11/4/ Discrete Quiz 14

$P(n)$: $3^1 + 3^2 + 3^3 + \dots + 3^n = \frac{3^{n+1} - 3}{2}$

$P(1)$ LHS $3^1 = 3$ $\frac{3^{1+1} - 3}{2}$
$n=1$

$\frac{3^2 - 3}{2} = \frac{6}{2}$
$=$
3 RHS

$P(2)$

$3 + 3^2 = 3 + 9 = 12$ LHS ✓

$\frac{3^3 - 3}{2} = \frac{27 - 3}{2} = \frac{24}{2} = 12$

2. Prove all $P(n)$ true all $n \geq 1$ by induction

Base: $P(1)$ is true ✓ by #1

Step

Induct hyp: Assume $P(n)$:

$$3+3^2+\ldots+3^n = \frac{3^{n+1}-3}{2}$$

Prove $P(n+1)$

NTS $\underline{3+3^2+\ldots+3^n+3^{n+1} = \frac{3^{n+2}-3}{2}}$

add 3^{n+1} to both sides

$$3+3^2+\ldots+3^n+3^{n+1} = \frac{3^{n+1}-3}{2}+3^{n+1}$$

$$\frac{3^{n+1}-3}{2}+\frac{2\cdot 3^{n+1}}{2} =$$

$$\frac{\boxed{3^{n+1}}-3+\boxed{2\cdot 3^{n+1}}}{2}$$

$$2\cdot 3^{n+1} + 1\cdot 3^{n+1} - 3$$

$$\frac{(2+1)3^{n+1} - 3}{2} = \frac{3(3^{n+1}) - 3}{2}$$

$$= \frac{3^{n+2} - 3}{2}$$

§2.4

§2.5

Thm 1 A, B finite sets

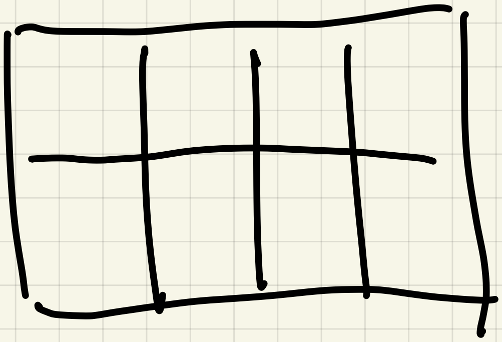
$f: A \to B$

$\boxed{1}$ $|A| > |B| \Rightarrow f$ is not 1-1

(2) $|A| < |B| \Rightarrow f$ is not onto

(Pigeonhole principle) ??

"If p pigeons are placed in h pigeonholes and $p > h$, then some pigeonhole has at least 2 pigeons"



Ex1 Show that among 50000 workers, at least 2 workers have the same last 4 digits of SSN.

Why?

Workers = pigeons

last 4 digits of SSN are pigeon holes

$p = 50000$

$h = 10^4 = 10000$ $\quad \overline{10}\ \overline{10}\ \overline{10}\ \overline{10}$

$p > h \Rightarrow$ at least two workers have same last 4 digits of SSN.

<u>Alternatives</u>:

$$f : \{\text{Workers}\} \to \{\text{last 4 digits}\}$$

$$\quad\ \ /A \qquad\qquad\qquad\qquad B$$

$|A| = 50000$

$|B| = 10000$ $\qquad |A| > |B|$

Thm $\Rightarrow$ f is not 1-1

<u>Follow up</u>: Can we say more?

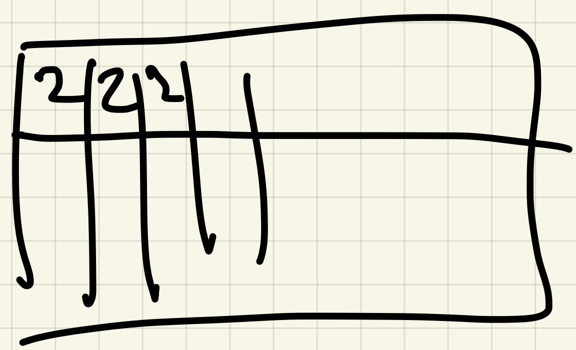
Must there be at least 3

workers with same last 4 digits

<u>YES</u>:

otherwise at most

SSN



20000 workers

$\boxed{4}$? yes

$\boxed{5}$? yes

$\boxed{6}$ no

i.e. exactly 5 workers have same last 4 digits.

But for 50001 workers, yes, at least 6.

Ex2 Show that there are $a \neq b$ in $\mathbb{N}$ so that

$$10 \mid 7^a - 7^b.$$

Pf: $10 \mid 7^a - 7^b$ if last digit of $7^a, 7^b$ are same

$10 \mid 7^6 - 5^6$, $\quad 10 \nmid 7^2 - 4^3$

Let $f : \mathbb{N} \to \{0, 1, 2, \ldots 9\}$

$a \mapsto$ last digit of 7^a

$|\mathbb{N}| > |\{0, \ldots 9\}| = 10$

so f is not 1-1 $\to$

$\exists\, a \neq b : 7^a, 7^b$ same last digit.

How does this play out?

$7^0 = 1 \quad 7^1 = 7 \quad 7^2 = 49$

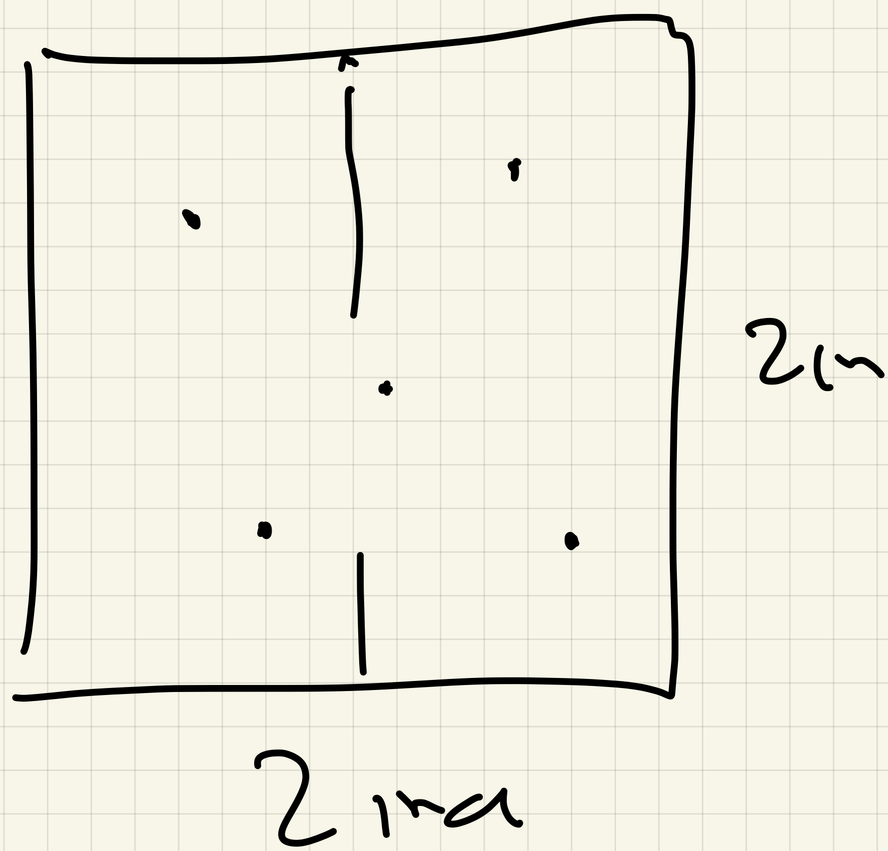
$7^3 = 343, \quad 7^4 = 2401,$

$10 \mid 7^4 - 7^0 = 2400$

$7^5 = 16807$

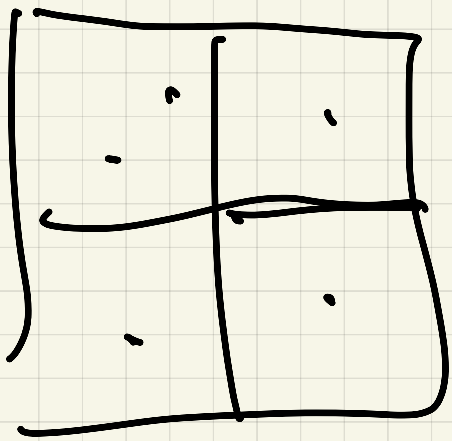
$10 \mid 7^5 - 7^1$

Ex 3 Given 5 points in a 2in × 2in square, show that two of them are within $\sqrt{2}$ of each other

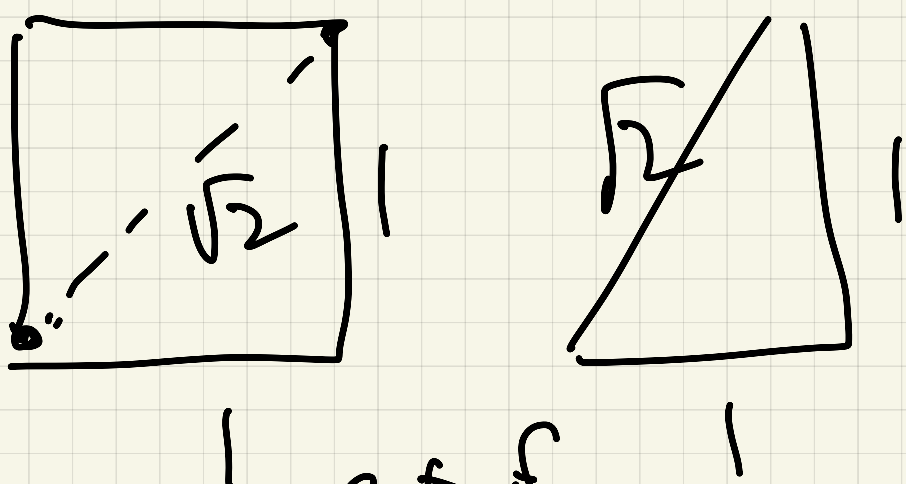


Idea: break square into 4

1in × 1in square



PHP → some 1×1 square has 2 points in it.



Ex 4 Among any set of 6 distinct integers in $[0, 10]$, there are two subsets with the same sum.

0 1 2 4 8 0

Pf: Take 6 integers

$a_1 \ldots a_6$

pigeons = set of all subsets of 6 ints

holes = sums

$p = \#\text{pigeons} = 2^6 = 64$

holes

$\underbrace{0 \le \text{sums} \le 60}_{h = 61 < 64}$

PHP $\to$ two subsets have same sum

§35 Division of integers:

Theorem 1 (Division algorithm)

Let $a, b \in \mathbb{Z}$ with $b > 0$. Then there exist unique integers $q, r \in \mathbb{Z}$ with

① $a = bq + r$

② $0 \le r < b$

Notation: q = quotient
r = remainder

Notation:
$$\boxed{\begin{array}{l} q = a \operatorname{div} b \\ r = a \bmod b \end{array}}$$

Ex1 $a = 50$, $b = 3$

$$50 = 3 \cdot 16 + 2$$

(a)

$q=16$, $r=2$

$50 \text{ div } 3 = 16$

$50 \text{ mod } 3 = 2$

(b) $a=50$, $b=7$

$50 \text{ div } 7 = 7$

$50 \text{ mod } 7 = 1$

$50 = 7\cdot 7 + 1$

(c) $a=-50$, $b=7$

$$-50 = \underbrace{7\cdot(-8)}_{-56} + 6$$

$-50 \text{ div } 7 = -8$

$-50 \text{ mod } 7 = 6$

(d) $a=-50$, $b=20$

$-50 \text{ div } 20 = -3$

$-50 \text{ div } 20 = 10$

$$-50 = (-3)(20) + 10$$

<u>Calculator</u>; $\begin{cases} q = \left\lfloor \frac{a}{b} \right\rfloor, \\ r = a - bq \end{cases}$

$$a = bq + r$$

$$\lfloor x \rfloor = \{\max\{n \in \mathbb{Z};\ n \le x\}\}$$

<u>Ex 2</u> $a = 45{,}980$

$b = 346$

$$\frac{a}{b} = 132.540\ldots$$

$$q = \left\lfloor \frac{a}{b} \right\rfloor = 132 = 45980 \text{ div } 346$$

$$r = 45{,}980 - (132)(346) = 308$$

$$\parallel$$

$$45980 \bmod 346$$

<u>Defn</u>: For $a, b, m \in \mathbb{Z}$, $m > 0$,

we say

"a is <u>equivalent to b mod m</u>"

if $m \mid a-b$

<u>Notation</u>: $a \equiv b \pmod{m}$

Recall handy divisibility tricks

$d = a_n a_{n-1} \dots a_0$

$d = 87{,}181$

a) $2 \mid d \iff 2 \mid a_0$

b) $5 \mid d \iff 5 \mid a_0$

c) $3 \mid d \iff 3 \mid \sum a_i$

d) $9 \mid d \iff 9 \mid \sum a_i$

$3 \mid 111111$ b/c $3 \mid 6$ ✓

$111111 = 3 \cdot 37{,}037$

e) $11 \mid d \iff 11 \mid \sum (-1)^i a_i$

$4 \mid 111{,}111$

$111{,}111 = 11 \times 10101$

<u>Ex3</u> $7 \equiv 10 \pmod 3$

$50 \not\equiv -50 \pmod 3$

$3 \equiv 21 \pmod 2$ $\quad 3 \mid \underbrace{50-(-50)}$

$3 \nmid 100$

$\vdots$