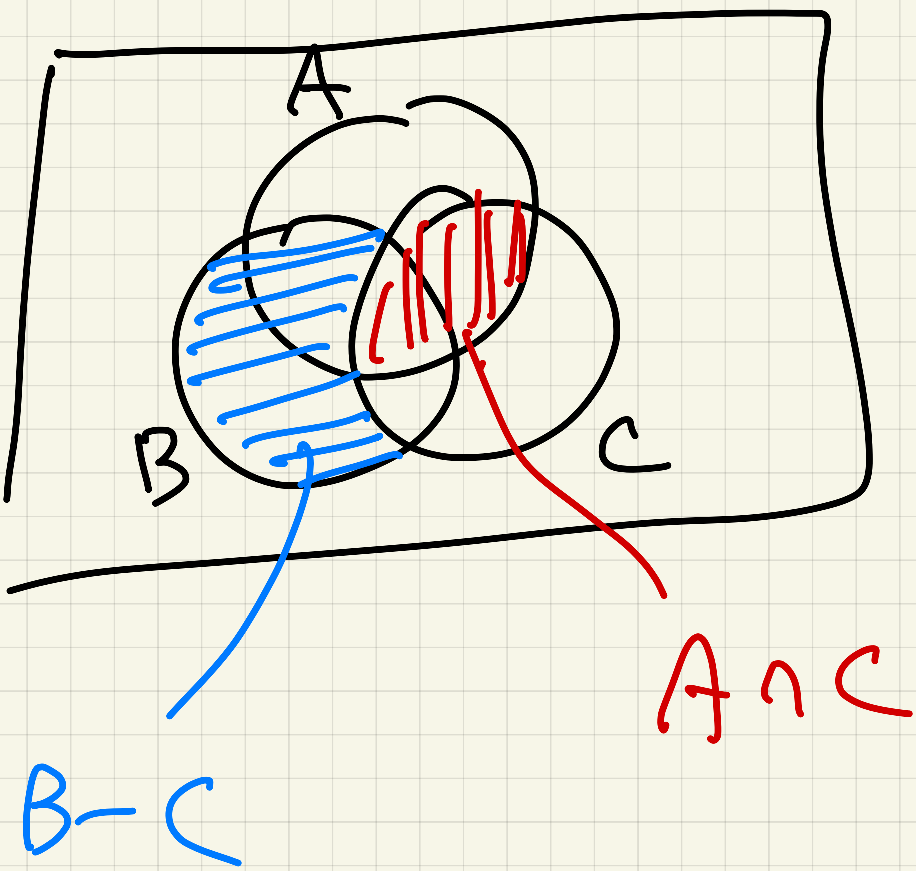
10/14/ Discrete

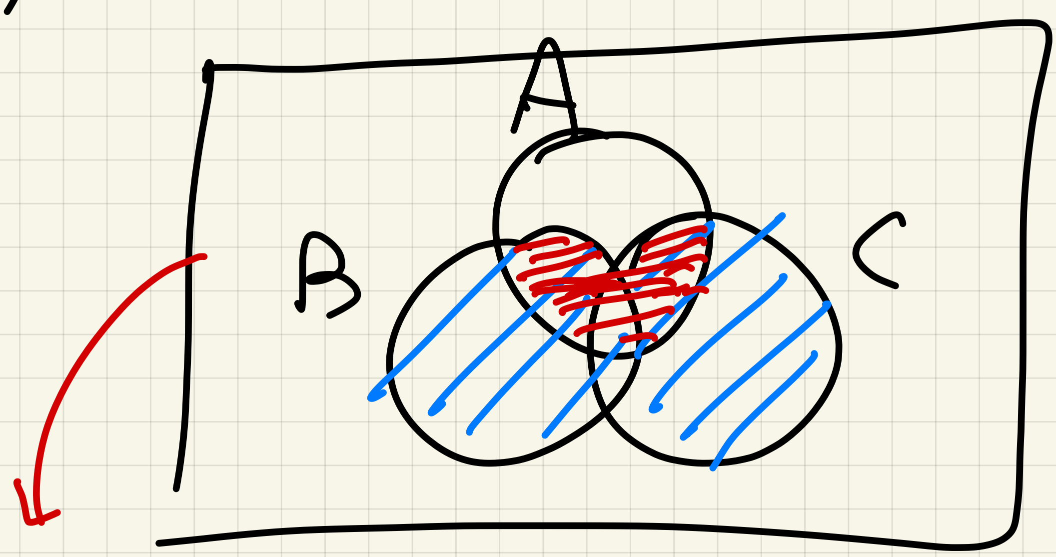
Quiz 10

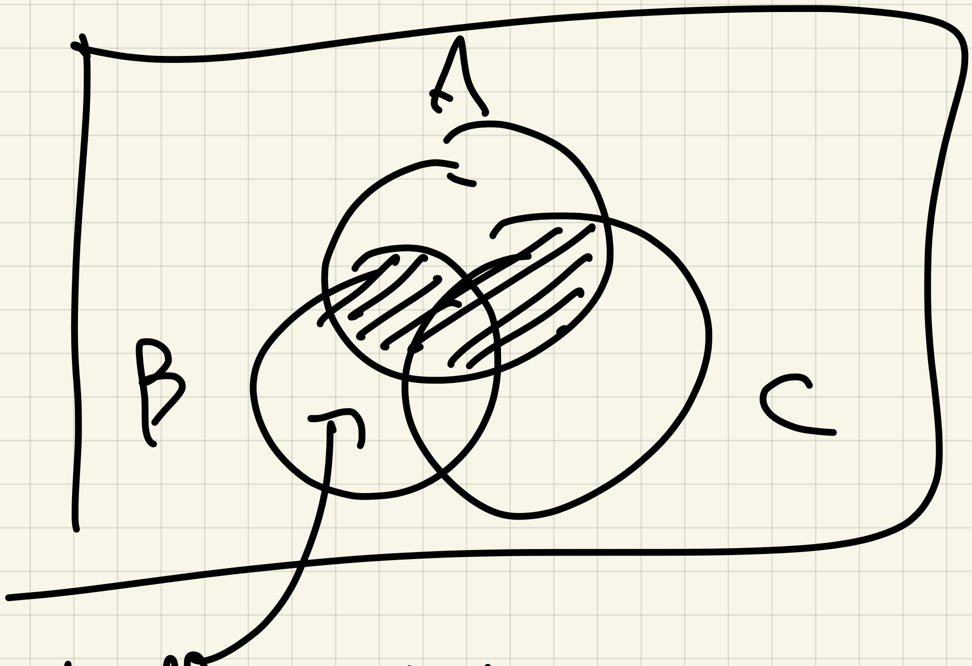
avg 83
med 85

1. $(B-C) \cup (A \cap C)$



2. $A \cap (B \cup C)$





different :

3. $A = \cancel{C} = \{1\}$

$B = \{1, 2\}$

4. $A = B = C = \{1\}$

Exam 2 $\to$ Oct 23

Last time §20

Two proof techniques:

To prove $A \Rightarrow B$

<u>Contrapositive</u> $\neg B \Rightarrow \neg A$

<u>Contradiction</u> $A \wedge \neg B \Rightarrow F$

(b/c $x \to y \equiv \neg y \to \neg x \equiv (x \wedge \neg y) \to F$)

<u>Ex)</u> If $\boxed{A \subseteq B}$ then $\boxed{A - B = \emptyset}$

<u>Proof</u> BWOC = By way of contradiction assume $A \subseteq B$ and $A - B \neq \emptyset$

Since $A - B \neq \emptyset$, $\exists x \in A - B$,

i.e., $x \in A$, $x \notin B$.

But $A \subseteq B$ means

$x \in A \to x \in B$

Look at x: ① $x \notin B$

but ② $x \in A \to x \in B$

$\Rightarrow\Leftarrow$

(contradiction)

<u>Note</u>: Ⓐ Contradiction good method to show sets are empty.

Ⓑ Contradiction also a good way to show uniqueness

<u>Ex2</u> If $a \in \mathbb{Z}$ and $0 < b \in \mathbb{Z}$

then $\boxed{\exists\, q, r \in \mathbb{Z}}$ such that

$$\begin{cases} a = bq + r \\ 0 \le r < b \end{cases}$$

q, r are <u>unique</u>

quotient and remainder

Given a, b, the q, r are unique!

i.e. If $q', r' \in \mathbb{Z}$ and

$$a = bq' + r'$$
$$0 \le r' < b$$

Then $q = q'$ and $r = r'$

Proof:

Claim 1: $r = r'$

Suppose not. Then

$a = bq + r$ $\quad\quad$ $a = bq' + r'$

$0 \le r < b$ $\quad\quad$ $0 \le r' < b$

$r \ne r'$

say $r < r'$.

1 $\boxed{0 < \boxed{r' - r} \le r' < b}$

$r \ge 0$

But also; $a = bq' + r'$

subtract $a = bq + r$

$$0 = b(q' - q) + (r' - r)$$

$r' - r = b(q - q')$

(2) so $\boxed{b \mid r' - r}$

(1) + (2) $\rightarrow$ $r' - r > 0$

and $b \mid r' - r$ $\longrightarrow$

$r' - r = bk$, $k > 0$

$r' - r \geq b$

$\Rightarrow \Leftarrow$

$r' - r < b$

Know $r = r'$

Claim 2 :

$a = bq + r$

$a = bq' + r'$

$$q = \frac{a - r}{b}$$

$$q' = \frac{a - r'}{b} = \frac{a - r}{b}$$

same ✓

§21 + §22

Ex) $\forall n \in \mathbb{N}$, n is odd or even (but not both)

Proof: Suppose not.

Then ~~there~~ $\exists n \in \mathbb{N}$: n neither odd nor even.

Then

$$C = \left\{ n \in \mathbb{N} : \begin{matrix} n \text{ not odd} \\ n \text{ not even} \end{matrix} \right\} \neq \emptyset$$

Set of counterexamples.

Let $c \in C$ be the smallest element of C.

Note $c \neq 0$ b/c $c = 2 \cdot 0$

So $c > 0$.

Since $c \in C$ is the smallest,

$c-1 \notin C$

So $c-1$ is odd or even

$c-1$ odd $\Rightarrow$ $c-1 = 2d+1$

$c = 2d+2 = 2(d+1)$

$\Rightarrow c$ even

$d \in \mathbb{Z}$

$c-1$ even $\Rightarrow$ $c-1 = 2d$

$\Rightarrow c = 2d+1$ odd

In either case, $c \notin C$

$\Rightarrow\Leftarrow$

<u>Idea</u>: <u>Least Counterexample</u>:

To prove statement $P(n)$

for all $n \in \mathbb{N}$

(or $n_0 \leq n \in \mathbb{N}$)

Let $C = \{n \in \mathbb{N} : P(n) \text{ false}\}$

BWOC assume $C \neq \emptyset$

Let $c \in C$ be smallest

Then argue as above:

$$c-1 \notin C \xrightarrow{\text{argue}} c \notin C$$

<u>Ex 2</u>: Show $n! > 3^n$ for all $n \geq 7$.

<u>Plausible</u>:

n	0	1	2	3	4	5	6	7	8
$n!$	1	1	2	6	24	120	720	5040	40320
3^n	1	3	9	27	81	243	729	2187	6561

<u>Claim</u> $n! > 3^n$ all $n \geq 7$

<u>Proof</u>: Let

$$C = \{n \in \mathbb{N} : n \geq 7 \text{ and } \underline{n! \leq 3^n}\}$$

Want $C = \phi$:

Suppose $C \neq \phi$.

Let $\boxed{c \in C \text{ be smallest in } C}$

Note $c \neq 7$ b/c

$$7! = 5040$$
$$3^7 = 2187$$

so $7 \notin C$.

Since c is small~~est~~,

$c-1 \notin C$, so

$$\boxed{(c-1)! > 3^{c-1}}$$

but

$$c! = c(c-1)! > c\,3^{c-1} > 3 \cdot 3^{c-1}$$

$c > 3$ $\qquad$ $c > 3$ $\qquad$ $3 \cdot 3^{c-1} = 3^c$

so $c! > 3^c$

so $c \notin C$ $\Rightarrow\Leftarrow$.

Remark Uses the

Well-ordering Principle (WOP)

Every nonempty subset $C \subseteq \mathbb{N}$ has a least element

This is a special fact about $\mathbb{N}$ = natural :

It's not true for others :

$C \subset \mathbb{Z}$: $C = \mathbb{Z}$ no smallest element

$C \subset \mathbb{Q}$: $C = \mathbb{Q}$ no smallest

$$C \subset \mathbb{Q}_+ = \{q \in \mathbb{Q}, q > 0\}$$

$C = \mathbb{Q}_+$ has no smallest element :

If q was smallest

then $q/2$ even smaller.

Important special case:

§22

To prove $P(0), P(1), P(2) \ldots P(n) \ldots$

prove $P(n)$ for all $n \in \mathbb{N}$

It's enough to prove two statements

ⓐ $P(0)$ true

ⓑ $P(n) \rightarrow P(n+1)$ true for $n \geq 0$

Called proof by induction

Induction base

Induction step

Variation: to prove

$P(n_0), P(n_0+1), \ldots P(n), \ldots$

enough to prove

Base $P(n_0)$ true

step $P(n) \to P(n+1)$, $n \geq n_0$

Ex 1 The sum of the first n odd natural numbers is n^2

$n=1$ $\quad 1 = 1^2$

$n=2$ $\quad 1+3 = 2^2$

$1+3+5 = 3^2$

$1+3+5+7+9+11 = 6^2$