9/11/ Calc3 Vector-valued functions

Last time

$$\vec{r}(t) = \langle \underset{x}{f(t)}, \underset{y}{g(t)}, \underset{z}{h(t)} \rangle$$

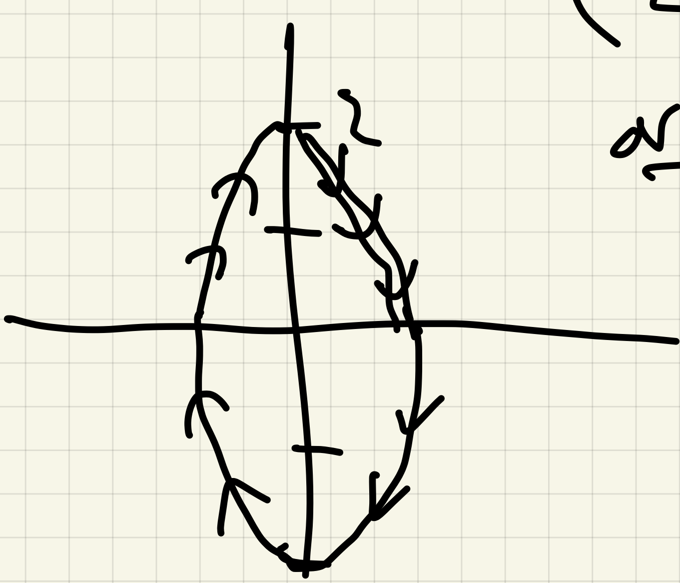
Visualize as position in $\mathbb{R}^3$ at time t.

Ex 0 Straight line parametric equation

Ex 1 $\vec{r}(t) = \langle \underset{x}{\sin t}, \underset{y}{2\cos t} \rangle$ in $\mathbb{R}^2$

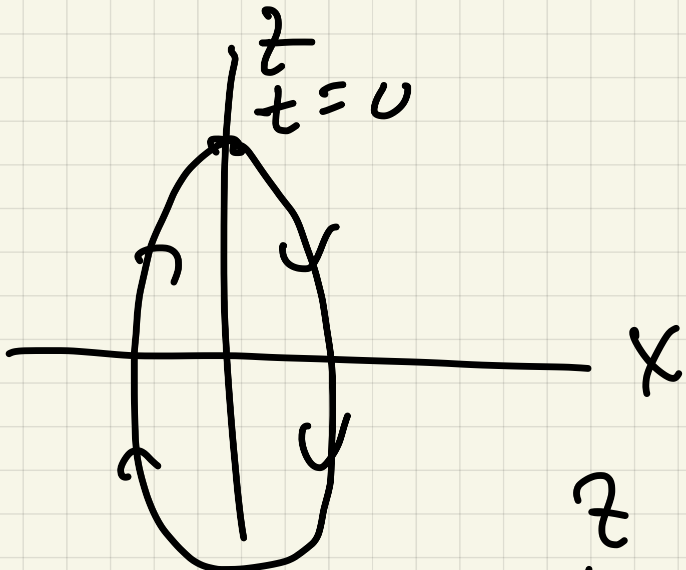
$$\sin^2 t + \cos^2 t = 1$$

$$x^2 + \left(\frac{y}{2}\right)^2 = 1$$



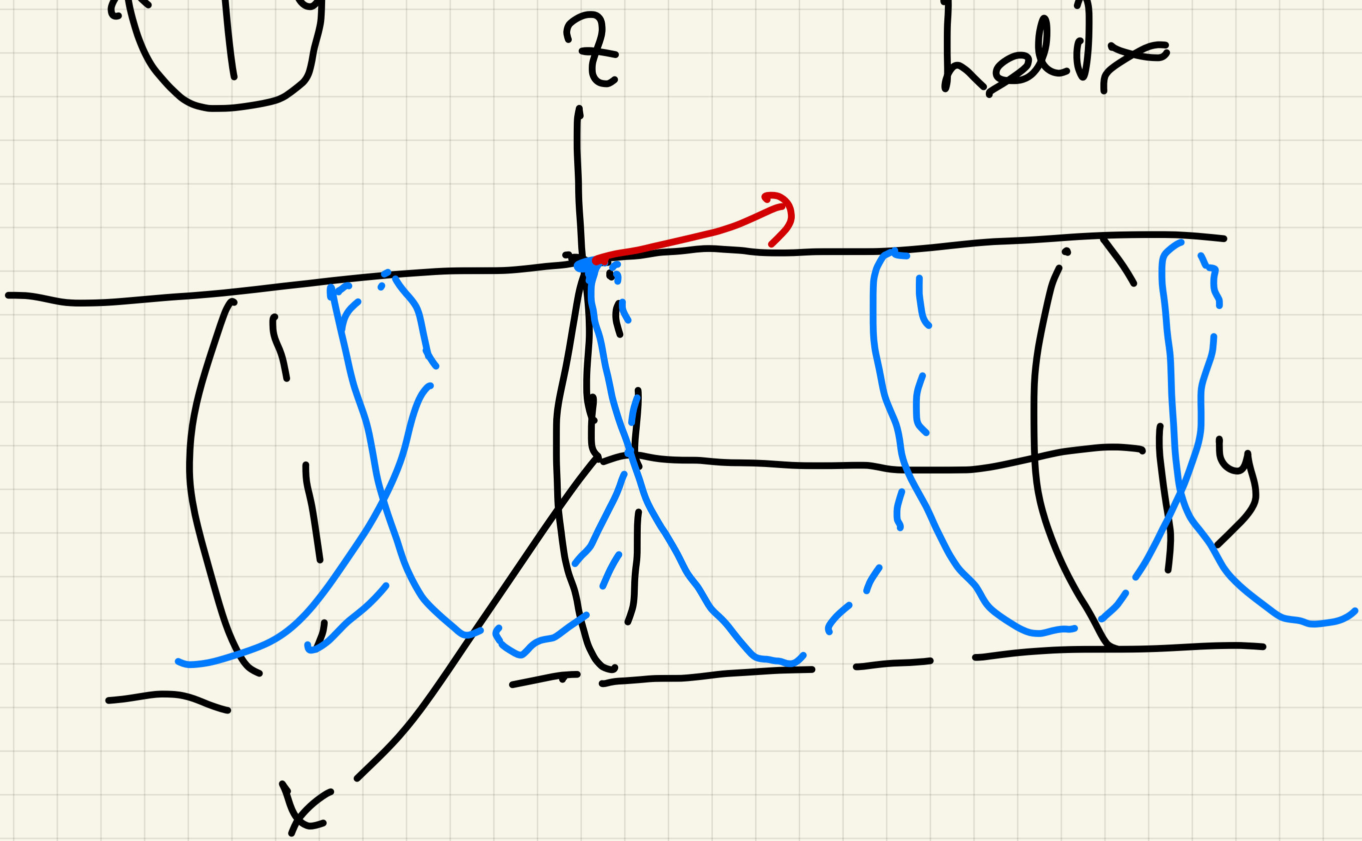
orientation: clockwise

Ex2 $\vec{r}(t) = \langle \sin t, t, 2\cos t \rangle$



(ignore y)

hold x



Ex 3 $\langle \underbrace{1+\cos t}_{x}, \underbrace{\sin t}_{y}, \underbrace{1-\cos t}_{z} \rangle$

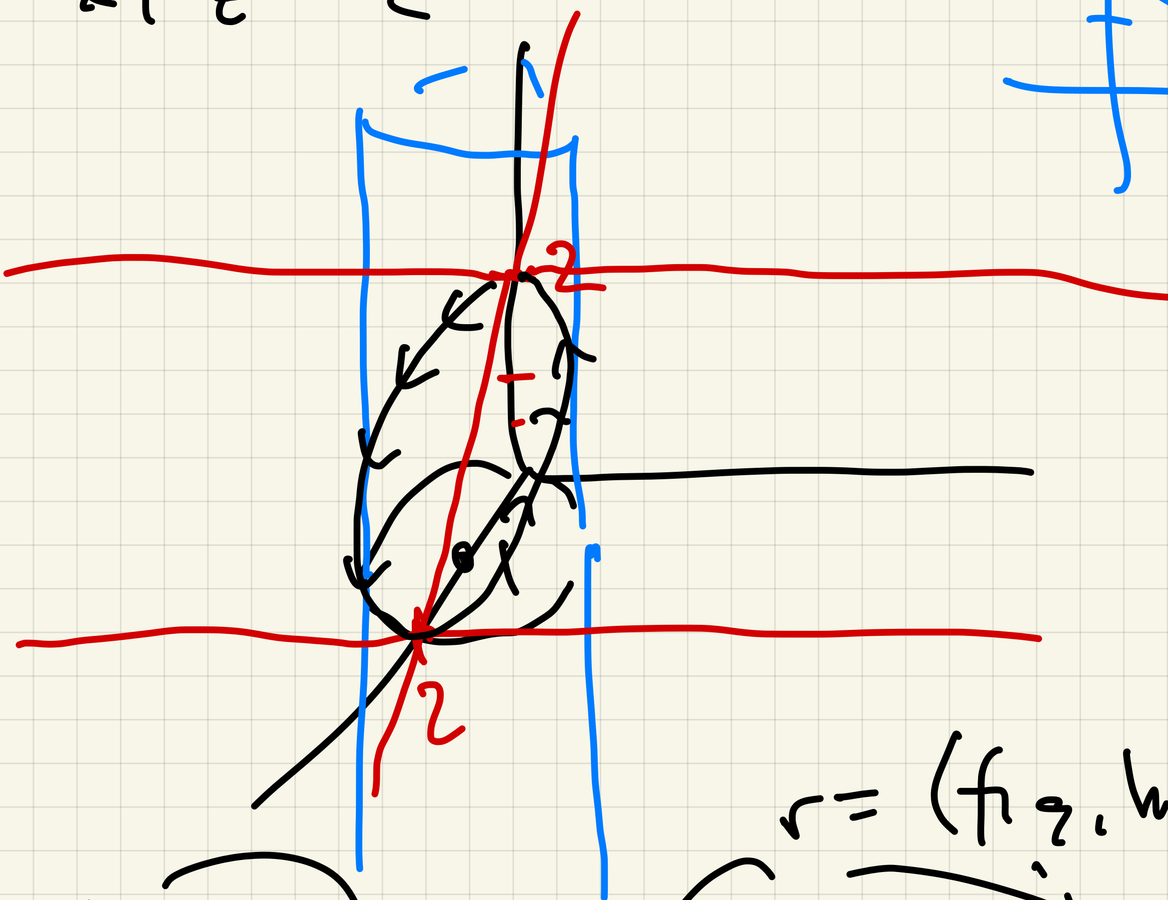
$\sin^2 t + \cos^2 t = 1$

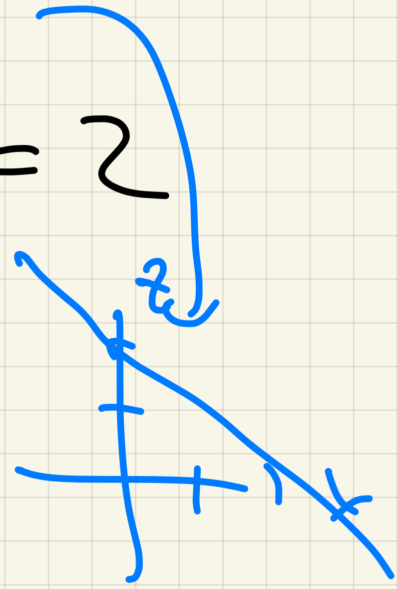
$\longrightarrow y^2 + (x-1)^2 = 1$ circle

$$x+z=2$$

$$(1+\cos t)+(1-\cos t)=2$$

$$x+z=2$$





$$r=(f,g,h)$$

Limits and continuity

Calc I: f cnt at $x=a$ if

$$\lim_{x\to a} f(x)=f(a)$$

① $\lim_{t\to a} \vec{r}(t)=\langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t)\rangle$

② $\bar{r}(t)$ is continuous at $t=a$ if $\lim_{t\to a} \bar{r}(t) = \bar{r}(a)$

<u>**Ex 4**</u> (cont) Is $\bar{r}(t) = \langle 1+\cos t, \sin t, 1-\cos t\rangle$ continuous at $t=0$?

$$\lim_{t\to 0} \bar{r}(t) = \langle \lim_{t\to 0} (1+\cos t), \lim_{t\to 0} \sin t, \lim_{t\to 0} 1-\cos t \rangle$$

$$\langle 2, 0, 0\rangle$$

$$= \bar{r}(0) \checkmark$$

yes

(b) How about

$$\bar{r}(t) = \langle 3\sin t, \frac{t^3+t}{7t}, e^t\rangle$$

$\frac{0}{0}$

<u>**No**</u>: $\bar{r}(0) = \langle 0, \text{undef.}, 1\rangle$

but $\lim_{t\to 0} \langle 3\sin t, \frac{t^3+t}{7t}, e^t\rangle$

$\langle 0, 1/7, 1\rangle$

Fails, but can fix this

(c)

$$\bar{r}(t) = \begin{cases} \left\langle 3\sin t,\ \frac{t^3+t}{7t},\ e^t \right\rangle & t \neq 0 \\ \left\langle 0,\ 1/7,\ 1 \right\rangle & t = 0 \end{cases}$$

Is continuous

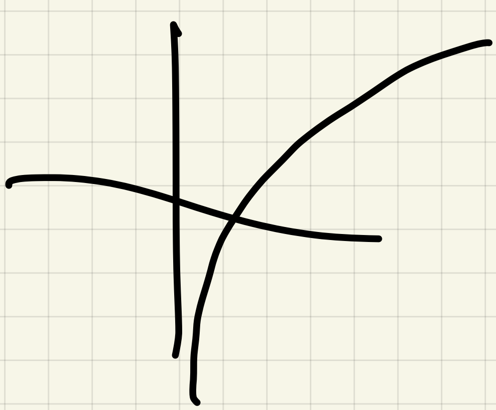
(d) For which t is

$$\bar{r}(t) = \left\langle t^3+7,\ \ln t,\ \sqrt{20-t^2} \right\rangle$$

continuous?

x: all t

y: $t > 0$



z: $20 - t^2 \geq 0$

$20 \geq t^2$

$\sqrt{20} \geq t \geq -\sqrt{20}$

$$0 < t \le \sqrt{20}$$

Derivatives

Recall Calc1: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

In Calc3:

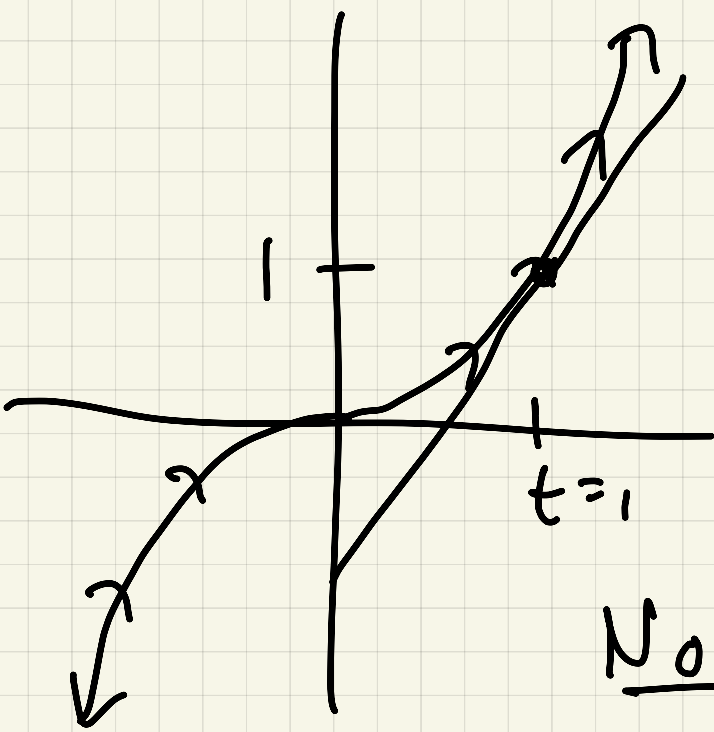
$$\bar{r}'(t) = \lim_{h \to 0} \frac{\bar{r}(t+h) - \bar{r}(t)}{h} =$$

$$\langle f'(t), g'(t), h'(t) \rangle$$

Ex5: $\bar{r}(t) = \langle t, t^3 \rangle$

x y

$y = x^3$



$$\bar{r}'(1) = \langle 1, 3 \rangle$$

$$\bar{r}'(t) = \langle 1, 3t^2 \rangle$$

Note: $\frac{d}{dx}(x^3) = 3x^2$

$$\frac{d}{dx}(x^3)\Big|_{x=1} = 3$$

$\bar{r}'(t)$ = tangent vector or

velocity vector

$|r'(t)|$ = speed

$\bar{r}(t)$ is smooth at t if

$\bar{r}'(t) \neq \bar{0} \iff |r'(t)| \neq 0$

Ex 6 (a) $\bar{r}(t) = \langle 6, \cos(t^3), e^{\tan t} \rangle$

chain

$\Downarrow$

$\bar{r}'(t) = \langle 0, -\sin(t^3) \cdot 3t^2,$

$e^{\tan t} \cdot \sec^2 t \rangle$

smooth for $-\frac{\pi}{2} < t < \pi/2$

(b) $\bar{r}(t) = \langle t^3, t^2 \rangle$

$\bar{r}'(t) = \langle 3t^2, 2t \rangle = (0, 0)$

at $t = 0$

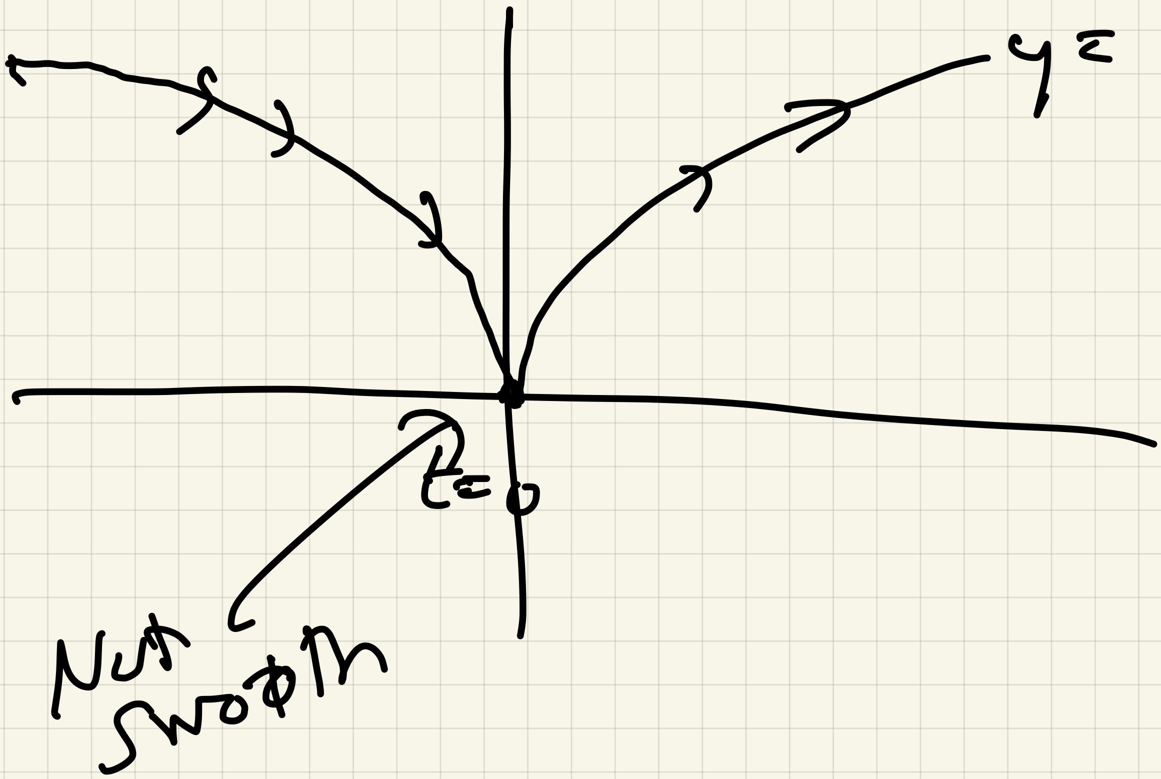
Not smooth $t = 0$ / smooth $t \neq 0$

Sketch: $\langle t^3, t^2 \rangle$

$\quad\quad\quad\;\; x \quad y$

$x^2 = y^3 \; (= t^6)$

$y = x^{2/3}$

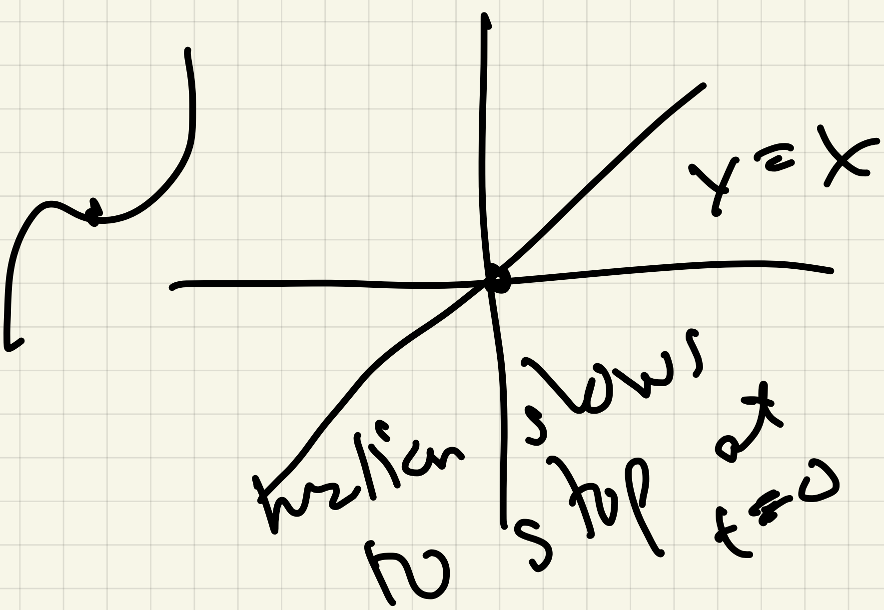


(c) $\bar{r}(t) = \langle t^3, t^3 \rangle$

$\quad\quad\quad\quad\quad\;\; x \quad y$

$\bar{r}'(t) = \langle 3t^2, 3t^2 \rangle = \langle 0, 0 \rangle$

$t = 0$



Ex 7 Find the tangent line to

$\bar{r}(t) = \langle \sin t, t, 2\cos t \rangle$ at $t=0$

Need point / direction;

<u>point</u>: $t=0$: $\bar{r}(0) = \underline{\underline{\langle 0, 0, 2 \rangle}}$

<u>direction</u>: $\bar{r}'(0)$

$$\bar{r}'(t) = \langle \cos t, 1, -2\sin t \rangle \Big|_{t=0}$$

$$= \underline{\underline{\langle 1, 1, 0 \rangle}}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 + t \\ 0 + t \\ 2 + 0t \end{pmatrix}$$

<u>Note</u>: $\bar{r}'(0) = \langle 1, 1, 0 \rangle$ tells us orientation of motion.

<u>More physics language</u>

<u>velocity</u> $\bar{r}'(t) = \bar{v}(t)$

speed $|\bar{r}'(t)| = |\bar{v}(t)|$

direction $\dfrac{\bar{v}(t)}{|\bar{v}(t)|}$ unit

acceleration : $\bar{a}(t) = \bar{v}'(t) = \bar{r}''(t)$