9/10 Calc3

Last time: vector valued functions

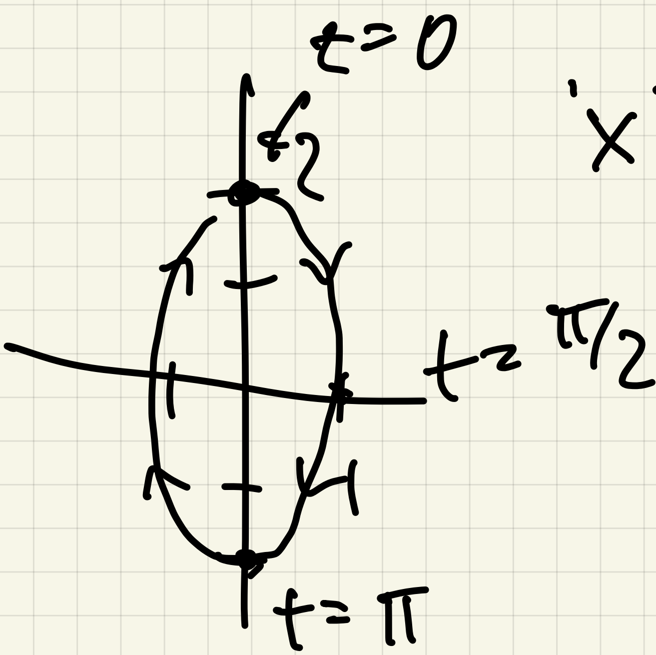
$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

(x, y, z)

position of particle in $\mathbb{R}^3$ at time t

Ex1 $\vec{r}(t) = \langle \sin t, 2\cos t \rangle$

(x, y)



$x^2 + \left(\frac{y}{2}\right)^2 = 1$

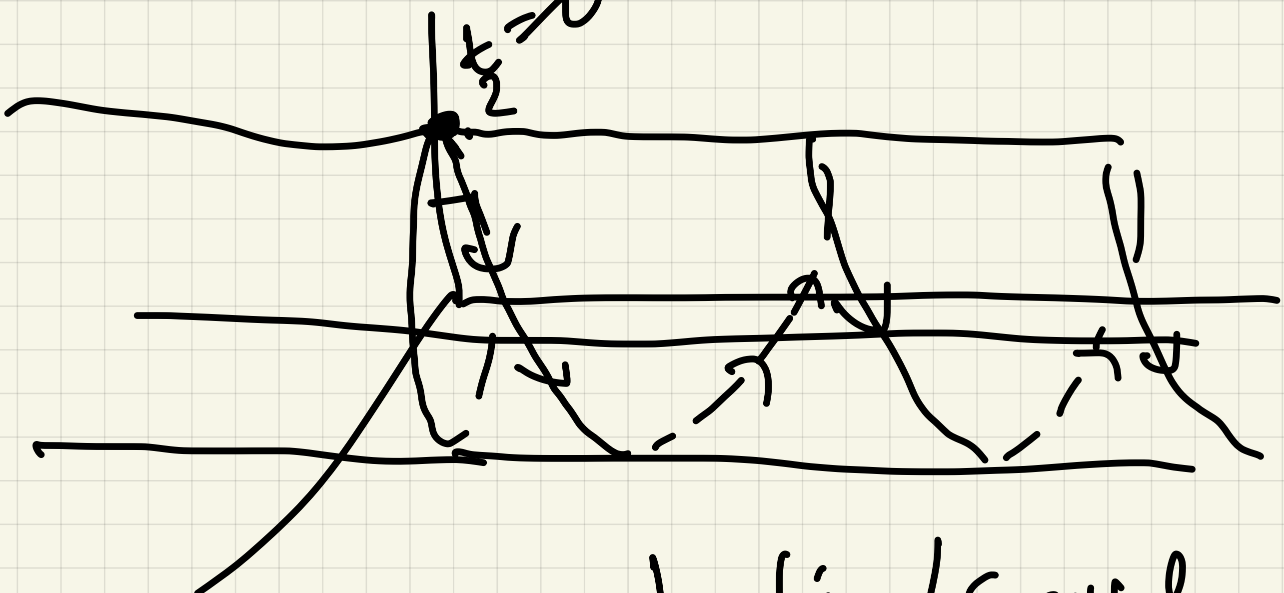
$$x^2 + \frac{z^2}{4} = 1$$

Ex2 $\vec{r}(t) = \langle \sin t, t, 2\cos t \rangle$

(x, y, z)

xz understand motion

but y-coordinate increases



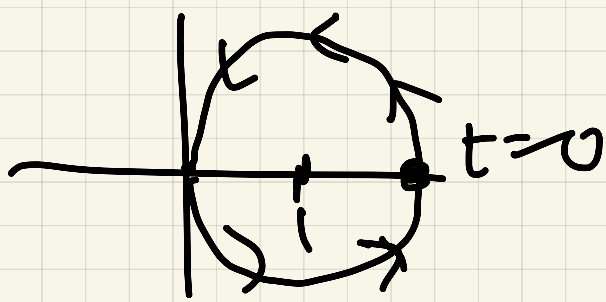
helix / spiral

Ex3 $\vec{r}(t) = \langle 1+\cos t, \sin t, 1-\cos t \rangle$

($x = 1+\cos t$, $y = \sin t$, $z = 1-\cos t$)

$(x-1)^2 + y^2 = 1$

$\cos^2 t + \sin^2 t = 1$

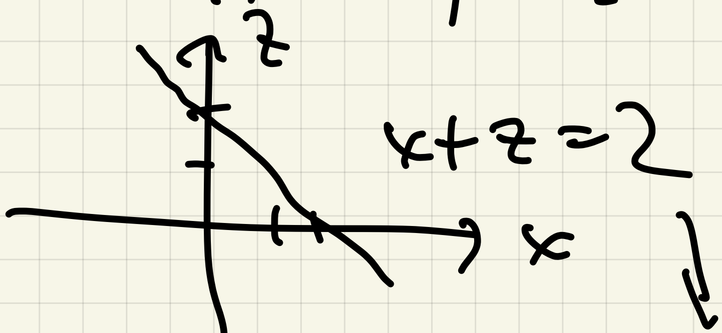


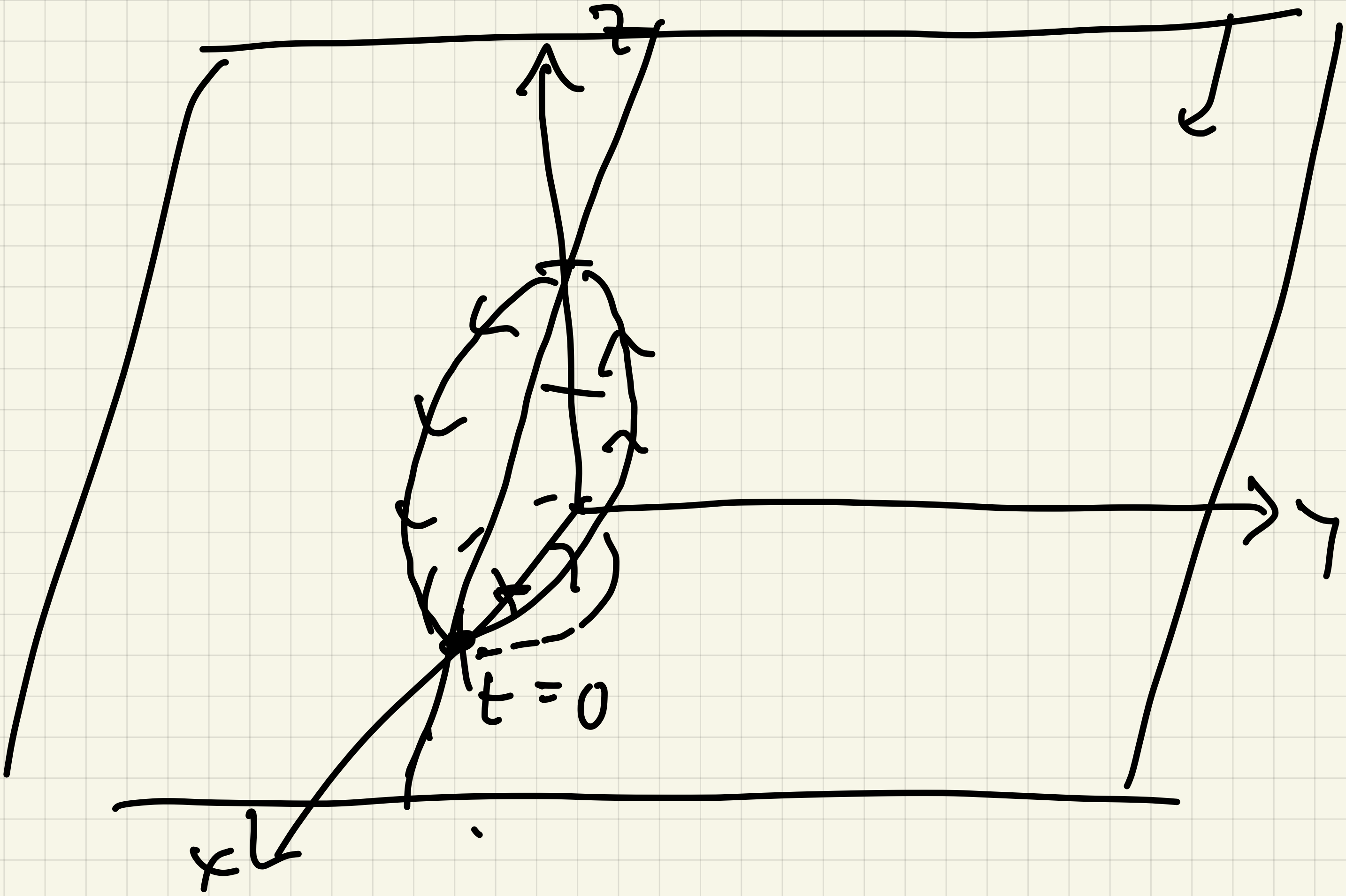
$x + z = 2$

plane

$\langle 1, 0, 1 \rangle$

$1x + 0y + 1z = 2$





Limits + continuity

If $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

① $\lim_{t \to a} \vec{r}(t) = \langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \rangle$

②

Def: f is continuous at a if $\lim_{t \to a} f(t) = f(a)$

⇓ sup

② $\vec{r}(t)$ is continuous at $t=a$ if $\lim_{t \to a} \vec{r}(t) = \vec{r}(a)$

<u>Ex</u> (a) Is $\vec{r}(t) = \langle 1+\cos t, \sin t, 1-\cos t \rangle$ continuous at $t=0$?

<u>yes</u>:

$$\lim_{t \to 0} \vec{r}(t) = \langle \lim_{t \to 0} 1+\cos t, \lim_{t \to 0} \sin t, \lim_{t \to 0} 1-\cos t \rangle$$

$$= \langle 2, 0, 0 \rangle = \vec{r}(0) \checkmark$$

(b) How about $\vec{r}(t) = \langle 3\sin t, \frac{t^3+t}{7t}, e^t \rangle$?

<u>NO</u> $\vec{r}(0)$ DNE ↑

$\frac{0}{0}$ DNE

(c) For which t is

$\bar{r}(t) = \langle t^3+7,\ \ln t,\ \sqrt{20-t^2} \rangle$

is continuous?

all t $\qquad t > 0 \qquad t^2 \le 20$

$-\sqrt{20} \le t \le \sqrt{20}$

<u>Ans</u>: $0 < t \le \sqrt{20}$

<u>Derivatives</u>:

<u>Calc I</u>: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

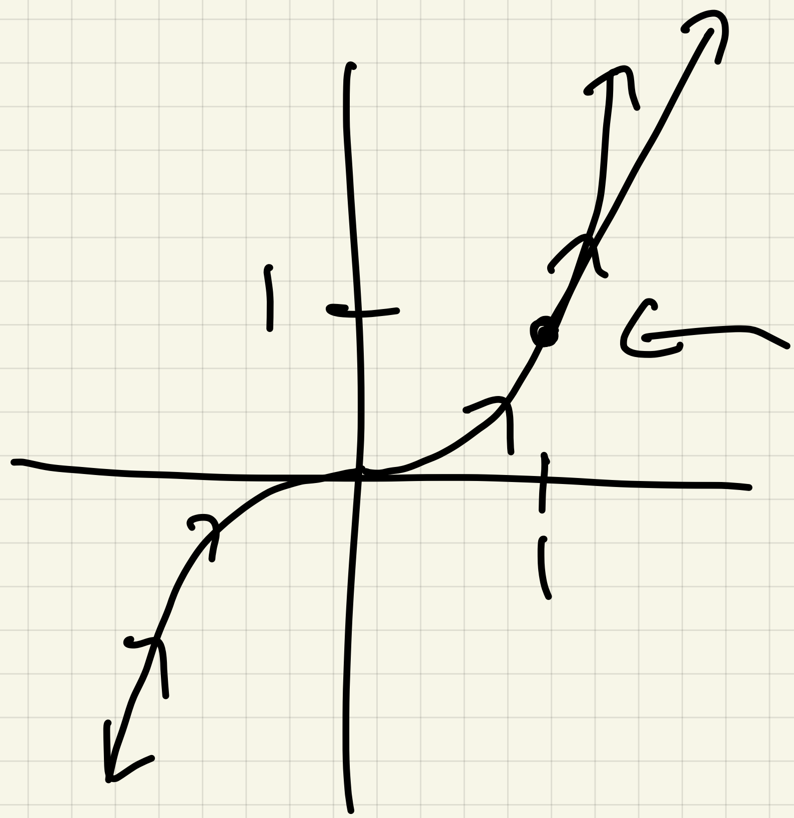
<u>Defn</u> $\bar{r}'(t) = \lim_{h \to 0} \frac{\bar{r}(t+h) - \bar{r}(t)}{h}$

$\Downarrow$

$\langle f'(t), g'(t), h'(t) \rangle$

Ex 5 $\bar{r}(t) = \langle t, t^3 \rangle$

x y



$y = x^3$

Calc 1

$\frac{dy}{dx} = 3x^2$

slope of tangent line is $3x^2\big|_{x=1} = 3$

Calc 3: $\bar{r}'(t) = \langle 1, 3t^2 \rangle$

$\bar{r}'(1) = \langle 1, 3 \rangle$

Notation $\bar{r}'(t)$ = tangent vector = velocity vector = $\bar{v}(t)$

$\bar{r}(t)$ is smooth at t if $\bar{r}'(t) = \bar{v}(t) \neq \bar{0}$

Ex 6

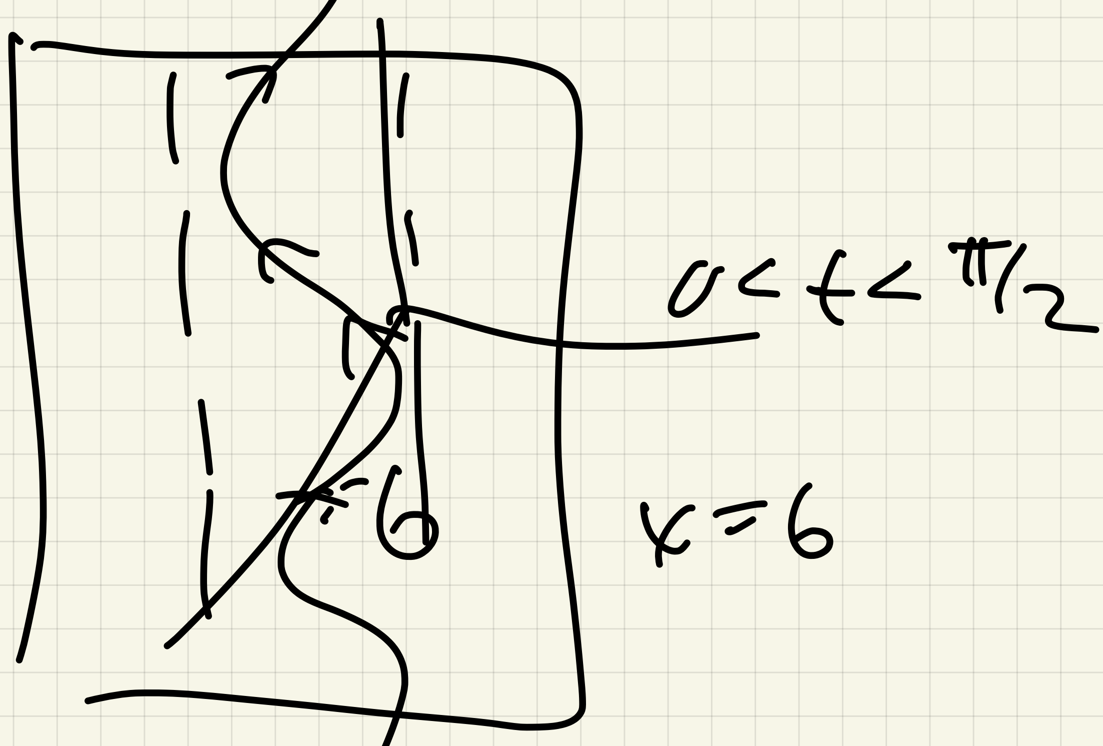
$$\vec{r}(t) = \langle 6,\ \cos(t^3),\ e^{(\tan t)} \rangle$$

$x \qquad y \qquad z$

$$\vec{r}\,'(t) = \langle 0, -3t^2 \sin(t^3),\ \sec^2 t\, e^{\tan t} \rangle$$

$\vec{r}\,'(t)$ smooth at $t \neq \pm \pi/2, \pm 3\pi/2,$

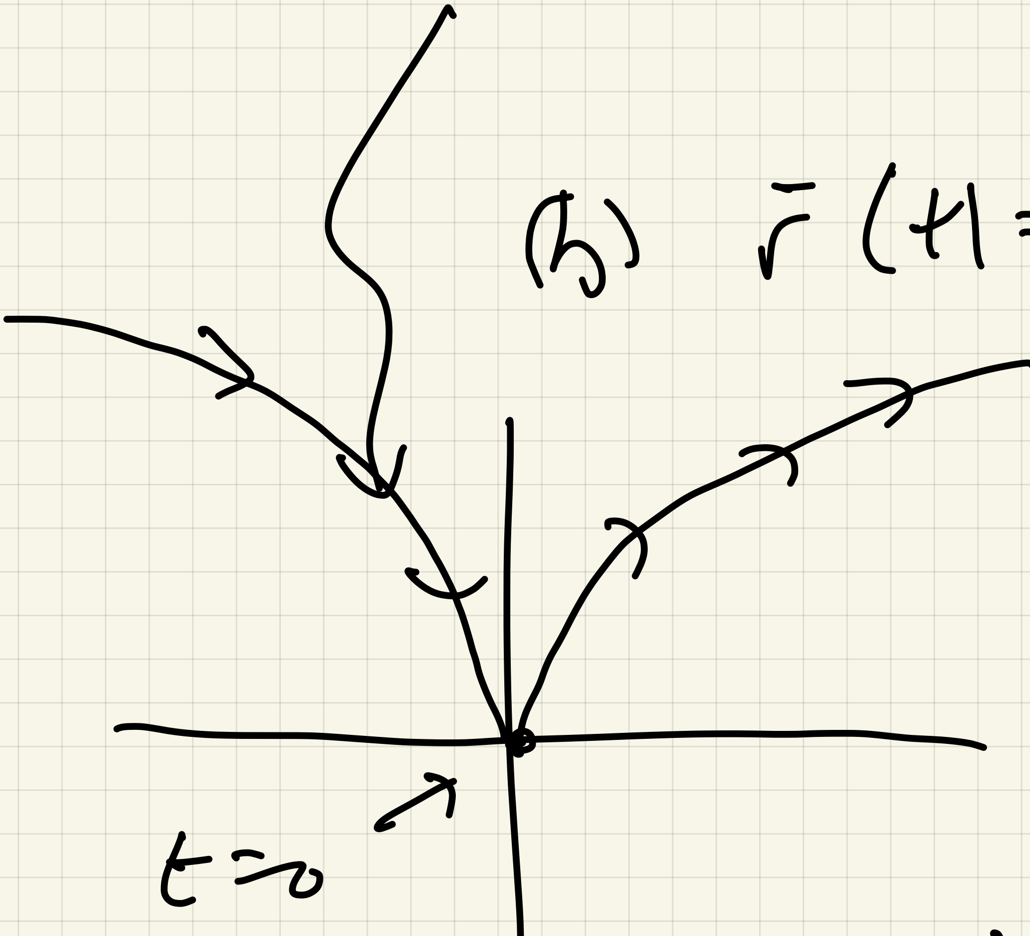
i.e. where $\vec{r}\,'(t)$ is defined



(b) $\bar{r}(t) = \langle t^3, t^2 \rangle$

$x \quad y$

$x^2 = y^3$



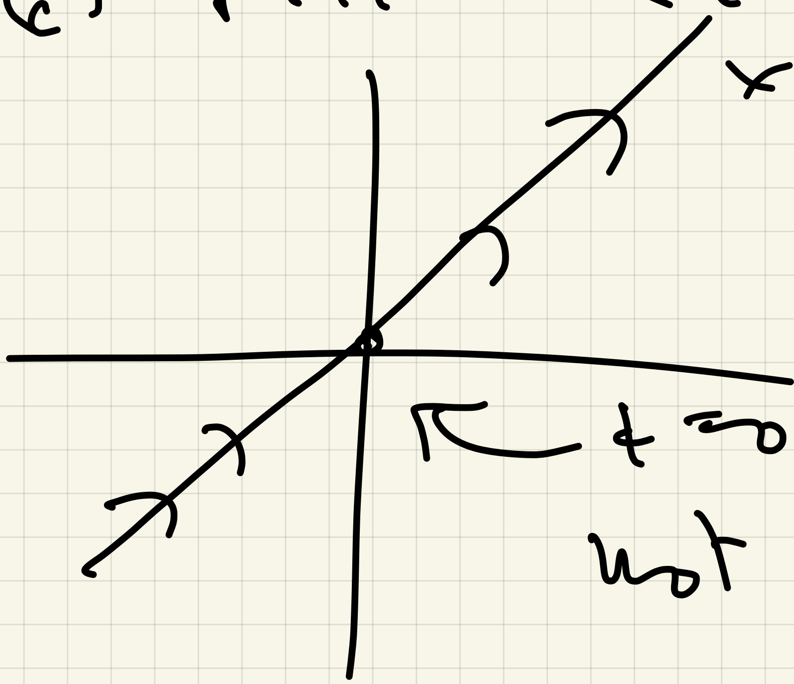
$\bar{r}'(t) = \langle 3t^2, 2t \rangle = \langle 0, 0 \rangle$

at $t = 0$

(c) $\bar{r}(t) = \langle t^3, t^3 \rangle$

$x \quad y$

$y = x$



$t = 0$, $\bar{r}'(0) = \langle 0, 0 \rangle$

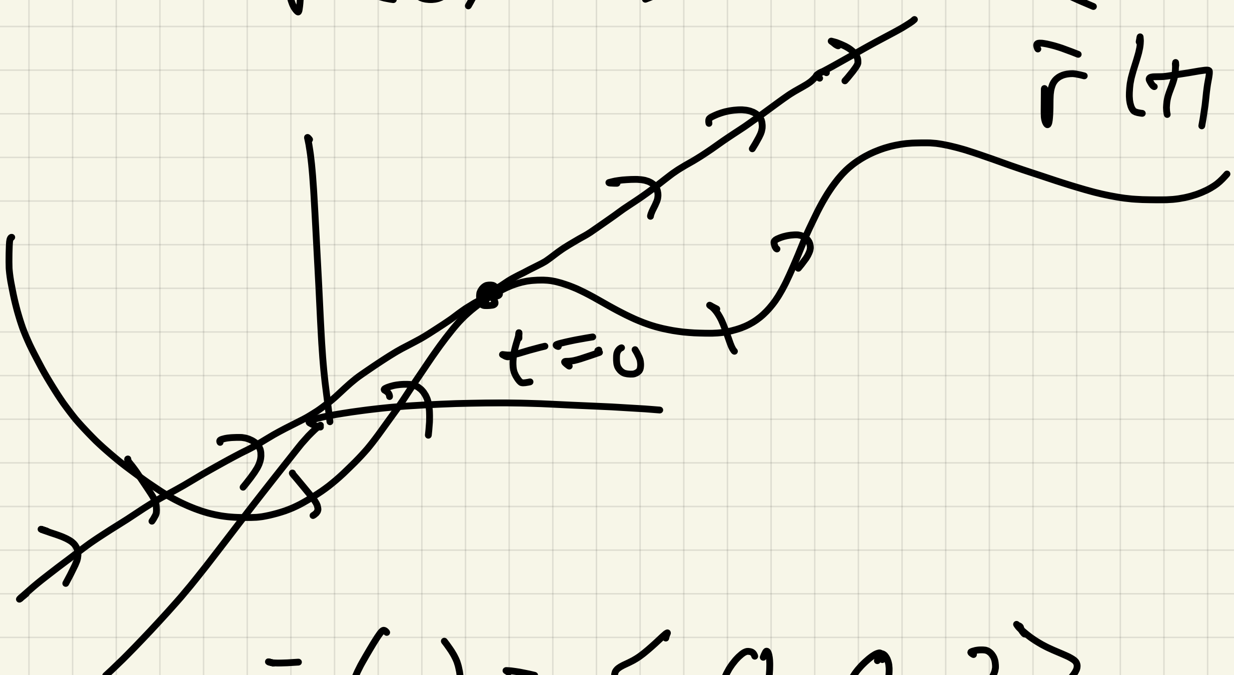
not smooth at $t = 0$

[Ex] $\bar{r}(t) = \langle \sin t, t, 2\cos t \rangle$

Find the line to $\bar{r}'(t)$

at $t=0$.

i.e. { direction is $\bar{r}'(0)$
$\bar{r}(0)$ is on line

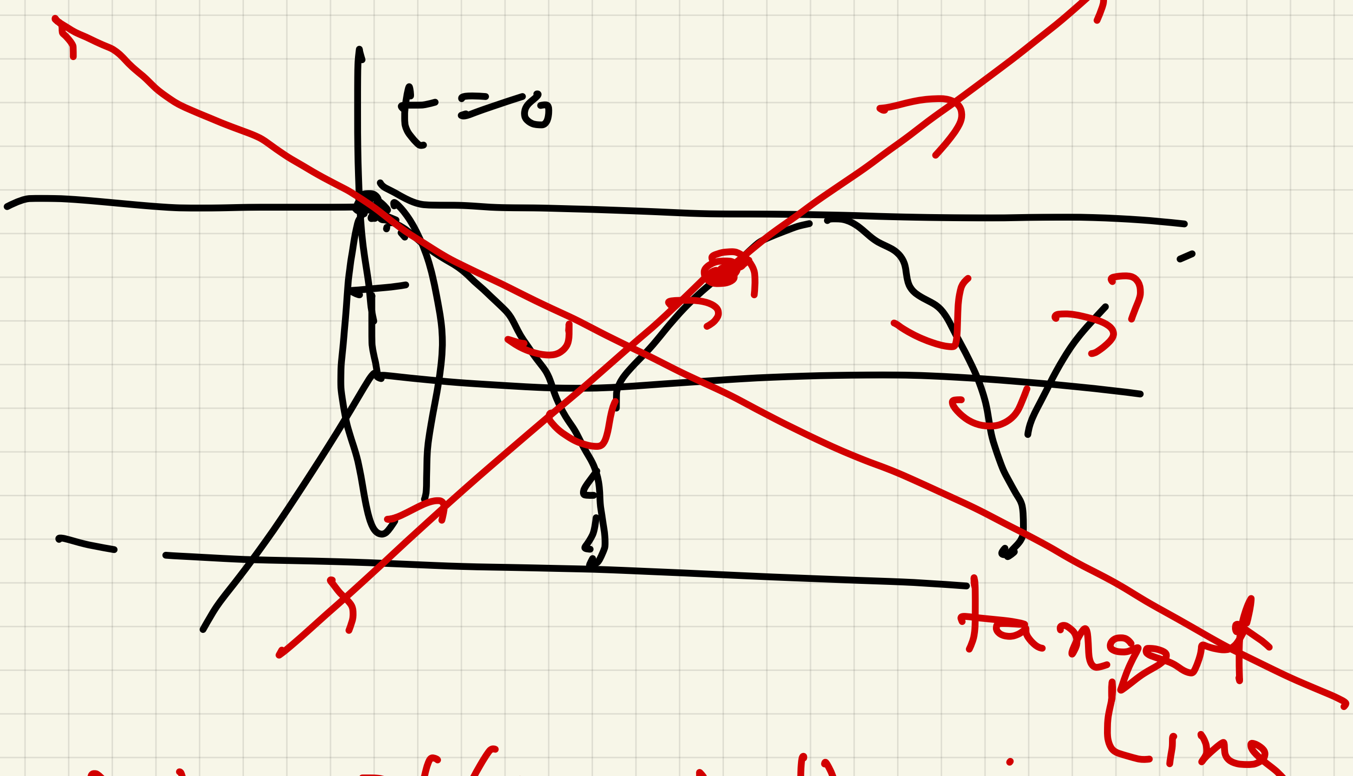


$\bar{r}(0) = \langle 0, 0, 2 \rangle$

$\bar{r}'(t) = \langle \cos t, 1, -2\sin t \rangle$

$\bar{r}'(0) = \langle 1, 1, 0 \rangle$

$$\text{So} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0+t \\ 0+t \\ 2 \end{pmatrix}$$



Note $\bar{r}'(t)$ tells you the orientation.

More physics:

$\bar{r}(t)$ vector valued function

$\bar{r}'(t) = \bar{v}(t) = \text{velocity}$

$|\bar{v}(t)| = \text{speed}$

$\dfrac{\bar{v}(t)}{|\bar{v}(t)|} = \text{direction}$

$\bar{a}(t) = \bar{r}''(t)$ = acceleration of motion

Ex 8 Find velocity, accel, speed for motion

$\bar{r}(t) = \langle 3\cos 2t, 3\sin 2t \rangle$

x y $t \geq 0$

sketch motion.

$x^2 + y^2 = 9$

draw velocity, accel vectors at $t = \pi/2$

$\bar{v}(t) = \bar{r}'(t) = \langle -6\sin 2t, 6\cos 2t \rangle$

$\bar{a}(t) = \bar{r}''(t) = \langle -12\cos 2t, -12\sin 2t \rangle$

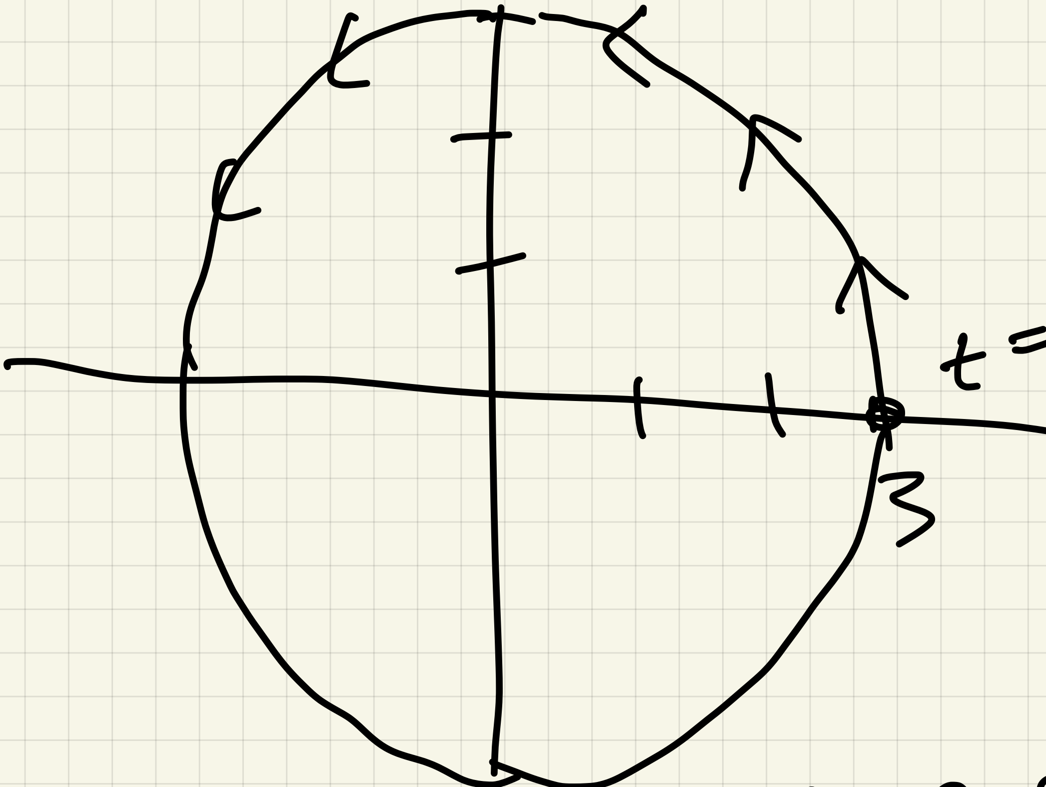
Speed: $|\bar{v}(t)| =$

$|\langle -6\sin 2t, 6\cos 2t \rangle| =$

$\sqrt{(-6\sin 2t)^2 + (6\cos 2t)^2} =$

$\sqrt{36\sin^2 + 36\cos^2} = \sqrt{36}$

$= 6$



At $t = 2\pi$: gone around circle twice, so distance traveled is $2(\text{circum}) =$

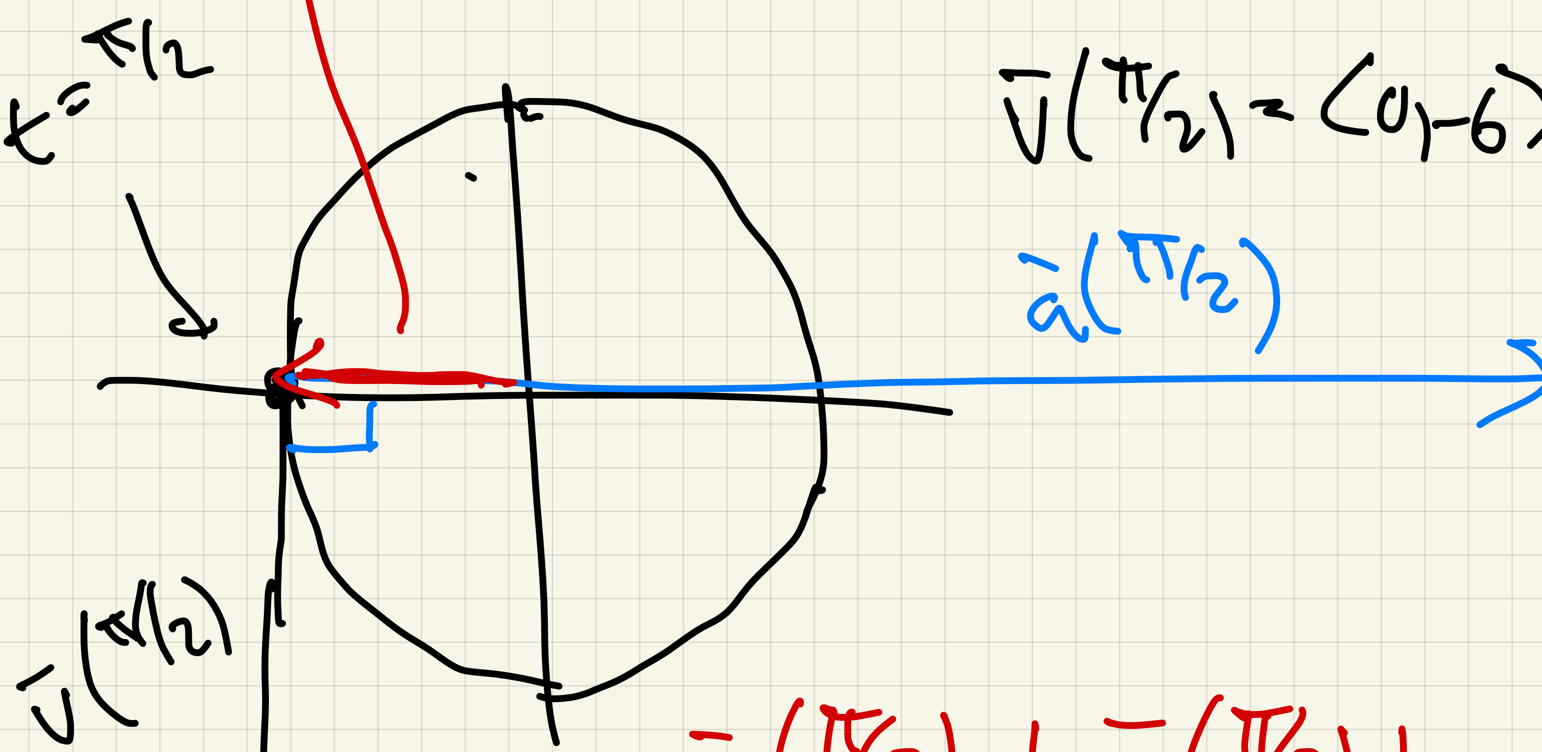
$2(2\pi \cdot 3) = 12\pi$

time elapsed is 2π

$\frac{12\pi}{2\pi} = 6$ ✓

$\sqrt{(\pi/2)}$

$\int$



$\bar{r}(\pi/2) \perp \bar{v}(\pi/2) \perp$

$\bar{a}(\pi/2) \perp$

reason: $\bar{r}(t) \perp \bar{v}(t)$

$\bar{r}'(t)$?!

$\|\bar{r}(t)\| = 3$

$\bar{r}(t) \cdot \bar{r}(t) = 9$

differentiate: || product rule

$\bar{r}'(t) \cdot r'(t) + \bar{r}(t) \cdot r'(t) = 0$

$2\bar{r}(t) \cdot \bar{r}'(t) = 0$

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