Home Work 8: Sections 17 and 19 **October, 2024**
Discrete Mathematics **Prof. Nollet**

1. $S = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : |x| + |y| \leq 22\}$ consists of integer points in $\mathbb{R}^2$ inside square bounded by the lines $y + x = \pm 20, y - x = \pm 20$.

   Solution 1: Like the class example, there are $\binom{22}{2} = 231$ points in each of the four quadrants not including the x and y axes: the x-axis has 45 as does y-axis, but origin is counted twice, so we get a total of $4 \times 231 + 45 + 45 - 1 = 1013$.

   Solution 2: Split into 4 triangles that fit together like a windmill but miss origin. Each has $\binom{23}{2} = 253$ points, so total is $4 \times 253 + 1 = 1013$.

   Solution 3: Triangle in quadrant I with both axes has $\binom{24}{2} = 276$ points, two triangles in quadrants II and IV have $\binom{23}{2} = 253$ points each and small triangle in quadrant III has $\binom{22}{2} = 231$ more for total of $276 + 253 + 253 + 231 = 1013$.

   Solution 4: Counting by the number in the 45 columns gives

   $$|S| = 1 + 3 + 5 + \cdots + 43 + 45 + 43 + \cdots + 3 + 1$$

   Group and add the first 20 and last 20 terms to get $22 \times 44 = 968$ and there are 45 more for the middle column giving 1013.

   Solution 5: From a diagonal perspective we see a 23×23 square with $23^2 = 529$ entries, but looking more closely reveals a square inside that is 22×22 for 484 more, a total of 1013.

2. (a) We choose 5 toppings from 8 for $\binom{8}{5} = 56$ possibilities.

   (b) Now we allow two of the five toppings to be the same, so we need to count the number of ways to have four toppings with one doubled and add to the answer from part (a).

   Solution 1: There are $\binom{8}{4} = 70$ ways to select four different toppings and for each there are 4 ways to choose the doubled topping, leading to $4 \times 70 = 280$ new possibilities. Adding to part (a) gives $280 + 56 = 336$.
   Solution 2: There are 8 ways to choose the doubled topping and for each such choice there are $\binom{7}{3} = 35$ ways to choose the remaining two, leading to $8 \times 35 = 280$ new possibilities. Adding those from part (a) gives $280 + 56 = 336$.

3. (a) There are $\binom{52}{5}$ hands of size 5 from a deck of 52.

   (b) A full house consists of three cards of one value and two of another. There are $\binom{13}{1} = 13$ ways to choose the tripled card value and $\binom{4}{3} = 4$ ways to choose their suits; there are $\binom{12}{1} = 12$ ways to choose the double card value and $\binom{4}{2} = 6$ ways to choose their suits. Therefore the total number is $13 \times 4 \times 12 \times 6 = 3744$.

   (c) A flush has 5 cards of the same suit. There are $\binom{4}{1} = 4$ ways to choose the suit and $\binom{13}{5} = 1287$ ways to choose the card values of that suit, so in total we have $4 \times 1287 = 5148$.

   (d) Three of a kind consists of three cards with the same value and two more cards with different values.

   Solution 1: As in part (b) there are $13 \times 4 = 52$ ways to choose 3 of a kind; there are $\binom{12}{2} = 66$ ways to choose two more values and $4 \times 4 = 16$ ways to choose their suits, so we have a total of $13 \times 4 \times 66 \times 4 \times 4 = 54,912$.

   Solution 2: There are $13 \times 4 = 52$ ways to choose 3 of a kind, 48 ways to choose the 4th card and 44 ways to choose the 5th card, but we have double counted (switching 4th and 5th cards gives the same hand), so total is $52 \times 48 \times 44/2 = 54,912$.

4. (a) There are $\binom{25}{7} = 480700$ ways to choose 7 marbles from 25.

   (b) Taking from the 11 yellow marbles gives $\binom{11}{7} = 330$.

   (c) There are $\binom{8}{4} = 70$ ways to choose 4 reds, $\binom{6}{3} = 20$ ways to choose 2 blue, so total number is $70 \times 20 = 1400$.

5. Let S_k be the set of integers $1 \leq n \leq 16,000$ that are multiples of k. Similar to the class example we have $|S_3| = \lfloor 16000/3 \rfloor = 5333$, $|S_5| = \lfloor 16000/5 \rfloor = 3200$ and $|S_7| = \lfloor 16000/7 \rfloor = 2285$. The intersections have sizes $|S_3 \cap S_5 = S_{15}| = \lfloor 16000/15 \rfloor = 1066$, $|S_3 \cap S_7| = \lfloor 16000/21 \rfloor = 761$, $|S_5 \cap S_7| = \lfloor 16000/35 \rfloor = 457$ and finally $|S_3 \cap S_5 \cap S_7| = \lfloor 16000/105 \rfloor = 152$, so the inclusion-exclusion formula gives $|S_3 \cup S_5 \cup S_7| = 5333 + 3200 + 2285 - 1066 - 761 - 457 + 152 = 8686$.

   (b) There are $|S_3 \cup S_5| = |S_3| + |S_5| - |S_3 \cap S_5| = 5333 + 3200 - 1066 = 7467$ multiples of 3 or 5 and we want to remove multiples of 7, which is the set $(S_3 \cup S_5) \cap S_7 = (S_3 \cap S_7) \cup (S_5 \cap S_7)$. Inclusion-exclusion on this last union gives $761 + 457 - 152 = 1066$ and subtraction gives $7467 - 1066 = 6401$.

6. We are counting the lists (a, b, c) with $1 \leq a, b, c \leq 20$ such that at least one of a, b, c is even.

   Solution 1: Let S_k be the set of such lists with the kth place an even number. Then $|S_i| = 10 \times 20 \times 20 = 4000$ and $|S_i \cap S_j| = 10^2 \times 20 = 2000$ and $|S_1 \cap S_2 \cap S_3| = 10^3 = 1000$ so inclusion-exclusion gives $|S_1 \cup S_2 \cup S_3| = 3 \times 4000 - 3 \times 2000 + 1000 = 7000$.

   Solution 2: There are $20^3 = 8000$ lists and $10^3 = 1000$ have a, b, c odd. Subtracting gives $8000 - 1000 = 7000$ possible lists.

7. Solution 1: If S is the set of all colorings, $|S| = 2^{25} = 33554432$ (two choices of color for each of 16 squares). Let $S_i \subset S$ be the colorings with row i of one color. Then $|S_i| = 2 \times 2^{20} = 2097152$ (2 ways to color row i and 2^{20} ways to color the remaining squares). Similarly $|S_i \cap S_j| = 2 \times 2 \times 2^{15} = 131072$ and $|S_i \cap S_j \cap S_k| = 2 \times 2 \times 2 \times 2^{10} = 8192$ and $|S_i \cap S_j \cap S_k \cap S_l| = 2 \times 2 \times 2 \times 2 \times 2^5 = 512$ and $|S_1 \cap S_2 \cap S_3 \cap S_4 \cap S_5| = 2^5 = 32$. Inclusion-exclusion gives $|S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5| = 5 \times 2097152 - 10 \times 131072 + 10 \times 8192 - 5 \times 512 + 32 = 9254432$ and so $|S - (S_1 \cup S_2 \cup S_3 \cup S_4)| = 33554432 - 9254432 = 24300000$.

   Solution 2: The number of ways to color one row without using all red or all yellow is $2^5 - 2 = 30$. By multiplication principle there are $30^5 = 24300000$ colorings of all 5 rows.

8. If P, C, J are the sets that have taken Precalculus, Calculus and Java, the information provided gives $|P| = 30, |C| = 18, |J| = 26, |P \cap J| = 16, |P \cap C| = 9, |C \cap J| = 8, |P \cup C \cup J| = 47$.

   (a) Only $50 - 46 = 4$ didn't take any of the courses.

   (b) Using inclusion-exclusion gives $46 = 30 + 18 + 26 - 16 - 8 - 9 + |P \cap C \cap J|$, so $|P \cap C \cap J| = 5$.

   (c) $|P \cap C| = 9$ and $|(P \cap C) \cap J| = 5$, so $|(P \cap C) - J| = 9 - 5 = 4$.

   (d) $|P \cap (C \cup J) = (P \cap C) \cup (P \cap J)| = 9 + 16 - 5 = 20$, so $|P - C - J = P - (P \cap (C \cup J))| = 30 - 20 = 10$.