9/3 | Calc 3 Quiz 3

$\bar{u} = \langle 2, -1, 1 \rangle$

$\bar{v} = \langle 2, 4, 2 \rangle = 2\langle 1, 2, 1 \rangle$

(a) $\bar{u} \cdot \bar{v} = 4 - 4 + 2 = 2$

(b) $\cos\theta = \dfrac{\bar{u} \cdot \bar{v}}{|\bar{u}||\bar{v}|} = \dfrac{2}{\sqrt{6}\; 2\sqrt{6}} = \dfrac{\cancel{2}}{\cancel{2} \cdot 6} = \dfrac{1}{6}$

(c) $\bar{w} = \langle 1, -1, 1 \rangle$

$\bar{u} \cdot \bar{w} > 0 \Rightarrow$ not $\perp$

$\bar{v} \cdot \bar{w} = 0 \Rightarrow \bar{v} \perp \bar{w}$

(d) $\text{Proj}_{\bar{v}} \bar{u} = \dfrac{\bar{u} \cdot \bar{v}}{|\bar{v}|^2} \bar{v}$

$= \dfrac{2}{24} \langle 2, 4, 2 \rangle =$

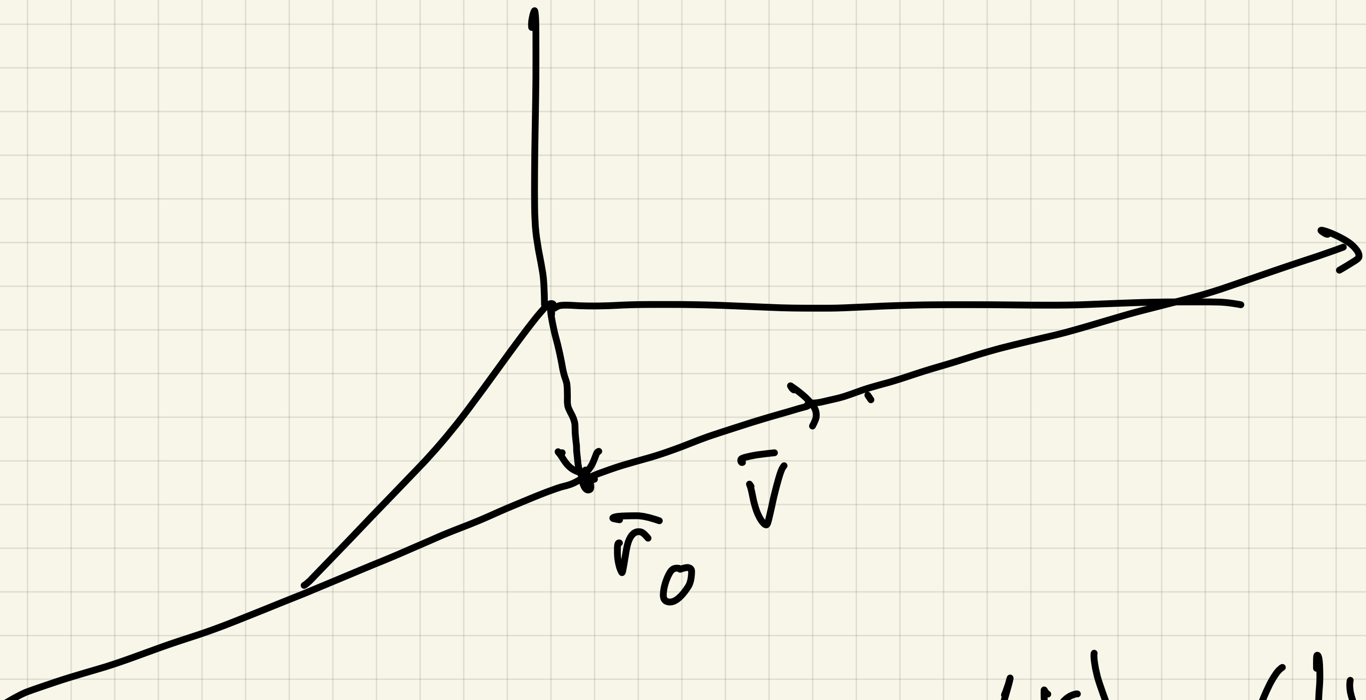
$\dfrac{1}{12} \langle 2, 4, 2 \rangle = \langle \frac{1}{6}, \frac{1}{3}, \frac{1}{6} \rangle$

Last time Parametric lines in $\mathbb{R}^3$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underbrace{\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}}_{\bar{r}_0 \text{ pt}} + t \underbrace{\boxed{\begin{pmatrix} a \\ b \\ c \end{pmatrix}}}_{\bar{v} \text{ direction}}$$

book

$$\bar{r}(t) = \bar{r}_0 + t\bar{v}$$



Ex] L_1 = is line $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1+2t \\ 2+3t \\ 4-5t \end{pmatrix}$

(Last time: $Q = \begin{pmatrix} -5 \\ -7 \\ 19 \end{pmatrix}$ is on L_1)
$t = -3$

(a) Does L_1 intersect L_2

L_2 is line $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3-4t \\ 7-6t \\ 1+10t \end{pmatrix}$

<u>Usually</u>, solve

$$\boxed{\begin{pmatrix} 1+2t \\ 2+3t \\ 4-5t \end{pmatrix}} = \begin{pmatrix} 3-4s \\ 7-6s \\ 1+10s \end{pmatrix}$$

for s & t

<u>Note</u> $-2\begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix} = \begin{pmatrix} -4 \\ -6 \\ 10 \end{pmatrix}$

<u>$L_1 \,//\, L_2$</u> or <u>$L_1 = L_2$</u>

<u>test</u>: $\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$ on L_1,
$(t=0)$

is $\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$ on L_2 ??

Solve $\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 3-4s \\ 7-6s \\ 1+10s \end{pmatrix}$??

No solution!! :

$\nexists$ $\quad 1 = 3 - 4s \quad s = \frac{1}{2}$

$2 = 7 - 6s \quad s = 5/6$

$4 = 1 + 10s \quad s = 3/10$

So Lines disjoint.

(b) Show that L_1 intersects

$$L_3 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3+3t \\ 13-4t \\ -3+t \end{pmatrix}$$

solve for s, t $\longrightarrow$

$$\underbrace{\begin{pmatrix} 1+2t \\ 2+3t \\ 4-5t \end{pmatrix}}_{L_1} = \underbrace{\begin{pmatrix} -3+3s \\ 13-4s \\ -3+s \end{pmatrix}}_{L_3}$$

$x: \; 1+2t = -3+3s \Rightarrow -3s+2t = -4$

y

z

$$\boxed{\begin{aligned} 4s+3t &= 11 \\ -s-5t &= -7 \end{aligned}}$$

$$\begin{aligned} 4s + 3t &= 11 \\ \text{add} \quad -4s - 20t &= -28 \\ \hline -17t &= -17 \end{aligned}$$

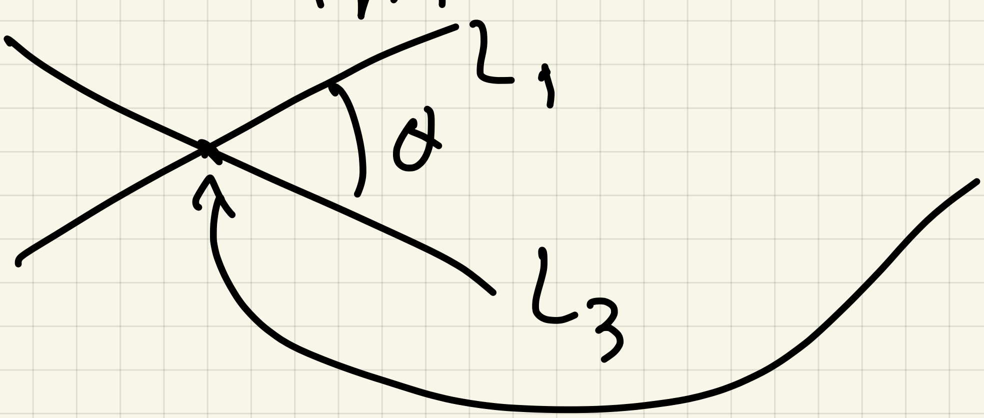
$t = 1, \; s = 2$

these also solve

$1 + 2t = -3 + 3s$

so lines

intersect at $\begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix}$



What is the angle of intersection between the lines?

directions $\quad \bar{u} = \langle 2, 3, -5 \rangle$

$\bar{v} = \langle 3, -4, 1 \rangle$

$$\cos\theta = \frac{\bar{u} \cdot \bar{v}}{|\bar{u}|\,|\bar{v}|} = \frac{6 - 12 - 5}{\sqrt{38}\,\sqrt{26}} =$$

$$\frac{-11}{\sqrt{988}} = \cos\theta \to \theta > 90^\circ$$

Want angle smaller than 90°

replace $\bar{u}$ with $-\bar{u}$

$$\cos\theta = \frac{11}{\sqrt{988}} \to$$

$$\theta = \cos^{-1}\frac{11}{\sqrt{988}} = 69.52^\circ$$

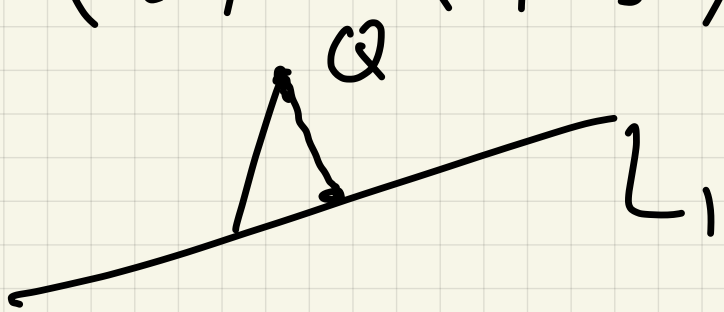
$$= 1.213 \text{ rad}$$

Ex2 Find the distance

from point $Q = \begin{pmatrix}1\\2\\3\end{pmatrix}$ to $r(t)$

line L_1: $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1+2t\\2+3t\\4-5t\end{pmatrix}$

Approach?

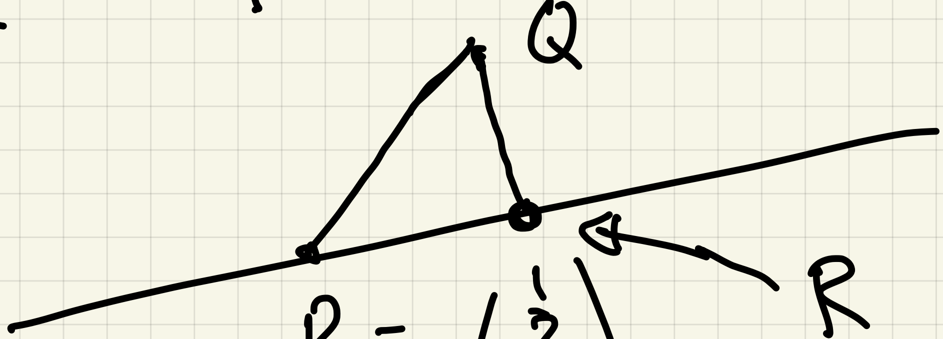


(A)

<u>minimize</u>: $\text{dist}(Q, r(t))^2 = f(t)$

<u>Calc1</u> $f'(t) = 0$

(B)

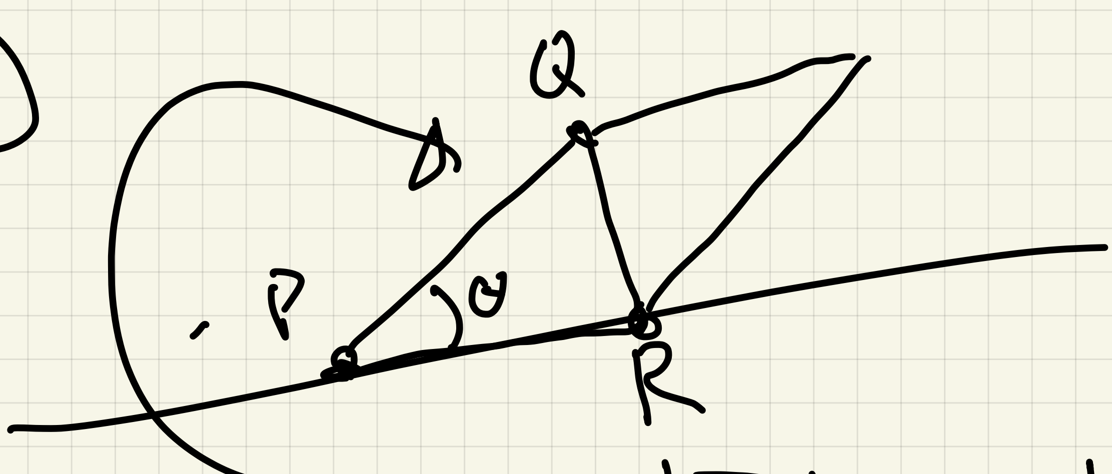


$P = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$

dist Q to L_1 is

$$\left|\overrightarrow{PQ} - \underline{\text{Proj}_{\vec{v}} \overline{PQ}}\right| \qquad \vec{v} = \begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix}$$

(C)



Area ▱ $= |\overline{PQ}| \sin\theta \, |PR|$

$|RQ| = $ ht ▱ $= \dfrac{\text{Area}}{\text{base}} =$

$$\frac{|PQ \times PR|}{|PR|}$$

$\overrightarrow{PR} \parallel \bar{v} = \begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix}$

There's a scalar c :
$\overline{PR} = c\langle 2, 3, -5\rangle$

$$\frac{|PQ \times cv|}{|c\bar{v}|} = \frac{|\cancel{c}||PQ \times v|}{|\cancel{c}||\bar{v}|} =$$

$$\frac{|PQ \times \bar{v}|}{|\bar{v}|}$$

(p. 643)

$P = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$
$Q = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

$\overline{PQ} = \langle 0, 0, -1\rangle$

$\bar{v} = \langle 2, 3, -5\rangle$

$$\overline{PQ} \times \bar{v} = \begin{vmatrix} i & j & k \\ 0 & 0 & -1 \\ 2 & 3 & -5 \end{vmatrix} =$$

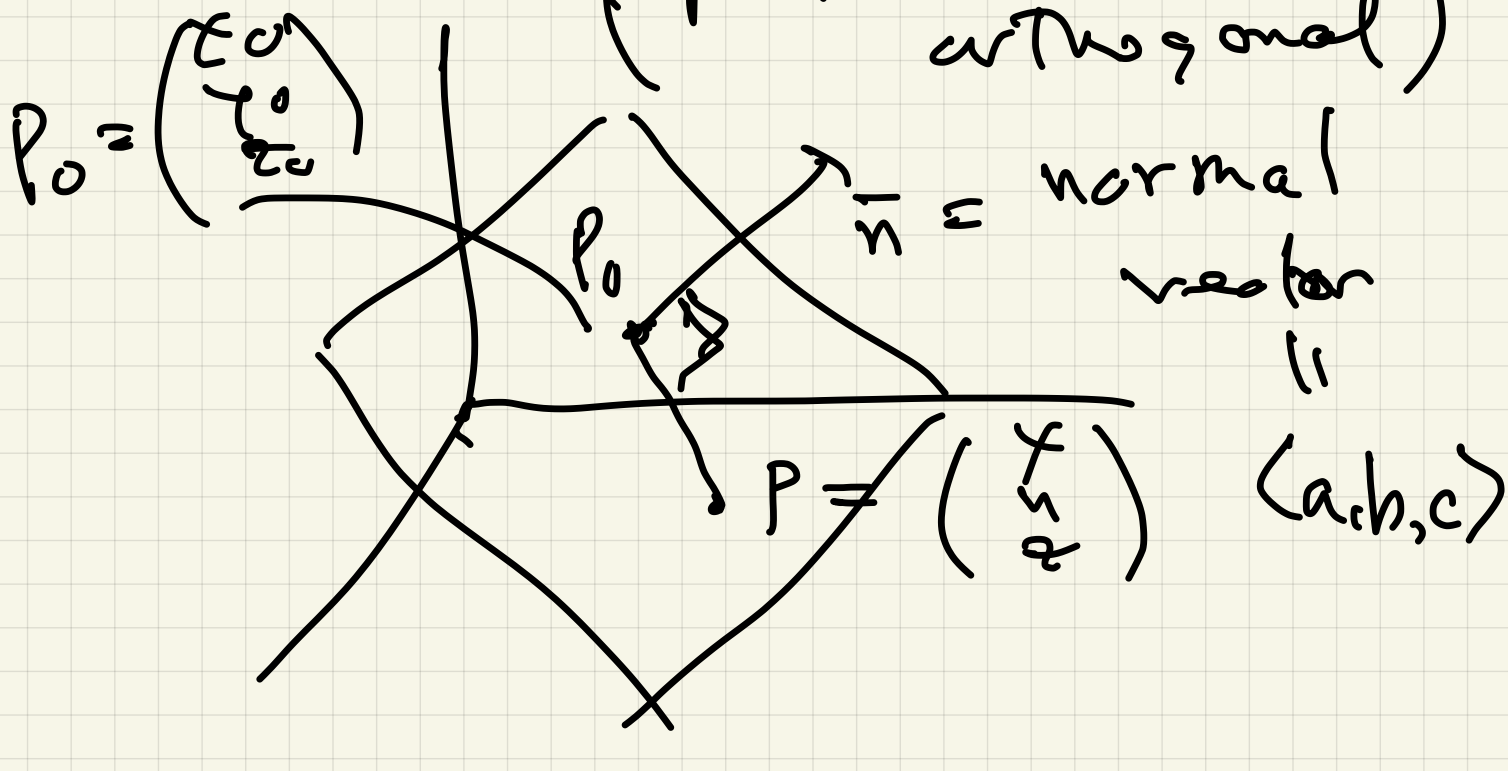
$\langle 3, -2, 0\rangle$

so $d(t) = \dfrac{|\langle 3,-2,0\rangle|}{|\langle 2,3,-5\rangle|} = \dfrac{\sqrt{13}}{\sqrt{38}} = \sqrt{\dfrac{13}{38}} < 1$ ✓

<u>Planes in $\mathbb{R}^3$</u>

A plane has no <u>direction</u>, but it <u>does</u> have a <u>normal</u> vector

(perpendicular, orthogonal)



P is on the plane $\iff$

$\overline{P_0P} \perp \bar{n} \iff$

standard form $\quad \overline{P_0P} \cdot \bar{n} = 0$

$\langle x-x_0, y-y_0, z-z_0 \rangle \cdot \langle a, b, c \rangle = 0$

$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$

Ex) The plane through point $P_0 = \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix}$ and normal to $\bar{n} = \langle 2, 3, 5 \rangle$ has equation:

$2(x-3) + 3(y-5) + 5(z+1) = 0$

$\Downarrow$

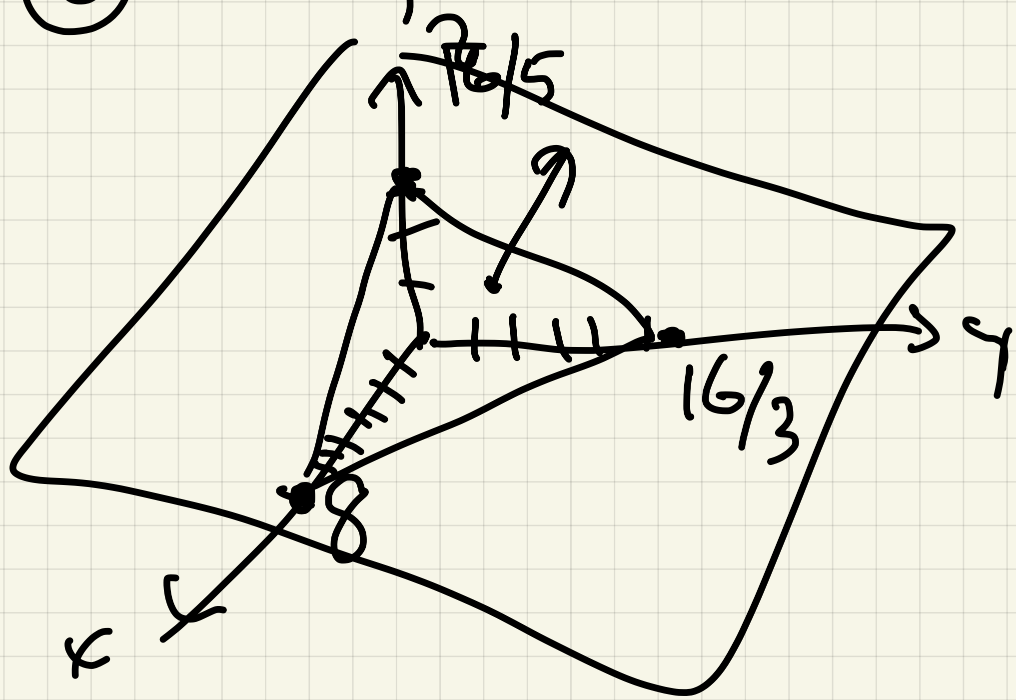
$2x + 3y + 5z - 6 - 15 + 5 = 0$

$$2x + 3y + 5z = 16$$

simplified form.

Note ① Easy to read off normal vector $\langle 2, 3, 5 \rangle$

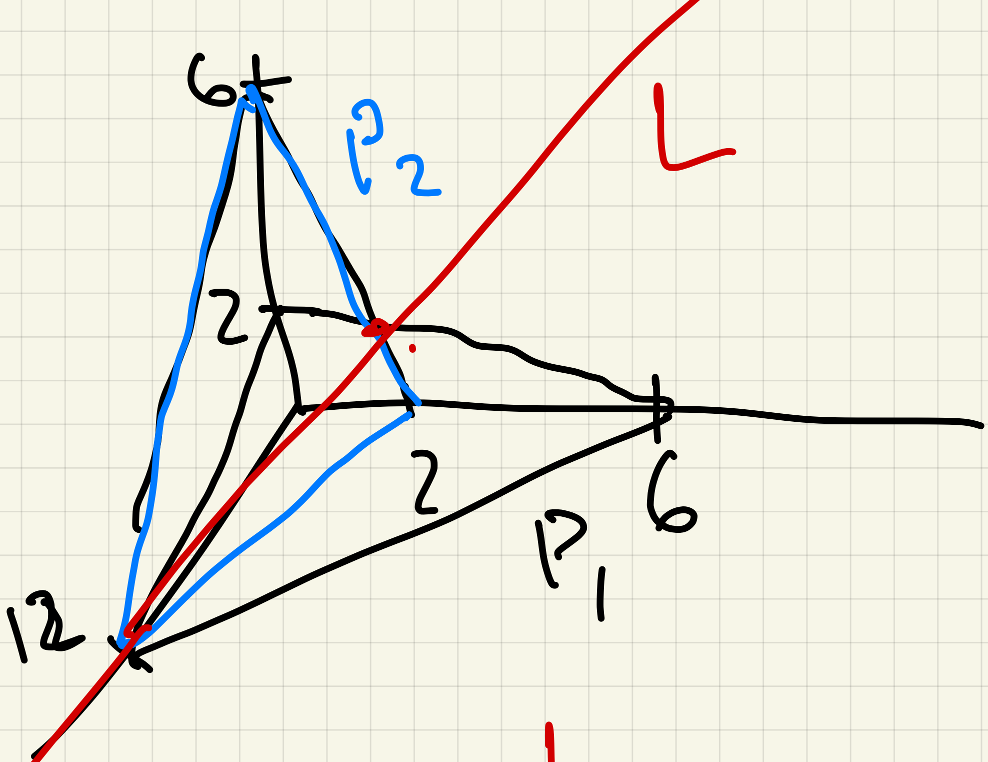
② Easy to sketch:



Ex2 Two planes

$P_1 \quad x + 2y + 6z = 12$

$P_2 \quad x + 6y + 2z = 12$



(a) find the line L of intersection of P_1 & P_2

<u>Note</u>: $P_0 = \begin{pmatrix} 12 \\ 0 \\ 0 \end{pmatrix} = (12, 0, 0)$

is on line, need <u>direction</u> of line L.

P_1: normal vector $n_1 = \langle 1, 2, 6 \rangle$

P_2 normal vector $n_2 = \langle 1, 6, 2 \rangle$

$\therefore$ direction of L is $\bar{n}_1 \times \bar{n}_2 =$

$$\begin{vmatrix} i & j & k \\ 1 & 2 & 6 \\ 1 & 6 & 2 \end{vmatrix} = \langle -32, 4, 4 \rangle$$

$$/\!/$$

$$\langle -8, 1, 1 \rangle \checkmark$$

$$\therefore L: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 12 - 8t \\ t \\ t \end{pmatrix}$$

(b) Find angle between the planes P_1 & P_2

Should be angle between normal vectors:

$$\cos\theta = \left| \frac{\bar{n}_1 \cdot \bar{n}_2}{|n_1||n_2|} \right| = \frac{25}{\sqrt{41}\sqrt{41}} =$$

$$\frac{25}{41} \Rightarrow \theta = \cos^{-1}\left(\frac{25}{41}\right)$$