10/31 | Calc 3

Quiz 14

$z = f(x,y) = x^3 + 3x^2 + y^3 - 3y$

$\nabla f = \langle 3x^2 + 6x, 3y^2 - 3 \rangle$

$3x(x+2)$ $\quad$ $3(y-1)(y+1)$

$0, -2$ $\quad$ ± 1

$(0, \pm 1)$

$(-2, \pm 1)$

$f_{xx} = 6x + 6$ $\quad$ $f_{yy} = 6y$

$f_{xy} = 0$

$$d = \det \begin{pmatrix} 6x+6 & 0 \\ 0 & 6y \end{pmatrix}$$

$\begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$

$(0,1)$ $\quad$ $d = 36 > 0$, $f_{xx} = 6 > 0$

local min

$(0,-1)$ $\quad$ $d = -36 < 0$ $\quad$ saddle

$(-2,1)$ $d = -36 < 0$ $\begin{pmatrix} -6 & 0 \\ 0 & 6 \end{pmatrix}$

saddle

$(-2,-1)$ $d = \begin{vmatrix} -6 & 0 \\ 0 & -6 \end{vmatrix}$ $36 > 0$

$f_{xx} = -6 < 0$

local max

Last time

Regions & endpoints

polar coordinates

$x = r\cos\theta$

$y = r\sin\theta$

$$\iint_R f(x,y)\,dA = \iint_G f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

Stat $\int$

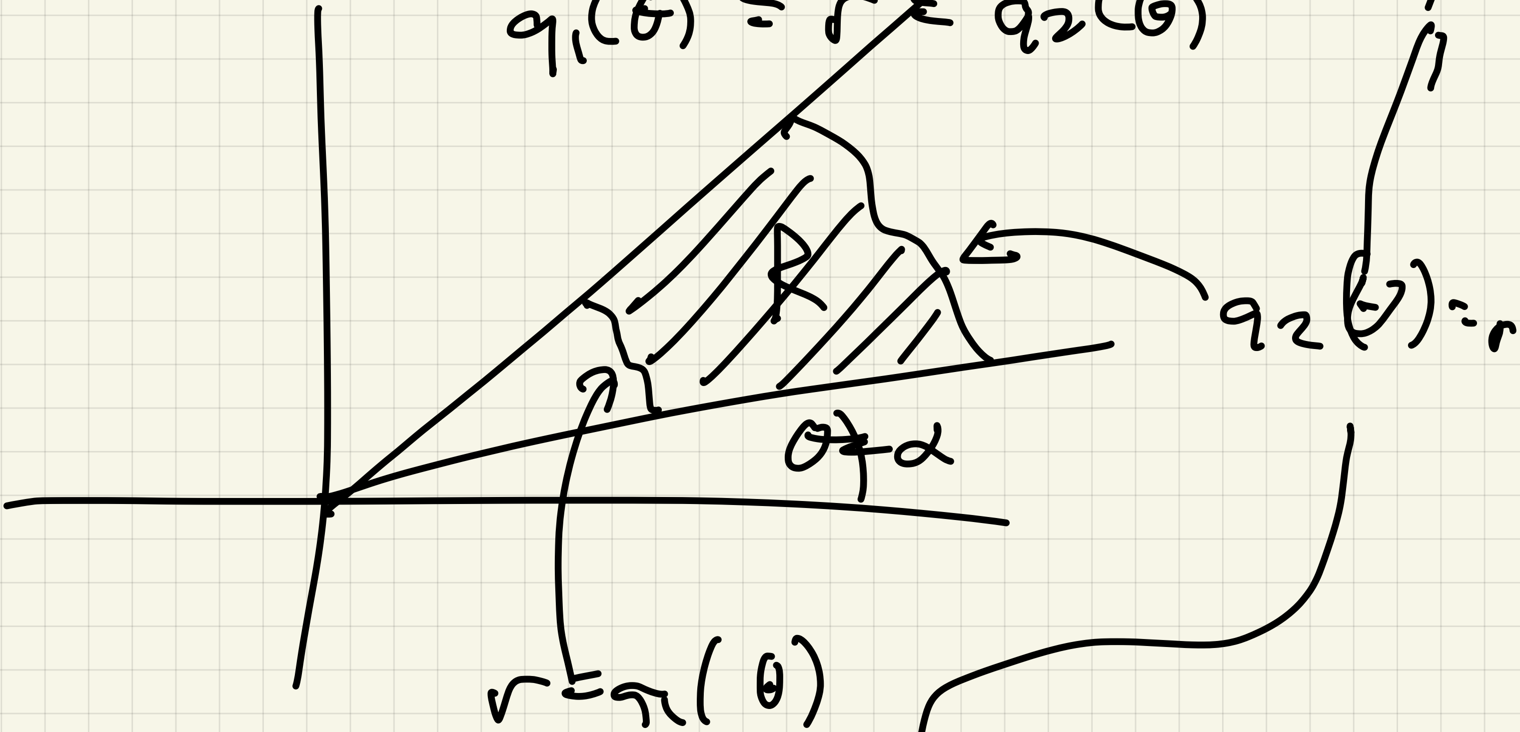
$$\int_{-\infty}^{\infty} e^{-x^2} = 2\sqrt{\pi} \;??$$

Specifically,

If R is described by

$$\alpha \le \theta \le \beta$$

$$g_1(\theta) \le r \le g_2(\theta)$$



$$\int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(\qquad\qquad) \, r \, dr \, d\theta$$

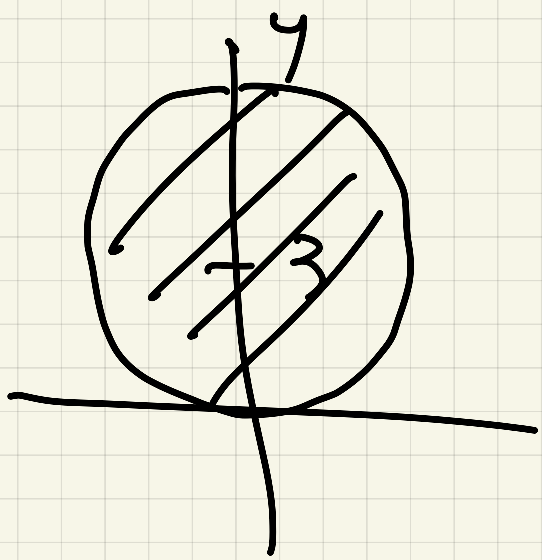
<u>Ex1</u> Find volume of the region bounded by xy plane, the cone $z = \sqrt{x^2+y^2}$

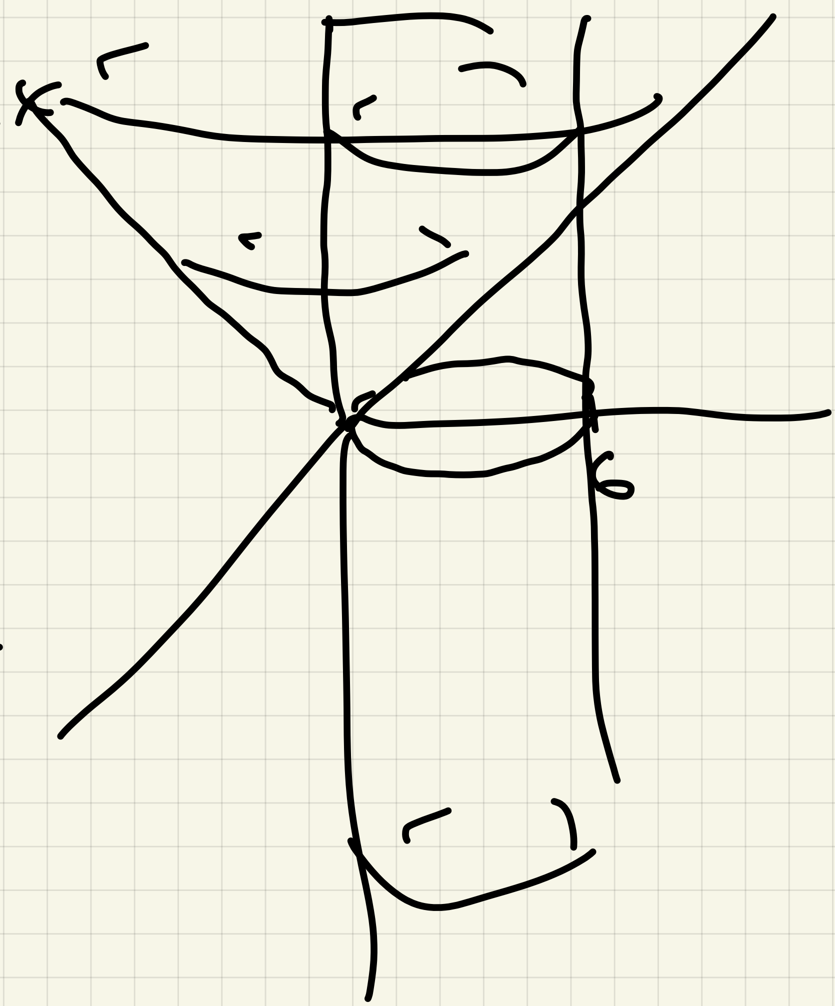
and cylinder $x^2+y^2-6y=0$

$$x^2+y^2-6y+9=9$$

$$x^2+(y-3)^2=3^2$$

$$x=\pm\sqrt{6y-y^2}$$





$$V=\iint_R \sqrt{x^2+y^2}\,dA$$

$$\int_0^6 \int_{-\sqrt{6y-y^2}}^{\sqrt{6y-y^2}} \sqrt{x^2+y^2}\,dx\,dy$$

$= r$

$x = y\tan\theta$

$dx = y\sec^2\theta\,d\theta$

$\vdots$

$$x^2+y^2-6y=0$$

$x = r\cos\theta$

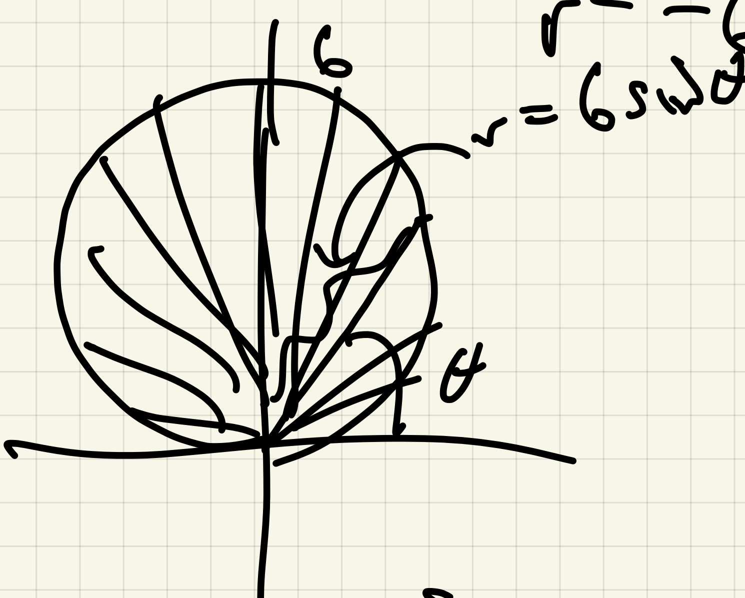
$y = r\sin\theta$

$$\underbrace{r^2\cos^2\theta + r^2\sin^2\theta}_{r^2} - 6r\sin\theta = 0$$

$$r^2 - 6r\sin\theta = 0$$

$$r - 6\sin\theta = 0$$

$$r = 6\sin\theta$$



so

$$V = \underbrace{\int_0^{\pi}\int_0^{6\sin\theta} r \cdot r \, dr \, d\theta}$$

$$\int r^2 = \frac{1}{3} r^3 \Big|_0^{6\sin\theta} =$$

$$\int_0^{\pi} \frac{1}{3}(6\sin\theta)^3 \, d\theta$$

$$6^3 = 216$$

$$72\int_0^{\pi} \sin^3\theta \, d\theta$$

$$72\int_0^{\pi} (1-\cos^2\theta)\,\underline{\sin\theta \, d\theta}$$

$$u = \cos\theta$$
$$du = -\sin\theta \, d\theta$$

$$\parallel$$

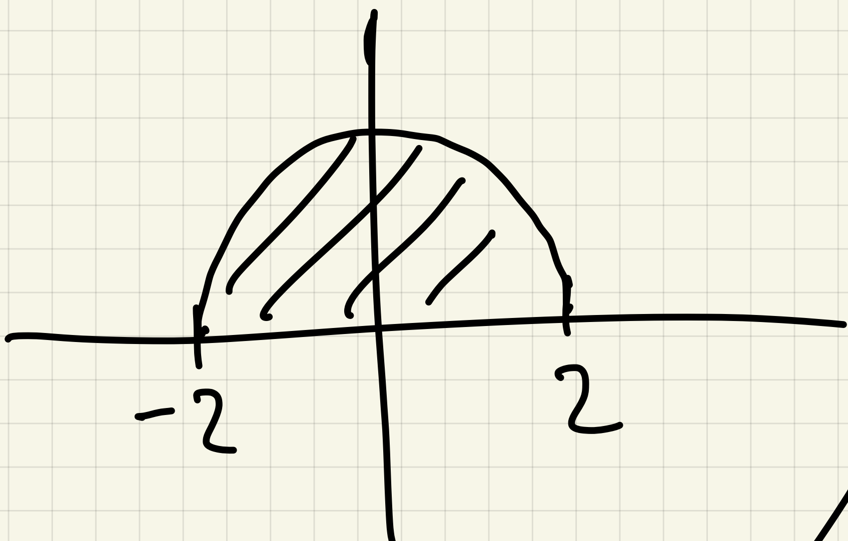
$$-72\int_1^{-1} (1-u^2)\,du = 72\int_{-1}^{1}(1-u^2)\,du =$$

$$144\int_0^1 (1-\underline{u^2})\,du \qquad (1-\underset{\text{even}}{u^2})$$

$$144\left(u - \tfrac{1}{3}u^3\right)\Big|_0^1 =$$

$$2/3\,(144) = 96$$

Ex2 $$\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} \cos(x^2+y^2)\,dy\,dx \qquad \leftarrow y = \sqrt{4-x^2}$$



$$\int_0^{\pi}\int_0^{2} \cos(r^2)\, r\, dr\, d\theta$$

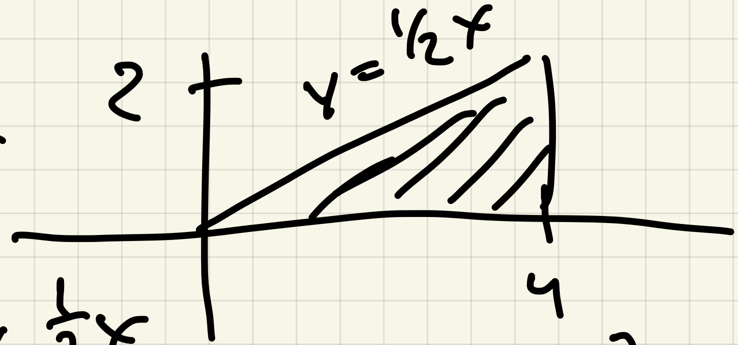
$u = r^2$

$$\frac{1}{2}\sin(r^2)\Big|_0^2 =$$

$$\int_0^{\pi} \frac{1}{2}\sin 4\, d\theta$$

$\frac{\pi}{2}\sin 4$

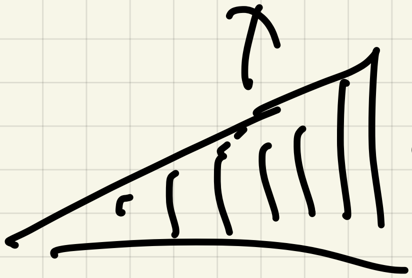
Ex 3



(Area = 4)

(a)

$$A = \int_0^4 \int_0^{\frac{1}{2}x} dy\,dx = \int_0^2 \int_{2y}^4 dx\,dy$$



polars?

$$x = 4$$
$$\| \quad \|$$
$$r\cos\theta$$

So $r = \frac{4}{\cos\theta} = 4\sec\theta$

$$A = \int_0^{\tan^{-1}\frac{1}{2}} \underbrace{\int_0^{4\sec\theta} r\,dr}_{\frac{1}{2}r^2\Big|_0^{4\sec\theta} =}\,d\theta$$

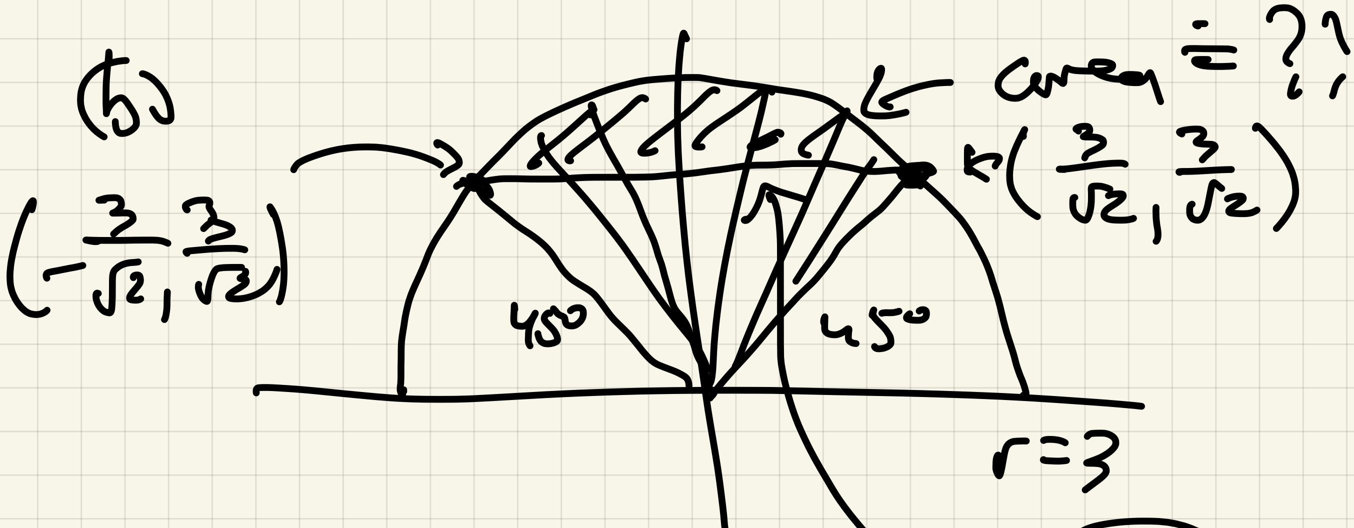
$$\frac{1}{2}(4\sec\theta)^2 = 8\sec^2\theta$$

$$\int_0^{\tan^{-1}1/2} 8\sec^2\theta\, d\theta =$$

$$8\tan\theta \Big|_0^{\tan^{-1}1/2} =$$

$$8\left(\tfrac{1}{2}\right) = 4 \checkmark$$

(5)

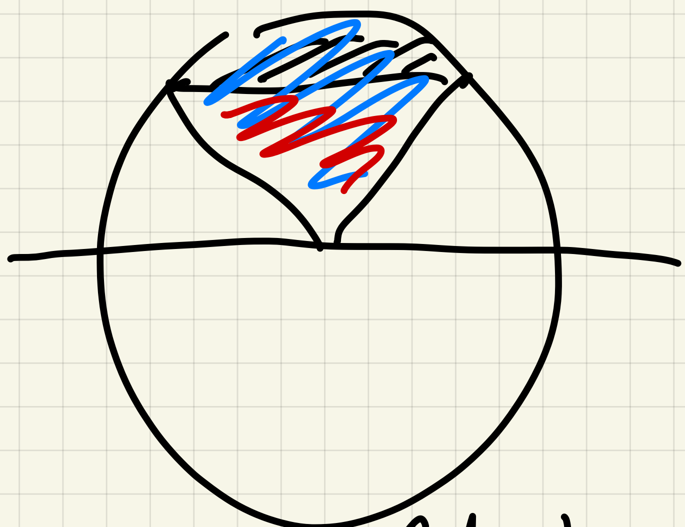


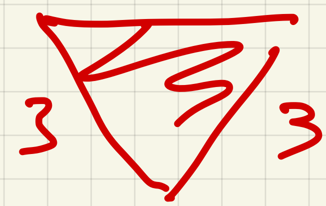
$$\text{Area} = \int_{-3/\sqrt{2}}^{3/\sqrt{2}} \int_{3\sqrt{2}}^{\sqrt{9-x^2}} dy\, dx$$

// polars

$$\int_{\pi/4}^{3\pi/4} \int_{\frac{3}{\sqrt{2}\sin\theta}}^{3} r\,dr\,d\theta = \_\ \_$$

$r\sin\theta = y = \frac{3}{\sqrt{2}} \rightarrow r = \frac{3}{\sqrt{2}\sin\theta}$





$A = \frac{1}{4}\begin{pmatrix}\text{disk of}\\ \text{rad } 3\end{pmatrix} - \frac{1}{2}\begin{pmatrix}\text{square}\\ \text{side } 3\end{pmatrix}$

$\frac{1}{4}(9\pi) - \frac{1}{2}(9)$

§14.5 Triple integrals

Definition: If $\overset{w=}{f(x,y,z)}$ is a function defined on a solid region B, the triple

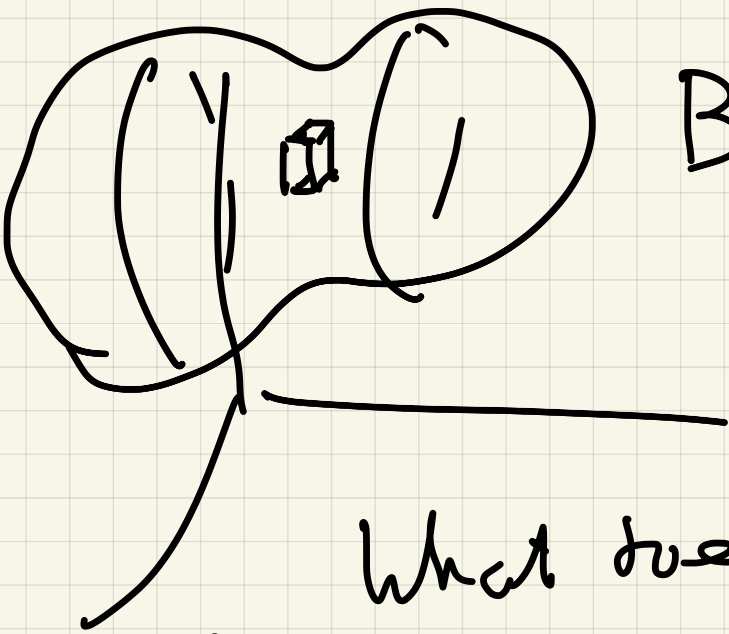
integral of f over B

$$\iiint_B f(x,y,z)\,dV =$$

$$\lim_{\|\Delta\|\to 0} \sum \boxed{f(x_i,y_i,z_i)\,dV_i}$$

$\Delta V_i = \Delta x_i \Delta y_i \Delta z_i$

volume of i^{th} 3d rectangle



B

What does it mean?

① $\iiint_B 1\,dV =$ Volume of B

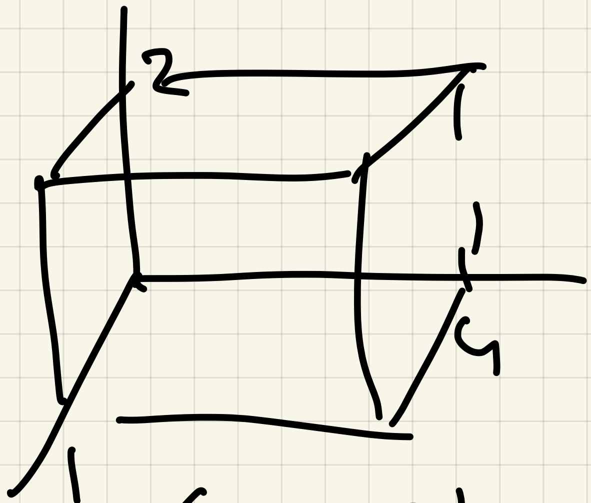
② $\iiint_B f(x,y,z)\,dV$

If $f(x,y,z) \geq 0$ is
density at position (x,y,z)

then $\displaystyle\iiint_B f(x,y,z)\,dV =$ mass of B

Ex1

B



(a) density at (x,y,z) is 5

then mass is

$$5\cdot(1\cdot 2\cdot 4) = 40$$

(b) What if density is

$$\rho\, f(x,y,z) = y+3$$

ρ

Can evaluate triple integrals as iterated integrals

$$B: \begin{array}{l} 0 \le x \le 1 \\ 0 \le y \le 4 \\ 0 \le z \le 2 \end{array}$$

So mass $= \displaystyle\int_0^2 \underbrace{\int_0^4 \underbrace{\int_0^1 y+3 \, dx}\, dy}\, dz$

$$(y+3)x \Big|_0^1 =$$

$$\int_0^2 \underbrace{\int_0^4 y+3 \; dy}\, dz$$

$$\frac{1}{2}y^2+3y \Big|_0^4 =$$

$$8+12$$

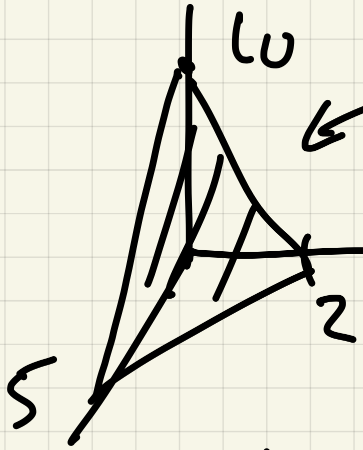
$dx\,dy$

$dy\,dx$

$$\int_0^2 20 \, dz = 20z \Big|_0^2 = 40$$

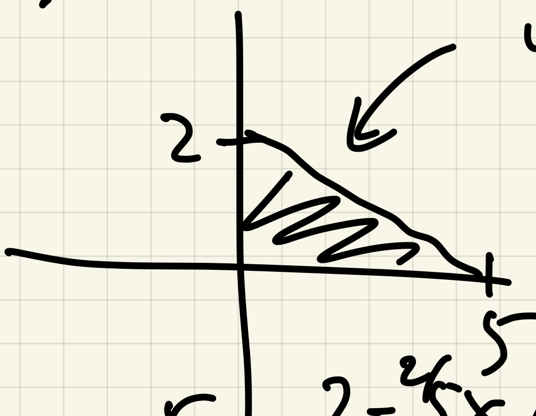
As before, can switch order of in~~tegr~~ation, but now there 6 orders of

Integration.



plane $\boxed{2x+5y+z=10}$

$y = 2 - \frac{2}{5}x$



$$\int_0^5 \int_0^{2-\frac{2}{5}x} \int_0^{10-2x-5y} 1 \, dz \, dy \, dx$$

$$\shortparallel$$

Vdump