10/8/ Calc 3

Last time $z = f(x,y)$

$(a,b) \in \mathbb{R}^2$

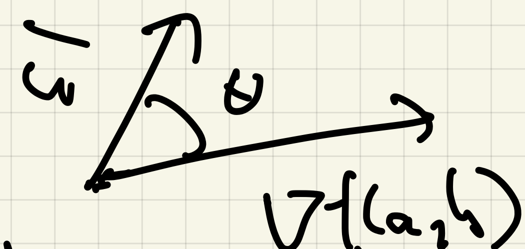
Gradient

$$\nabla f(a,b) = \langle f_x(a,b), f_y(a,b) \rangle$$

u = unit vector

Ⓐ $D_{\vec{u}} f(a,b) = \nabla f(a,b) \cdot \vec{u}$

directional derivative



$$D_u f(a,b) = |\nabla f(a,b)| \cos\theta$$

Ⓑ $\nabla f(a,b)$ – direction of maximal increase

$(\theta = 0, \; D_u f(a,b) = |\nabla f(a,b)|)$

- $-\nabla f(a,b)$: direction of maximal decrease

ⓒ $\nabla f(a,b)$ is $\perp$ to level set $f(x,y) = c = \text{const}$

<u>Ex1</u> $f(x,y) = x^2 - 9y^2$ ←

(a) Find $\nabla f(4,-1)$

(b) Find $D_u f(4,-1)$

$u_1 = \langle 1,1 \rangle$
$u_2 = \langle 1,-1 \rangle$

(c) What is direction of maximal incr/decrease

(d) Sketch level ~~set~~ (curve) $f(x,y) = 7$

Find the tangent line to curve at $(4,-1)$.

(a) $\nabla f = \langle 2x, -18y \rangle$

$\nabla f(4,-1) = \langle 8, 18 \rangle$

(b) $u_1 = \langle 1, 1 \rangle$

$D_{u_1} f(4,-1) = \langle 8, 18 \rangle \cdot \dfrac{\langle 1, 1 \rangle}{\sqrt{2}} =$

$\dfrac{26}{\sqrt{2}}$

$u_2 = \langle 1, -1 \rangle$

$D_{u_2} f(4,-1) = \langle 8, 18 \rangle \cdot \dfrac{\langle 1, -1 \rangle}{\sqrt{2}} = \dfrac{-10}{\sqrt{2}}$

(c) Direction ~~a~~f max incr.

$\nabla f = \langle 8, 18 \rangle =$

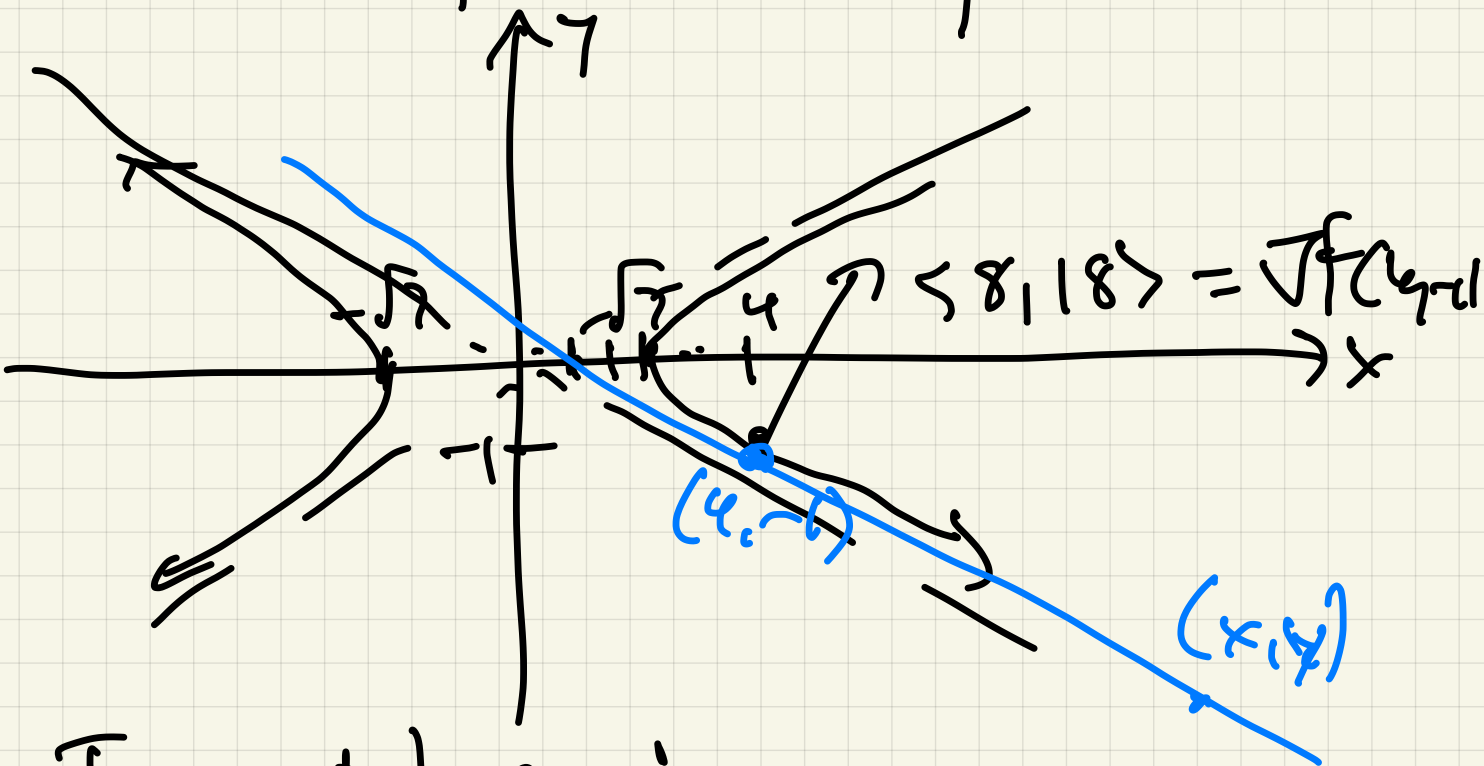
Direction: $\dfrac{\langle 8, 18 \rangle}{|\langle 8, 18 \rangle|} =$

$\dfrac{\langle 4, 9 \rangle}{|\langle 4, 9 \rangle|} = \dfrac{\langle 4, 9 \rangle}{\sqrt{97}}$

Direction of max decrease

$$-\frac{\langle 4,9\rangle}{\sqrt{97}}$$

(d) $x^2-9y^2=7$ hyperbola



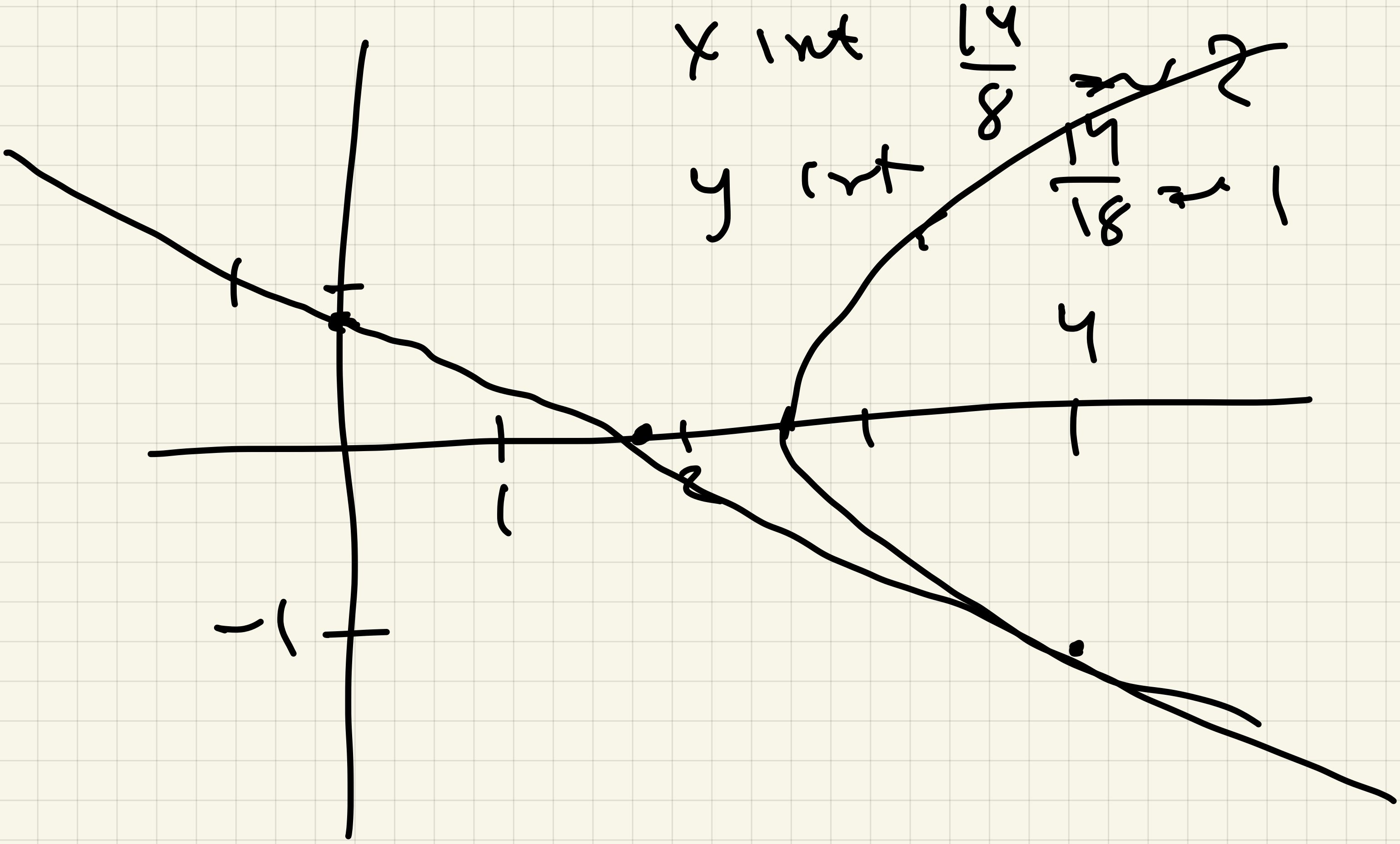
Tangent line:

$$\langle (x,y)-(4,-1)\rangle \cdot \langle 8,18\rangle = 0$$

$$\langle x-4,\, y+1\rangle \cdot \langle 8,18\rangle = 0$$

$$8(x-4)+18(y+1) = 0$$

$$8x+18y = 14$$



In general, if (a,b) is on level curve $f(x,y) = c = \text{const}$

then the tangent line to level curve through (a,b) is

$$f_x(a,b)(x-a) + f_y(a,b)(y-b) = 0$$

Fact (A), (B), (C) valc

just as well in 3D.

$w = f(x,y,z)$

$\nabla f = \langle f_x, \ f_y, \ f_z \rangle$

$D_u f(a,b,c) = \nabla f(a,b,c) \cdot u$

$= |\nabla f(a,b,c)| \cos\theta$

Ex 2 $\quad w = f(x,y,z) = \boxed{xe^{yz} - 2}$

(a) $\nabla f(2,0,-4)$

(b) $D_u f(2,0,-4)$ in direction $\langle 1,2,3 \rangle$

(c) Find direction of max incr/decr

(a) $\quad \nabla f(x,y,z) = \langle e^{yz}, xze^{yz}, xye^{yz} \rangle$

$$\nabla f(2,0,-4) = \langle 1, -8, 0 \rangle$$

(b) $$u = \frac{\langle 1,2,3 \rangle}{\sqrt{14}}$$

$$D_u f(2,0,-4) = \langle 1,-8,0 \rangle \cdot \frac{\langle 1,2,3 \rangle}{\sqrt{14}}$$

$$= -\frac{15}{\sqrt{14}}$$

(c) direction max increase

$$\frac{\langle 1,-8,0 \rangle}{\sqrt{65}}$$

direction of max decrease

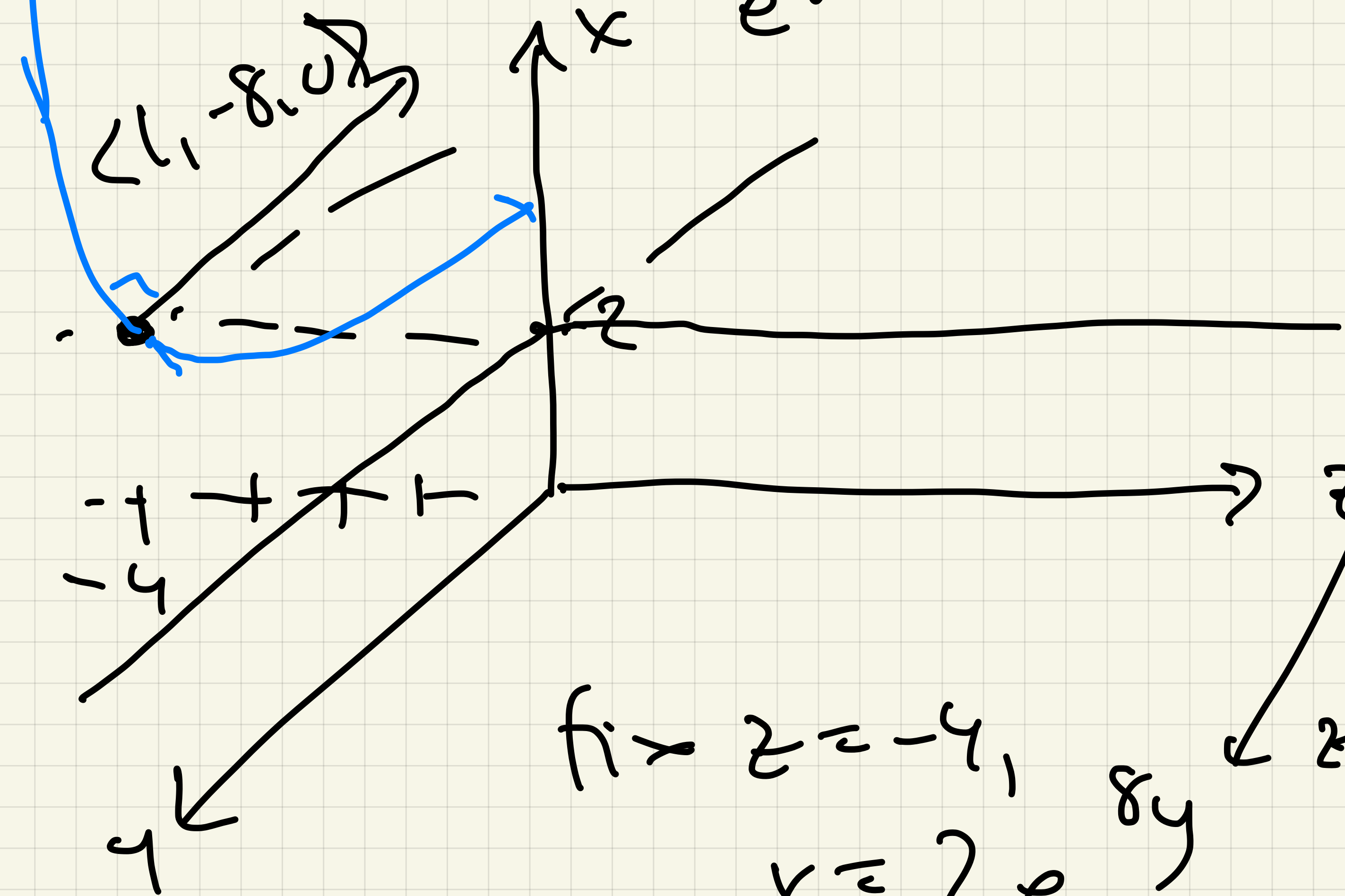
$$\frac{\langle -1,8,0 \rangle}{\sqrt{65}}$$

$(2,0,-4)$ on level surface

$$xe^{yz} - 2 = 0$$

$$xe^{yz}=2$$

$$x=\frac{2}{e^{yz}}=2e^{-yz}$$



fix $z=-4$, $\quad z=-4$

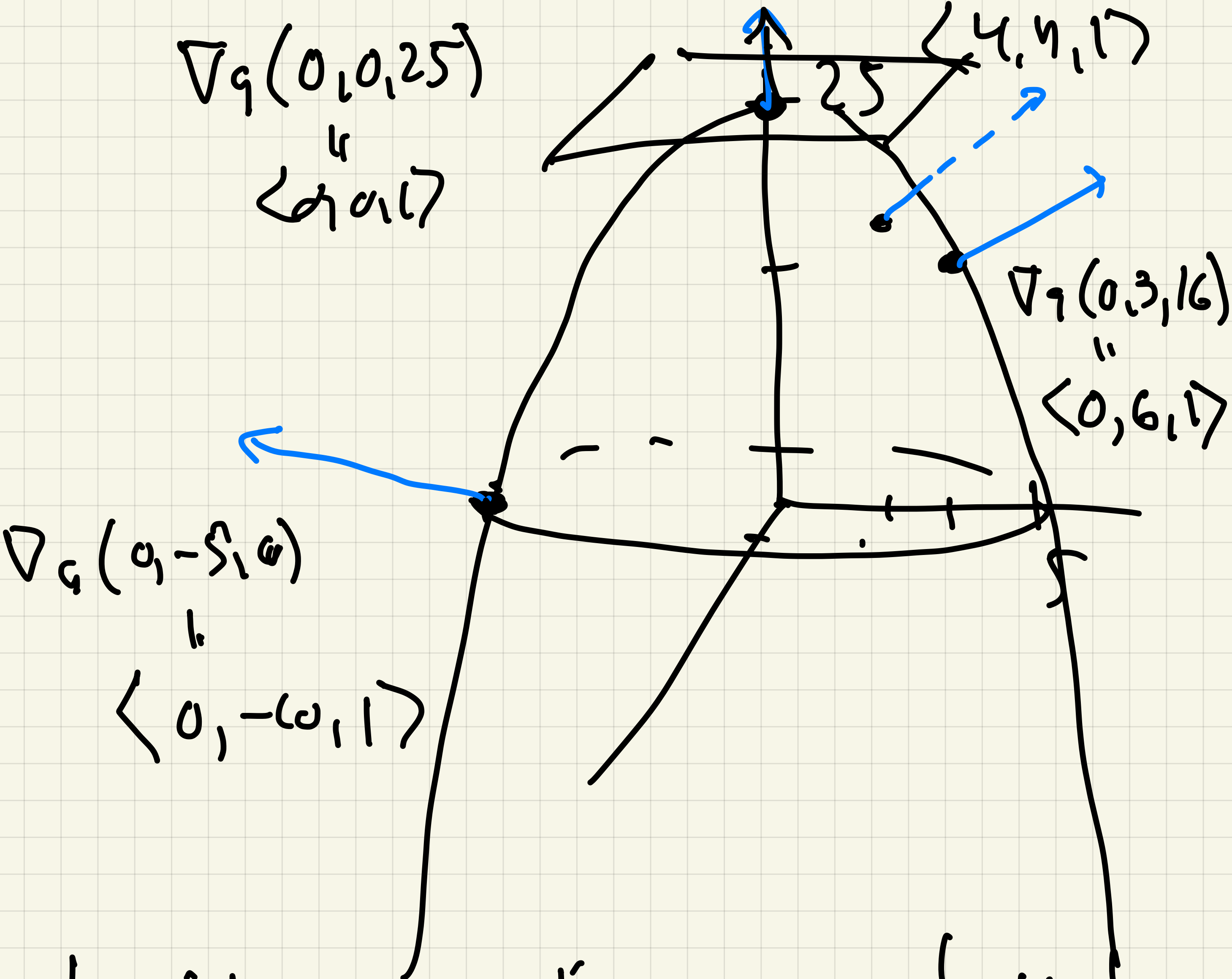
$$x=2e^{8y}$$

Ex 3 Find normal vectors to surface $z=25-x^2-y^2$

at $(0,0,25)$ $\quad (0,3,16)$

$(0,-5,0)$ $\quad (2,2,17)$

$\uparrow \nabla g(2,2,17)$



Write equation as a level set : $f(x,y,z) = \text{const}$

$$\underbrace{x^2+y^2+z}_{g(x,y,z)} = 25 \qquad g = 25$$

$$\nabla g = \langle 2x, 2y, 1\rangle$$

§ 13.6 If we can compute

normal vectors in Ex3,

then we can find tangent planes

Defn; If S is a level surface $f(x,y,z) = c$ = constant and $\nabla f(a,b,c) \neq 0$. Then

① The tangent plane to S passes through (a,b,c) and has normal vector $\nabla f(a,b,c)$

② The normal line has direction $\nabla f(a,b,c)$

Ex4 find tangent plane at $(2,2,17)$ and $(0,3,16)$ and normal line

At $(2,2,17)$, $\nabla g = \langle 4,4,1 \rangle$

so tangent plane is

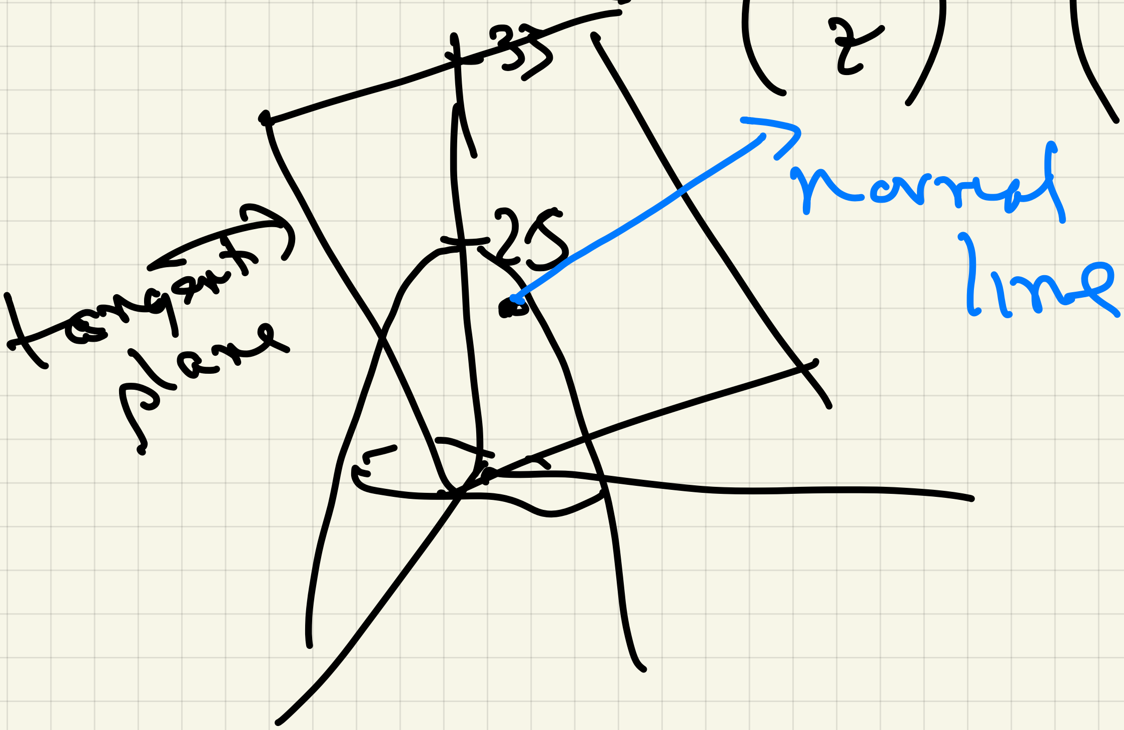
$$\langle x-2, y-2, z-17 \rangle \cdot \langle 4,4,1 \rangle = 0$$

$$4(x-2) + 4(y-2) + 1(z-17) = 0$$

$$4x + 4y + z = 33 \leftarrow$$

<u>Normal line</u>

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2+4t \\ 2+4t \\ 17+t \end{pmatrix}$$



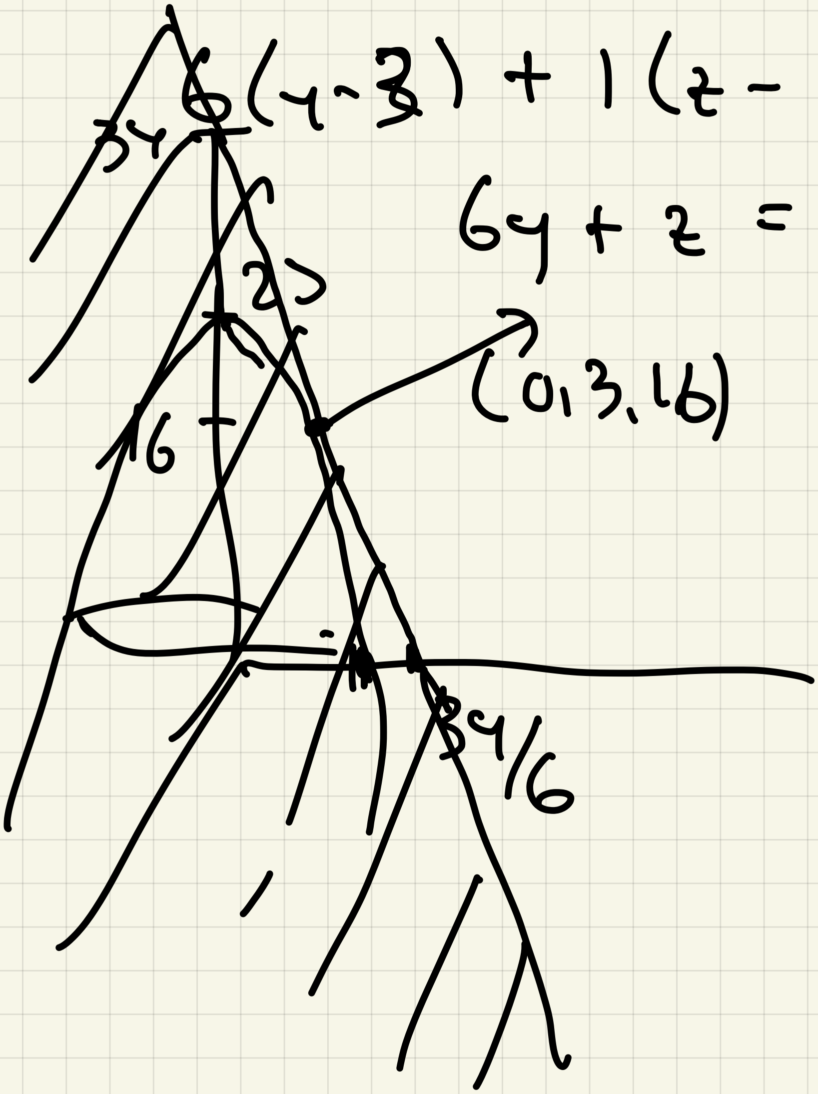
At $(0,3,16)$, $\nabla g(0,3,16) = \langle 0,6,1 \rangle$

<u>tangent plane</u>:

$\langle x, y-3, z-16 \rangle \cdot \langle 0, 6, 1 \rangle = 0$

$6(y-3) + 1(z-16) = 0$

$6y + z = 34$



<u>normal line</u>:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 + 0 \\ 3 + 6t \\ 16 + t \end{pmatrix}$$