{\|}} > \underset{2187}{\overset{3^7}{\|}}$$

<u>Step</u>: Need to show

$P(n) \Rightarrow P(n+1)$ $\quad$ <u>for $n \geq 7$</u>

$n! > 3^n \Longrightarrow (n+1)! > 3^{n+1}$

Assume $n \geq 7$, $n! > 3^n$

(Need to show $(n+1)! > 3^{n+1}$)

$(n+1)! = (n+1) \cdot n! > (n+1) \cdot 3^n =$

(mult $n+1$ by $n! > 3^n$)

$(n+1)3^n > 3 \cdot 3^n = 3^{n+1}$

$n \geq 7$, $n+1 \geq 8$

$n+1 > 3$

so $(n+1)! > 3^{n+1}$

$\therefore$ $P(n)$ holds for all $n \geq 7$ by induction.

Ex1 The sum of the first n positive odd integers is n^2

$$1 = 1^2$$
$$1+3 = 2^2$$
$$1+3+5 = 3^2 \ldots$$

Proof: By induction,

$P(n)$: $1+3+5+\ldots+(2n-1) = n^2$

(n a pos int)

Base: $n=1$ $\quad 1 = 1^2$ ✓

Step: $P(n) \Rightarrow P(n+1)$ for $n \geq 1$

Assume $n \geq 1$ and $P(n)$ true

(Induction hypothesis)

$$\boxed{1+3+5+\ldots+(2n-1) = n^2}$$

Need to show $P(n+1)$:

$$\underbrace{1+3+5+\dots+\underbrace{(2(n+1)-1)}_{2n+1}}_{\text{LHS}} = \underbrace{(n+1)^2}_{\text{RHS}}$$

$P(n)$: $1+3+5+\dots+(2n-1) = n^2$

add $2n+1$ to both

$$\underbrace{1+3+5+\dots+(2n-1)+(2n+1)}_{\text{LHS}} = \underbrace{n^2+(2n+1)}$$

$$= n^2+2n+1$$

$$= (n+1)^2 \checkmark$$

So $P(n)$ holds for all $n \geq 1$ by induction.

Remark: Can also prove directly

LHS: $1+3+5+\ldots+2n-1$

$$= \underline{1}+\underline{3}+\underline{5}+\ldots+\underline{2(n-1)+1}$$

$$= 0+1+(\underline{2}+1)+(\underline{4}+1)+(\underline{6}+1)+\ldots$$

$$2(\underline{n-1})+1$$

$$\left(\underbrace{1+1+1+\ldots+1}_{n}\right)+\left(\underbrace{0+2+4+\ldots+2(n-1)}_{n}\right)$$

$$\left(\underbrace{1+1+\ldots}_{n}+1\right)+2\left(\underbrace{0+1+2+3+\ldots+(n-1)}_{\shortparallel}\right)$$

$$\uparrow$$

$$n+2\binom{n}{2}$$

$$\left(\left(\underline{\text{recall}}:\ 1+2+3+\ldots+k=\binom{k+1}{2}\right)\right)$$

$$n + 2\binom{n}{2} =$$

$$n + 2\left(\frac{n!}{(n-2)!\,2!}\right)$$

$$n + \cancel{2}\left(\frac{n(n-1)}{\cancel{2}}\right)$$

$$= n + n(n-1) =$$

$$\cancel{n} + n^2 - \cancel{n} = n^2 \checkmark$$

Ex 2 Prove

$$1 \times 1! + 2 \times 2! + 3 \times 3! + \dots n \cdot n! = (n+1)! - 1$$

for all $n \geq 1$

Proof: Induction

base: $1 \times 1! = (1+1)! - 1$

$$1 = 2! - 1 = 1 \checkmark$$

Step: Assume $n \geq 1$ and

$1 \times 1! + \ldots + n \cdot n! = (n+1)! - 1$ $\cdot P(n)$

Need to show = NTS:

$$1 \times 1! + \ldots + n \cdot n! + (n+1)(n+1)! = (n+2)! - 1$$

// NTS

LHS

$$1(n+1)! - 1 + (n+1)(n+1)!$$

$$(n+1)!(1 + (n+1)) - 1$$

$$(n+1)!(n+2) - 1$$

$$= (n+2)! - 1 \checkmark$$

Ex 3 Prove that $4 \mid 5^n - 1$

for all $n \geq 0$

$n = 0$ $\quad 5^0 - 1 = 1 - 1 = 0$

$n = 1$ $\quad 5^1 - 1 = 4 = 4 \cdot 1 \checkmark$

$n=2 \quad 5^2-1 = 24 = 4\cdot 6$
$n=3 \quad 5^3-1 = 125-1 = 124 =$
$4\cdot(31)$

Proof: Induction:

$P(n): \; 4 \mid 5^n - 1$

Base $P(0): \; 4 \mid \underbrace{5^0-1}$
$0 = 4\cdot 0$ ✓

Step: $P(n) \Rightarrow P(n+1)$

$4 \mid 5^n-1 \Rightarrow 4 \mid 5^{n+1}-1$

Assume $4 \mid 5^n - 1$ (Ind. hyp.)

(NTS: $4 \mid 5^{n+1}-1$)

Since $4 \mid 5^n - 1$, $\exists c \in \mathbb{Z}$:

$$5^n - 1 = 4c$$

$$5^{n+1} - 1 = ??$$

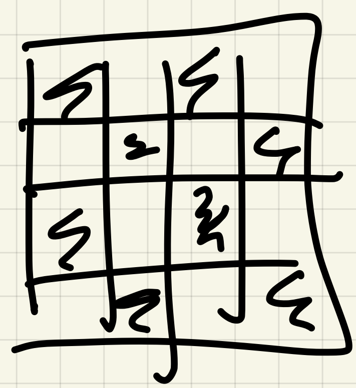
$$\underbrace{5^{n+1} - 5^n}_{\parallel} + 5^n - 1$$

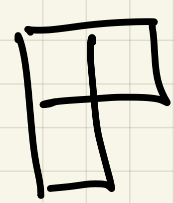
$$5^n(5-1) + \underbrace{5^n - 1}_{\parallel}$$

$$5^n \cdot 4 + 4c$$

$$\parallel$$

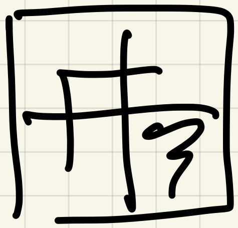
$$4(\underbrace{5^n + c}_{\text{int}}) \checkmark$$



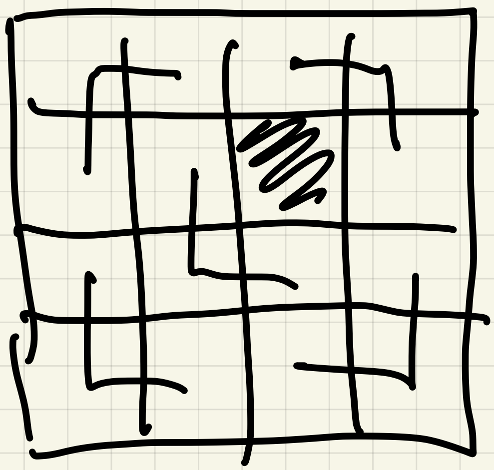
<u>Ex 4</u>: A trimono is a figure 

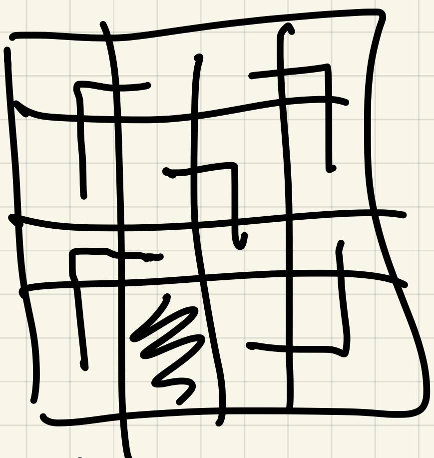
Show that $2^n \times 2^n$ chessboard with $\boxed{\text{any}}$ square removed can be tiled with trimonos. for $n \geq 1$

$\underline{n=1}$



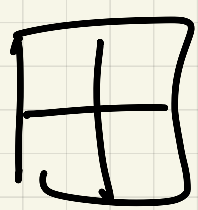
$\underline{n=2}$



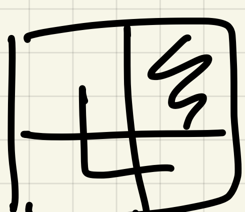


In general ??

Proof by induction:



Base $\underline{n=1}$

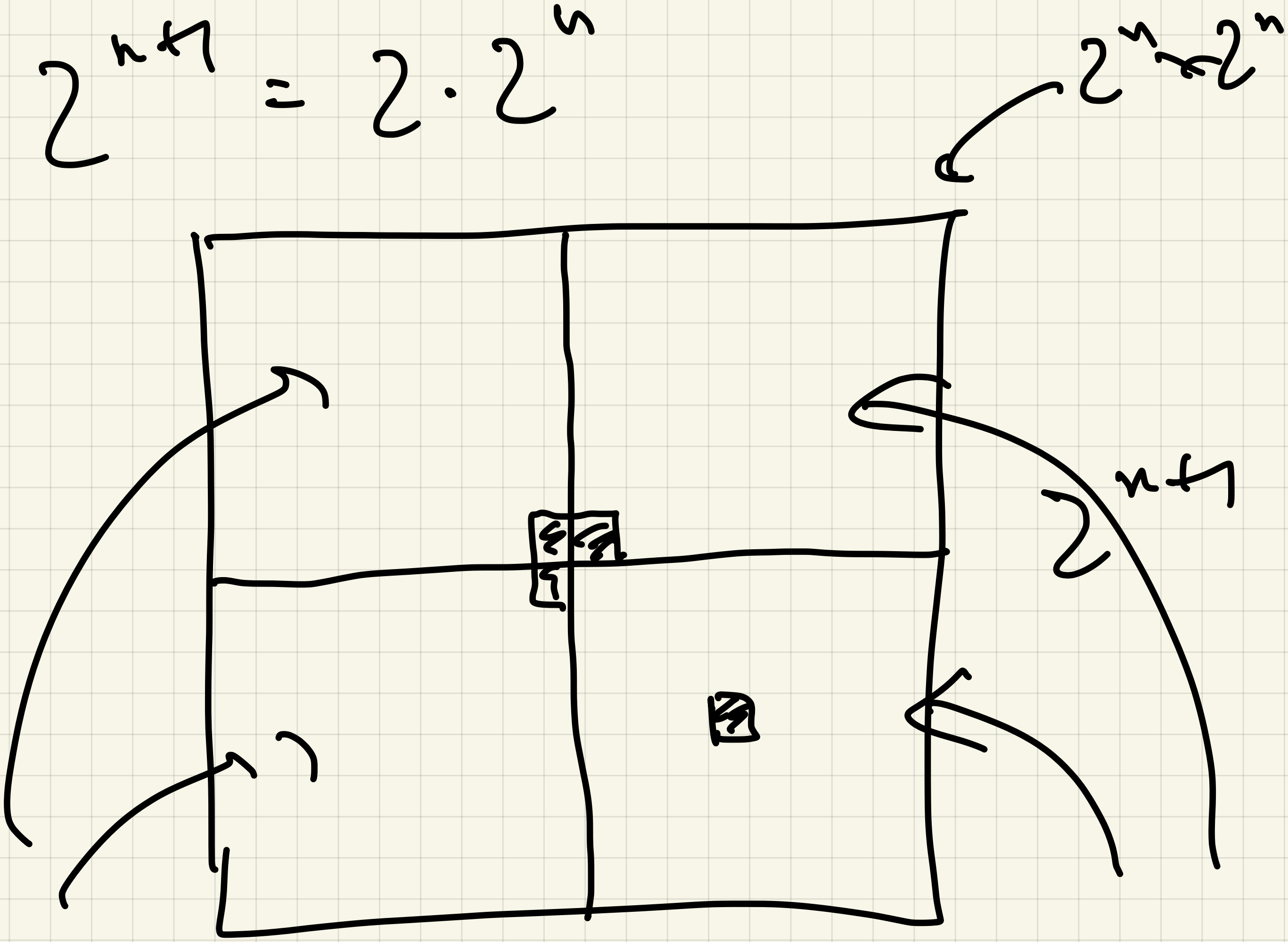


✓

Step : Assume that any $2^n \times 2^n$ board with one square removed can be tiled by trimonos.

NTS : Also for $2^{n+1} \times 2^{n+1}$ board !

$2^{n+1} = 2 \cdot 2^n$



$$2^{n+1} = 2 \cdot 2^n$$

By induction hypothesis, can tile

each of the

4 $2^n \times 2^n$ boards (-1 square)

by trominoes.

So can

tile all of it.

<u>Consequence</u> $3 \mid 2^n \cdot 2^n - 1$