9/12/ Calc3

Exam 1 → 9/19

Quiz 5 — avg 86%, med 95%

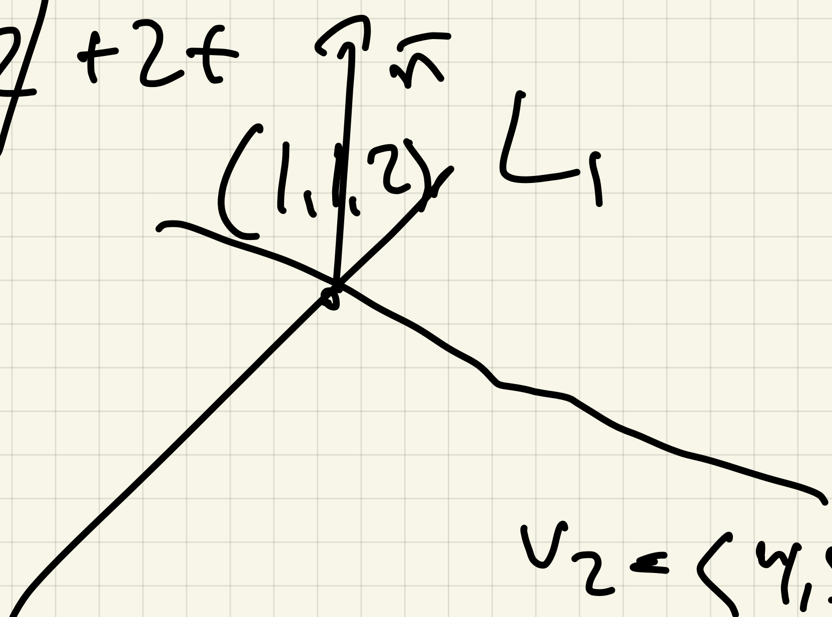
1. $\boxed{\begin{aligned} P &= (1,1,2) \\ Q &= (4,-1,4) \end{aligned}}$ $\quad \bar{v}_1 = \langle -3, 2, -2 \rangle$

$\langle 3, -2, 2 \rangle$

$$x = 1 + 3t$$
$$y = 1 - 2t$$
$$z = 2 + 2t$$

$\bar{r}_0 + t\bar{v}$

2.



$v_2 = \langle 4, 5, 1 \rangle$

$$\bar{n} = v_1 \times v_2 = \begin{vmatrix} \hat{i} & j & k \\ 3 & -2 & 2 \\ 4 & 5 & 1 \end{vmatrix} =$$

$\vec{n} = \langle -12, 5, 23 \rangle$

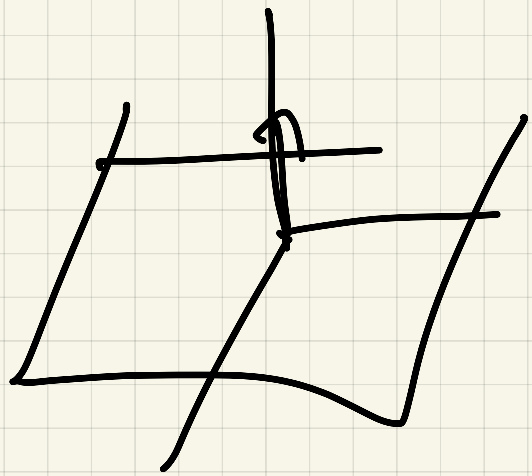
$r_0 = (1, 1, 2)$

$-12(x-1) + 5(y-1) + 23(z-2) = 0$

$-12x + 5y + 23z = 39$

$+12 \quad -5 \quad -46$

3. $n_1 = P_1$



$P_2 : z = 0$

$n_2 = \langle 0, 0, 1 \rangle$

$0x + 0y + 1z = 0$

$\cos\theta = \frac{|n_1 \cdot n_2|}{|n_1||n_2|} = \frac{23}{\sqrt{698} \cdot 1}$

$n_1 = (-12, 5, 23)$

$144 + 25 + 529 = 698$

<u>Last time</u> Vector valued functions

$$\bar{r}(t) = \langle f(t), g(t), h(t) \rangle$$

sketching

limits

continuity

derivative

smooth : $\bar{r}'(t) \neq \bar{0}$

<u>Physics</u> $\bar{r}'(t) = \frac{d}{dt}\bar{r}(t) = \frac{d\bar{r}}{dt}$

$\bar{r}'(t) = \bar{v}(t) =$ velocity

$\bar{r}''(t) = \bar{a}(t) =$ acceleration

$|\bar{r}'(t)| =$ speed

$\frac{\bar{r}'(t)}{|\bar{r}'(t)|} =$ direction

(Usual) Calc I derivatives

1. $\frac{d}{dt}[\bar{u}(t) + \bar{v}(t)] = \frac{d\bar{u}}{dt} + \frac{d\bar{v}}{dt}$

Product rules

2. $\frac{d}{dt}[\underbrace{f(t)}_{\text{scalar}}\ \underbrace{\bar{v}(t)}_{\text{vector}}] =$

$$f'(t)\,\bar{v}(t) + f(t)\,\bar{v}'(t)$$

3. $\frac{d}{dt}[\bar{u}(t) \cdot \bar{v}(t)] = \bar{u}'(t) \cdot \bar{v}(t)$

$$+ \bar{u}(t) \cdot \bar{v}'(t)$$

4. $\frac{d}{dt}[\bar{u}(t) \times \bar{v}(t)] =$

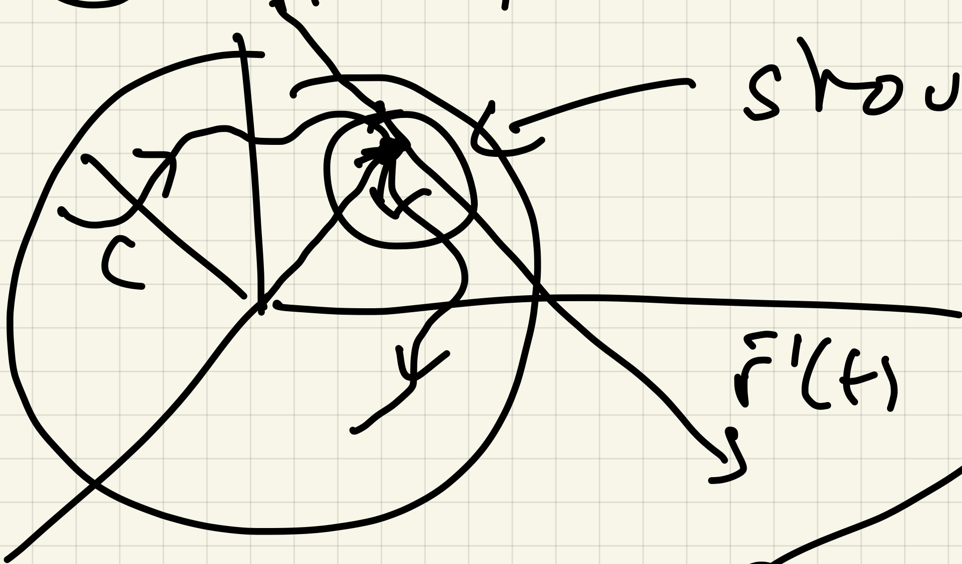
$$\bar{u}'(t) \times \bar{v}(t) + \bar{u}(t) \times \bar{v}'(t)$$

5. $\frac{d}{dt}[\bar{u}(f(t))] = f'(t) \cdot \bar{u}'(f(t))$

Consequences of 3

Ⓐ If $|\bar{r}(t)| = c$ = constant

should be $\perp$:



It is:

$\frac{d}{dt}$ $|\bar{r}(t)| = c \Rightarrow \bar{r}(t)\cdot\bar{r}(t) = c^2$

$\Rightarrow \bar{r}'(t)\cdot\bar{r}(t) + \bar{r}(t)\cdot\bar{r}'(t) = 0$

$2\,\bar{r}(t)\cdot\bar{r}'(t) = 0$

Similarly

$\bar{r}(t) \perp \bar{r}'(t)$

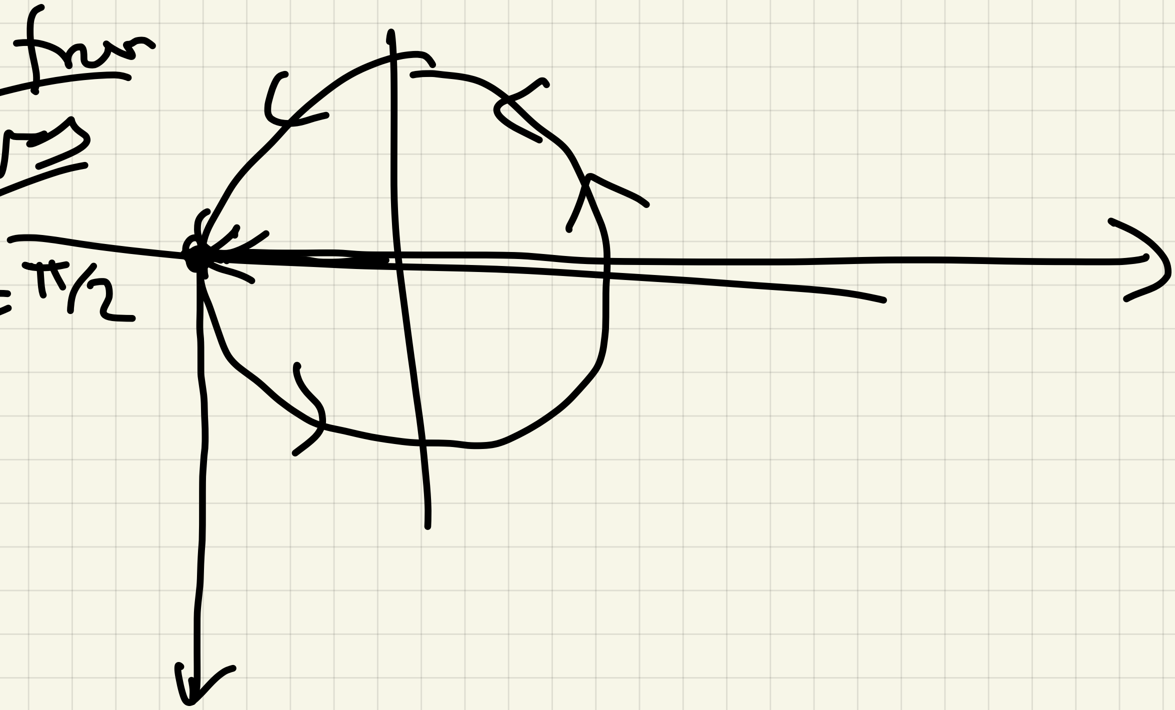
Ⓑ If speed constant $\Rightarrow$

$\bar{r}'(t) \perp \bar{r}''(t)$

Idea : $|\bar{r}'(t)| = s \Rightarrow$

$\bar{r}'(t)\cdot\bar{r}'(t) = s^2$

Ex from Tue



$t = \pi/2$

Ex2 $\quad \bar{r}(t) = \langle \sin^2 t, \sin^2 t, \cos 2t \rangle$

$\qquad\qquad\qquad\qquad x \qquad y \qquad z$

$x = y$

$z = \cos 2t = \underline{\cos^2 t} - \sin^2 t$

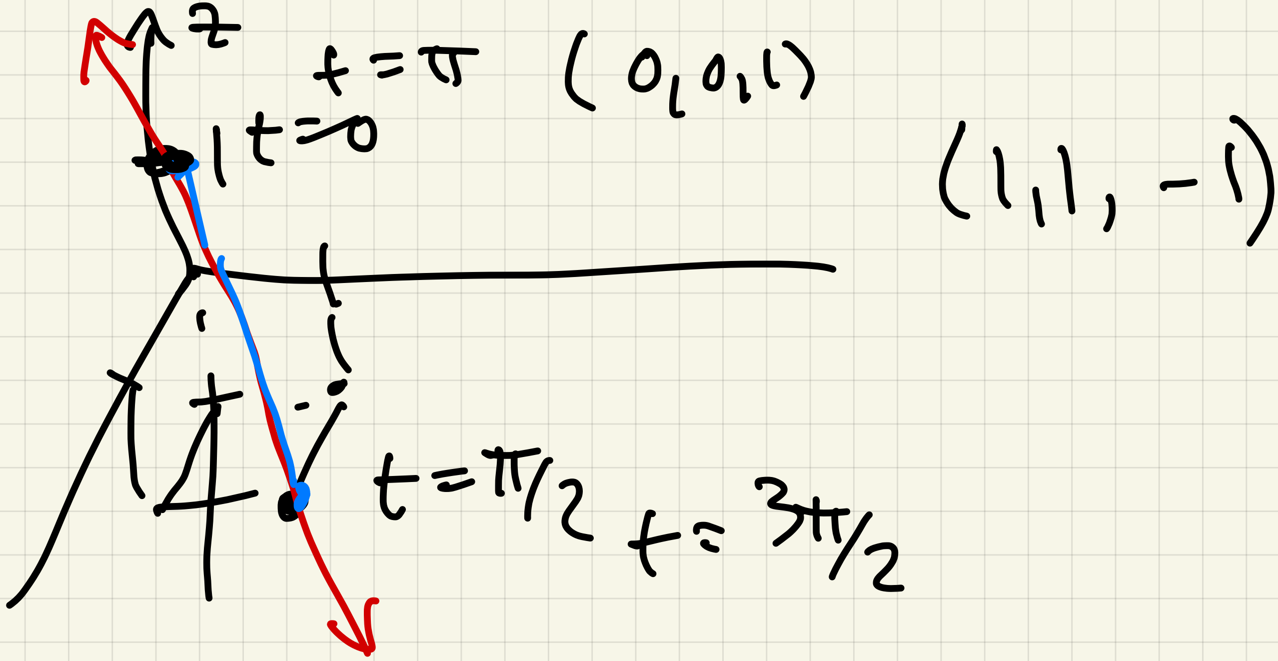
$= 1 - 2\,\underline{\underline{\sin^2 t}}$

$= 1 - 2x = 1 - 2y$

planes $\quad x = y$

$\qquad z = 1 - 2x$

Intersection is a line.



$$\bar{r}'(t) = \langle 2\sin t \cos t,\ 2\sin t \cos t,\ -2\sin 2t \rangle$$

$$= \bar{0} \quad \text{at}$$

$$t = 0, \pi/2, \pi, 3\pi/2,$$

<u>Integrals</u>:

$$\bar{r}(t) = \langle f(t), g(t), h(t) \rangle$$

(A) <u>Indefinite integral</u> =

set of all antiderivatives for <u>$\bar{r}(t)$</u> $= \{ \bar{S}(t) : \frac{d}{dt}\bar{S}(t) = \bar{r}(t) \}$

If $\bar{R}(t)$ is one antiderivative then

$$\int \bar{r}(t)\,dt = \bar{R}(t) + \bar{C}$$

$\bar{C}$ = constant (vector)

Notation $\int \bar{r}(t)\,dt$

(B) Definite integral:

$$\int_a^b \bar{r}(t)\,dt = \left\langle \int_a^b f(t)\,dt, \int_a^b g(t)\,dt, \int_a^b h(t)\,dt \right\rangle$$

Ex) $\bar{r}(t) = \langle t, \frac{1}{t}, 0 \rangle$

(a) $\int \bar{r}(t)\,dt =$

$$\langle \tfrac{1}{2}t^2, \ln|t|, 0 \rangle + \langle C_1, C_2, C_3 \rangle$$

$$\int_1^2 \vec{r}(t)\,dt = \left\langle \tfrac{1}{2}t^2, \ln|t|, 0 \right\rangle\Big|_1^2$$

$$\langle 2, \ln 2, 0\rangle - \langle \tfrac{1}{2}, \underbrace{\ln 1}_{0}, 0\rangle =$$

$$\langle 3/2, \ln 2, 0\rangle$$

(b) $$\int_0^{\pi} \langle \underline{\sin t}, \underline{\cos 3t}, e^{6t}\rangle\,dt$$

$$\left\langle -\cos t, \tfrac{1}{3}\sin 3t, \tfrac{1}{6}e^{6t}\right\rangle\Big|_0^{\pi}$$

$$\langle 1, 0, \tfrac{1}{6}e^{6\pi}\rangle - \langle -1, 0, \tfrac{1}{6}\rangle =$$

$$\langle 2, 0, \tfrac{1}{6}(e^{6\pi} - 1)\rangle$$

<u>Initial Value Problems</u> (<u>IVP</u>):

<u>Ex2</u> Find a function $\vec{r}(t)$

with $\begin{cases} \dfrac{d\bar{r}}{dt} = \langle \cos t, \sin t, e^t \rangle \\ \bar{r}(0) = \langle 10, 10, 0 \rangle \end{cases}$

$$\bar{r}(t) = \int \langle \cos t, \sin t, e^t \rangle \, dt$$

$$= \langle \sin t, -\cos t, e^t \rangle + \langle c_1, c_2, c_3 \rangle$$

$$= \bar{r}(t)$$

$$\bar{r}(0) = \langle 10, 10, 0 \rangle$$

$$\Downarrow \qquad =$$

$$\langle c_1, c_2 - 1, 1 + c_3 \rangle$$

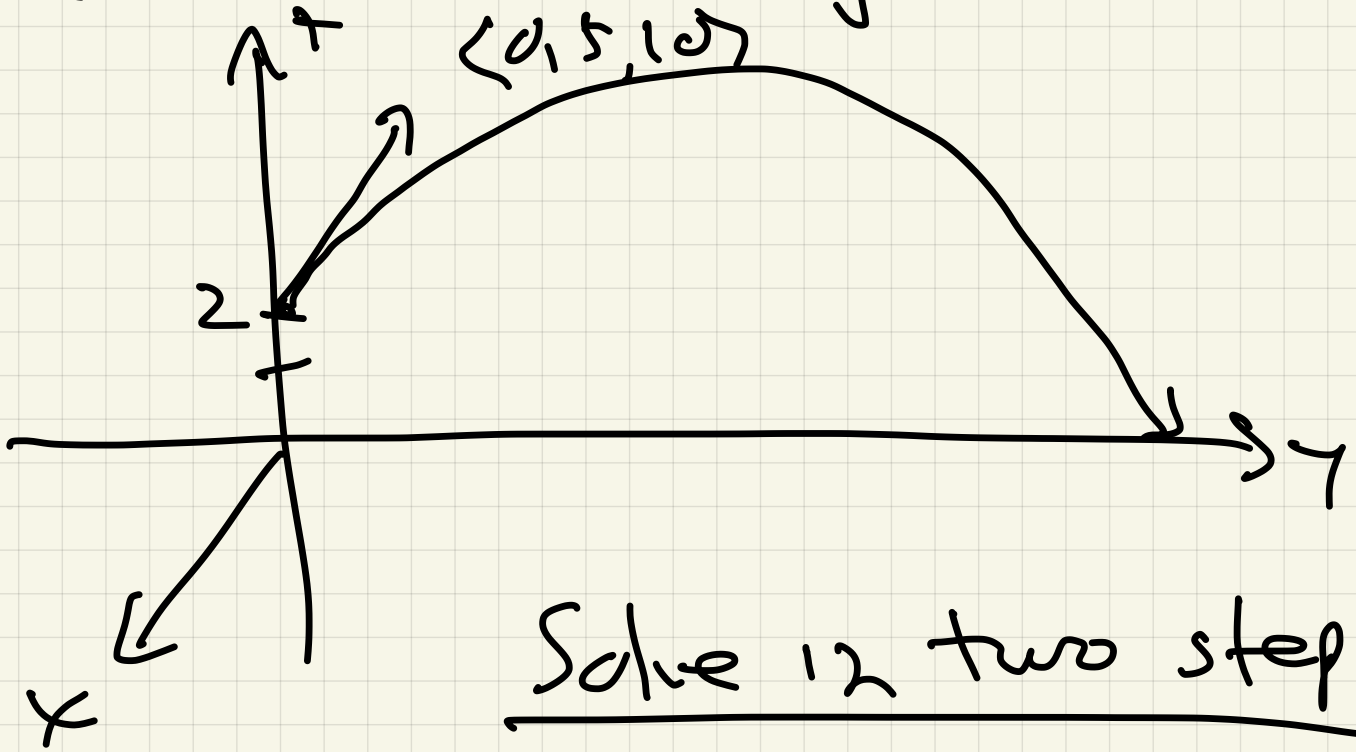
$$c_1 = 10, \quad c_2 = 11, \quad c_3 = -1$$

so $\bar{r}(t) = \langle 10 + \sin t, 11 - \cos t, -1 + e^t \rangle$

<u>Important Case</u>:

<u>Projective Motion</u>

Ex3 A rock is thrown from position of $\langle 0,0,2\rangle$ m at time $t=0$ with initial velocity $\bar{v}_0 = \langle 0,5,10\rangle$ and experiences gravity acceleration $\bar{g} = \langle 0,0,-10\rangle$



Solve in two steps:

$$\bar{g} = \langle 0,0,-10\rangle = \bar{a}(t) = \bar{v}'(t)$$

$$\Downarrow$$

$$\bar{v}(t) = \int \langle 0,0,-10\rangle dt =$$

$$\int \langle 0,0,-10t\rangle + \langle c_1, c_2, c_3\rangle$$

$$\bar{v}(0) = \langle 0, 5, 10\rangle$$

$c_1 = 0$, $c_2 = 5$, $c_3 = 10$

$$\Downarrow$$

$$\bar{v}(t) = \langle 0, 5, 10-10t\rangle$$

$$\| \qquad \Downarrow$$

$$\bar{r}'(t)$$

$$\Downarrow$$

$$\bar{r}(t) = \int \langle 0, 5, 10-10t\rangle \, dt$$

$$= \langle 0, 5t, 10t-5t^2\rangle + \langle D_1, D_2, D_3\rangle$$

$$\bar{r}(0) = \langle 0, 0, 2\rangle$$

$$\Downarrow \quad D_1 = 0$$

$$D_2 = 0$$

$$D_3 = 2$$

$$\Downarrow$$

$$\bar{r}(t) = \langle 0, \underset{y}{5t}, \underset{z}{10t-5t^2+2}\rangle$$

$$z = -\frac{1}{5}y^2 + 2y + 2$$