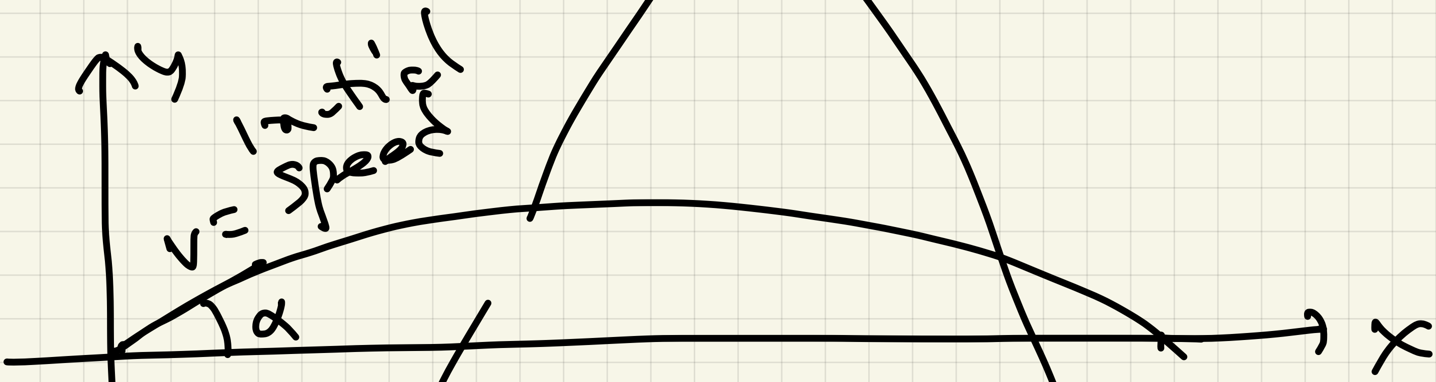
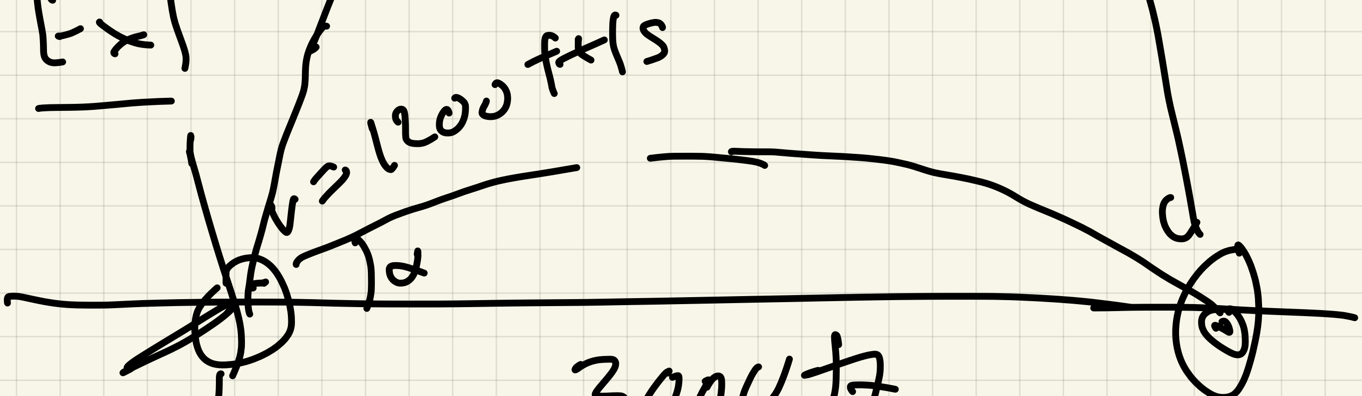
9/12 Calc 3

Exam 1



$$\vec{r}(t) = \left\langle \underbrace{v(\cos\alpha)t}_{x},\ v(\sin\alpha)t - \tfrac{1}{2}gt^2 \right\rangle$$

Ex 1



3000 ft

$v = 1200$ ft/sec. $g = 32\ \text{ft}/s^2$

$$\vec{r}(t) = \left\langle \underline{1200(\cos\alpha)t},\ \underbrace{1200(\sin\alpha)t - 16t^2}_{y} \right\rangle$$

If T = time when bullet hits target

$y(T) = 0 \Rightarrow 1200(\sin\alpha)T - 16T^2$

$\parallel$
0

$T = 0$ (not interesting)

$$1200(\sin\alpha) - 16T = 0$$

$$T = \frac{1200\sin\alpha}{16}$$

$$= 75\sin\alpha$$

At time $T = 75\sin\alpha$,

$$x = 1200\cos\alpha\, T$$

$$3000 = 1200\,\underline{\cos\alpha}\,75\,\underline{\sin\alpha}$$

$$\underline{\underline{\cos\alpha\sin\alpha}} = \frac{3000}{1200\cdot 75} = \frac{1}{30}$$

$$\boxed{\sin 2\alpha = 2\,\underline{\underline{\sin\alpha\cos\alpha}}}$$

$$\frac{1}{2}\sin 2\alpha = \frac{1}{30} \qquad \sin 2\alpha = \frac{1}{15}$$

$$2\alpha = \arcsin\frac{1}{15} \cong 3.822^\circ$$

$$\alpha = 1.911^\circ$$

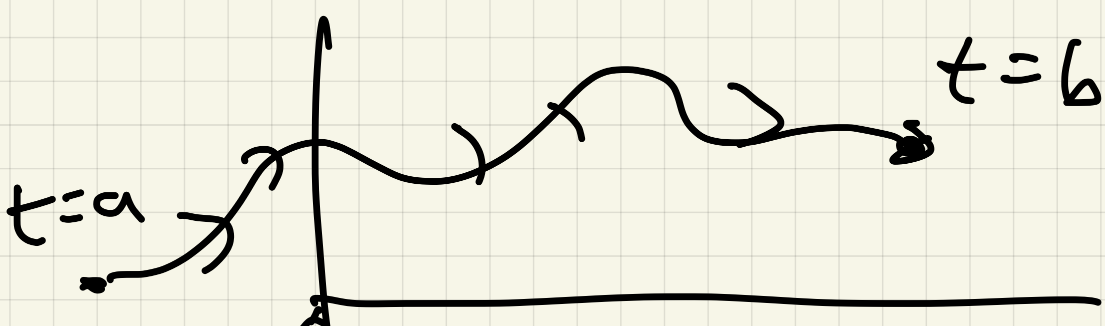
(Also $2\alpha = 180^\circ - 3.822$

$\alpha = 90 - 1.911 = 88.089^\circ$)

§12.3-4 Arclength & Curvature

Suppose a curve C in $\mathbb{R}^3$ is traced $\boxed{\text{once}}$ by a vector valued function (one direction)

$\vec{r}(t)$; $a \le t \le b$



Know dist = speed · time

$\Downarrow$ Calc

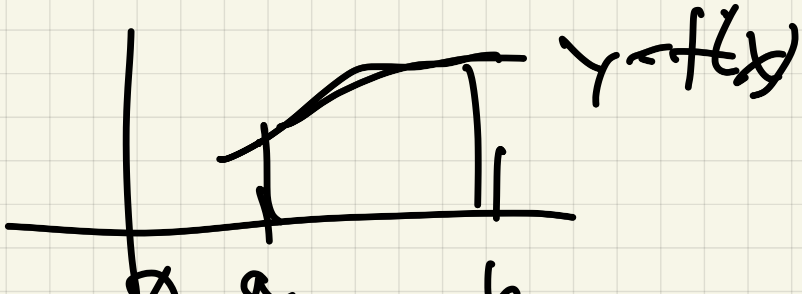
$$\text{dist} = \int (\text{speed})\, dt$$

$\Downarrow$

Define Arclength of C =

$$L = \int_a^b |\vec{r}'(t)|\, dt$$

In Calc2: saw $y = f(x)$,



$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

parametric

$$\vec{r}(t) = \langle \underset{x}{t}, \underset{y}{f(t)} \rangle$$

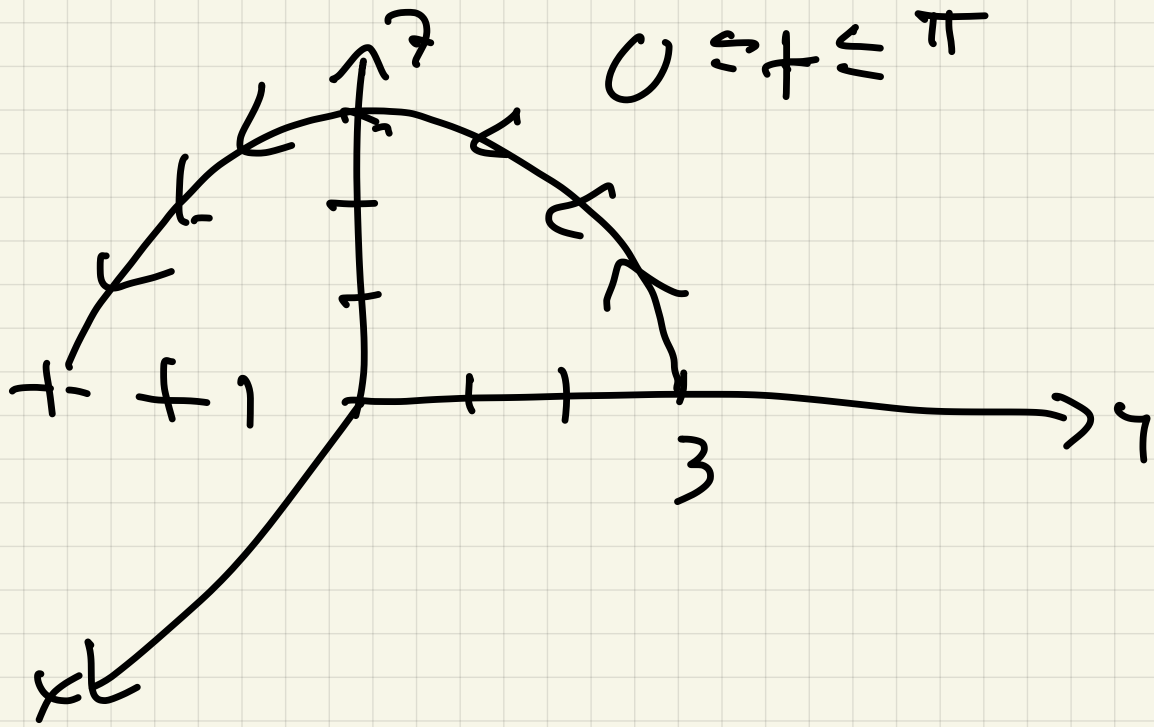
$$\int_a^b |\vec{r}'(t)|\, dt = \int_a^b |\langle 1, f'(t) \rangle|\, dt$$

$$\parallel$$

$$\int_a^b \sqrt{1^2 + (f'(t))^2}\, dt$$

Ex $\bar{r}(t) = \langle 0, 3\cos t, 3\sin t \rangle$

$0 \le t \le \pi$



Arc length = L =

$$\int_0^{\pi/2} |\bar{r}'(t)|\, dt =$$

$$\int_0^{\pi/2} |\langle 0, -3\sin t, 3\cos t \rangle|\, dt$$

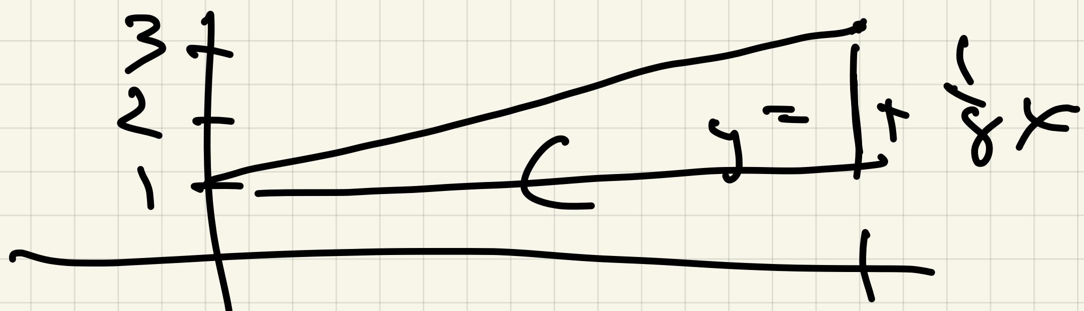
$$\int_0^{\pi/2} \sqrt{0^2 + 9\sin^2 t + 9\cos^2 t}\, dt$$

$$\int_0^{\pi/2} \sqrt{9} = \int_0^{\pi/2} 3\,dt =$$

$$3t \Big|_0^{\pi/2} = \frac{3\pi}{2}$$

$\frac{1}{2}$ circum of rad 3 circle ✓

Ex 2



Find arclength :

(1) dist between pts

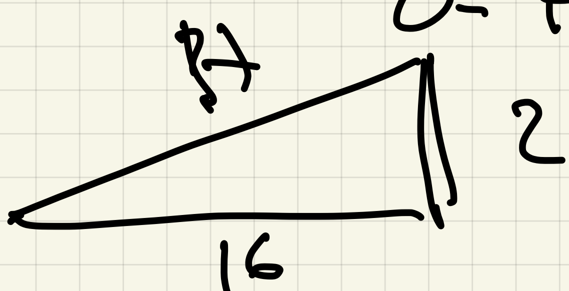
(2) $\vec{r}(t) = \langle t, 1 + \frac{1}{8}t \rangle$

$\boxed{0 \le t \le 16}$

(3) $\boxed{\vec{r}(t) = \langle t^2, 1 + \frac{1}{8}t^2 \rangle}$

$0 \le t \le 4$

(1)



$$H = \sqrt{2^2 + 16^2} =$$

$$2\sqrt{1 + 8^2}$$

$$2\sqrt{65}$$

(2) $\int_0^{16} |\vec{r}\,'(t)|\,dt$

$$\int_0^{16} \left|\left\langle 1, \frac{1}{8} \right\rangle\right| dt =$$

$$\int_0^{16} \sqrt{1 + \left(\frac{1}{8}\right)^2} =$$

$$\int_0^{16} \sqrt{\frac{64+1}{8^2}} = \int_0^{16} \sqrt{\frac{65}{64}} = \int_0^{16} \frac{\sqrt{65}}{8}$$

$$\frac{\sqrt{65}}{8} t \Big|_0^{16} = \frac{\sqrt{65}}{8} \cdot 16 = 2\sqrt{65}$$

(3) $\int_0^{4} |\vec{r}\,'(t)|\,dt =$

$$\int_0^4 \left|\langle 2t, t/4 \rangle\right| dt =$$

$$\int_0^4 \sqrt{4t^2 + \frac{t^2}{16}}\, dt =$$

$$\int_0^4 \sqrt{\frac{64t^2 + t^2}{16}} = \int_0^4 \sqrt{\frac{65t^2}{16}} =$$

$$\int_0^4 \frac{\sqrt{65}}{4}\, t\, dt =$$

$$\frac{\sqrt{65}}{4}\,\frac{t^2}{2}\Bigg|_0^4 =$$

$$\frac{\sqrt{65}}{4} \cdot \frac{16}{2} = 2\sqrt{65} \checkmark$$

Ex 3

$$\vec{r}(t) = \langle \underbrace{2+\sin t}_{x}, \underbrace{2-\sin t}_{y}, \underbrace{\sqrt{2}\cos t}_{z} \rangle$$

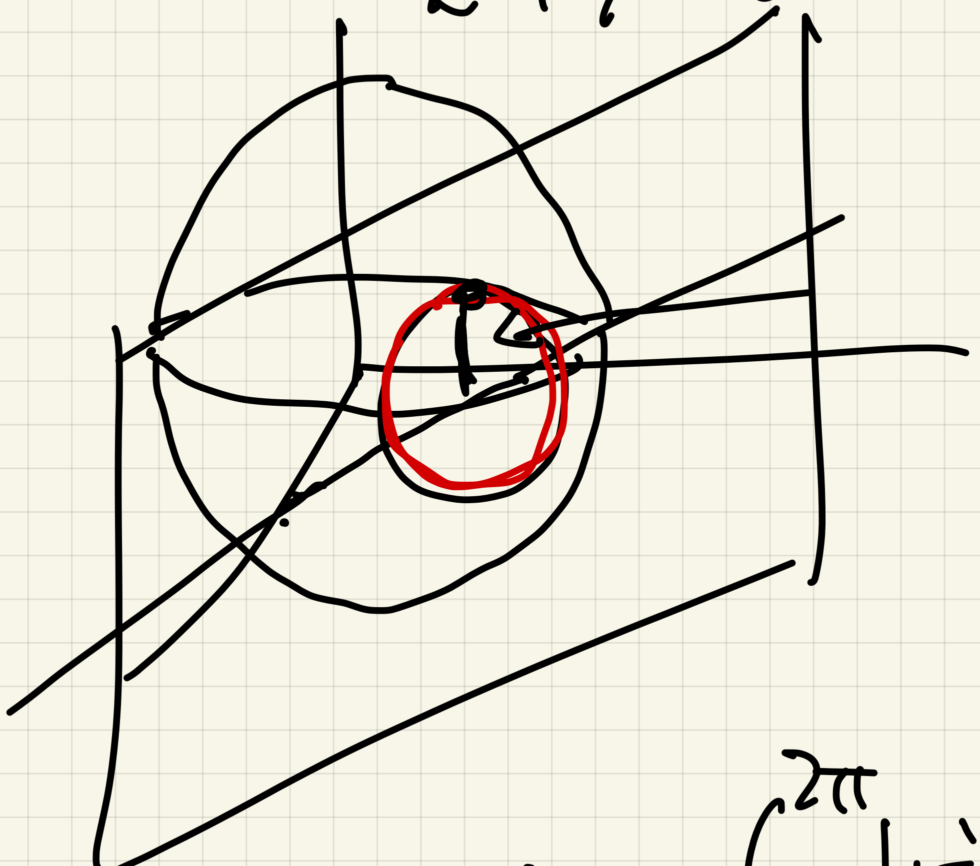
Notice: $x + y = 4$ $\quad 0 \le t \le 2\pi$

Less obvious:

$x^2+y^2+z^2=$

$$\underbrace{4+4\sin t+\sin^2 t}_{x^2} + \underbrace{4-4\sin t+\sin^2 t}_{y^2} + \underbrace{2\cos^2 t}_{z^2} = 10$$

$x^2+y^2+z^2=10$



radius $=\sqrt{2}$

Arclength $= \int_0^{2\pi} \|r'(t)\|\,dt =$

$$\int_0^{2\pi} \left|\langle \cos t, -\cos t, -\sqrt{2}\sin t\rangle\right| dt$$

$$\int_0^{2\pi} \sqrt{\cos^2 t + \cos^2 t + 2\sin^2 t}\, dt$$

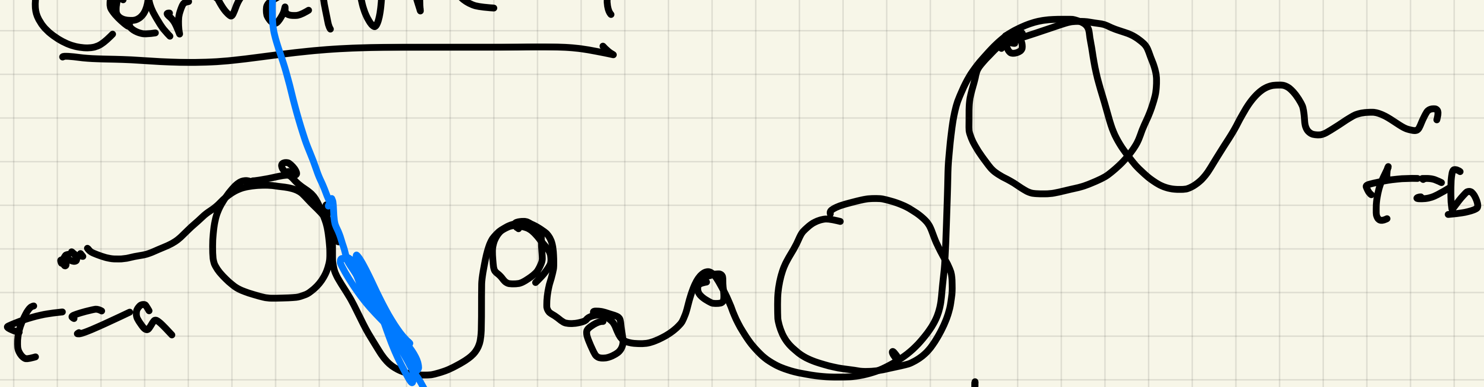
$$\int_0^{2\pi} \sqrt{2}\, dt = 2\sqrt{2}\pi$$

<u>12.4</u>

$$v = |v| \cdot \frac{v}{|v|}$$

mag direct

<u>Curvature</u> :



How tight are curves?
what is the radius of

circle that fits curves best?

Answer: $R = \frac{1}{K}$, where

K is the curvature of $\bar{r}(t)$,

$$K = \frac{|T'(t)|}{|r'(t)|}$$

$T(t)$ = unit tangent vector

// direction of velocity

Ex4 In Ex3, find the unit tangent $T(t)$ and curvature at the $t = \pi/2$ and radius R of curvature

$\bar{r}(t) = \langle 2+\sin t, 2-\sin t, \sqrt{2}\cos t \rangle$

$\bar{r}'(t) = \langle \cos t, -\cos t, -\sqrt{2}\sin t \rangle$

$|r'(t)| = \sqrt{2}$

$$T(t) = \frac{r'(t)}{|r'(t)|} = \frac{\langle \cos t, -\cos t, -\sqrt{2}\sin t \rangle}{\sqrt{2}}$$

$$T'(t) = \frac{1}{\sqrt{2}} \langle -\sin t, \sin t, -\sqrt{2}\cos t \rangle$$

$$|T'(t)| = \frac{1}{\sqrt{2}} \left| \langle -\sin t, \sin t, -\sqrt{2}\cos t \rangle \right|$$

$$\frac{1}{\sqrt{2}} \cdot \sqrt{2} = 1$$

So $K = \dfrac{|T'(t)|}{|r'(t)|} = \dfrac{1}{\sqrt{2}}$

So $\rho = \sqrt{2}$

But usually computing K is not nice:

$$\bar{r}(t) = \langle \sqrt{4-t^2}, t, 4-t^2 \rangle$$

$$\bar{r}'(t) = \left\langle \frac{-t}{\sqrt{4-t^2}}, 1, -2t \right\rangle$$

$T(t) =$

$$\underbrace{\frac{\left\langle \frac{-t}{\sqrt{4-t^2}}, 1, -2t \right\rangle}{\sqrt{\frac{t^2}{4-t^2} + 1 + 4t^2}}}_{T'(t)}$$

<u>A better formula</u>:

$$K = \frac{|r'' \times r'|}{|r'|^3}$$

<u>Ex5</u> $\bar{r}(t) = \langle \sqrt{4-t^2}, t, 4-t^2 \rangle$

find curvature at $t = 0$

$$\bar{r}'(t) = \left\langle \frac{-t}{\sqrt{4-t^2}}, 1, -2t \right\rangle$$

$$\bar{r}''(t) = \left\langle \frac{-1}{\sqrt{4-t^2}} + \frac{t^2}{(4-t^2)^{3/2}}, 0, -2 \right\rangle$$

$$\boxed{r'(0) = \langle 0, 1, 0 \rangle}$$

$$r''(0) = \left\langle -\frac{1}{2}, 0, -2 \right\rangle$$

So $K = \dfrac{|r' \times r''|}{|r'|^3} =$

$|r'| = 1$

$$\begin{vmatrix} i & j & k \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 2 \end{vmatrix} = 2i + \tfrac{1}{2}k = \langle 2, 0, \tfrac{1}{2} \rangle$$

$$K = \frac{|\langle 2, 0, \frac{1}{2} \rangle|}{|\langle 0, 1, 0 \rangle|^3} = \frac{\sqrt{4\,{}^1\!/_4}}{1^3} = \frac{\sqrt{17/4}}{} = \frac{\sqrt{17}}{2}$$

Radius of circle is $\dfrac{2}{\sqrt{17}}$