10/31 Calc3

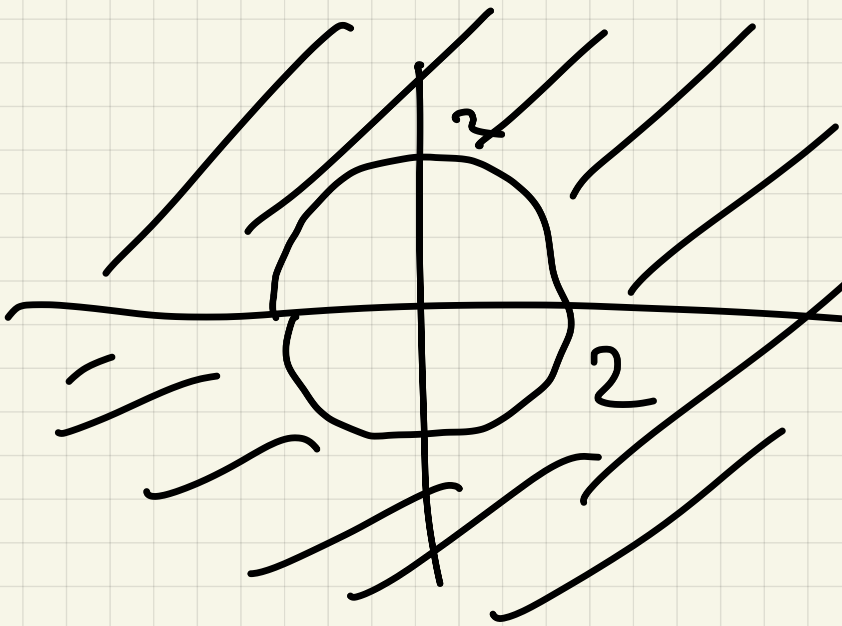
Quiz 9

avg 88%

med 100%

$$z = -\sqrt{\underbrace{x^2+y^2-4}}$$

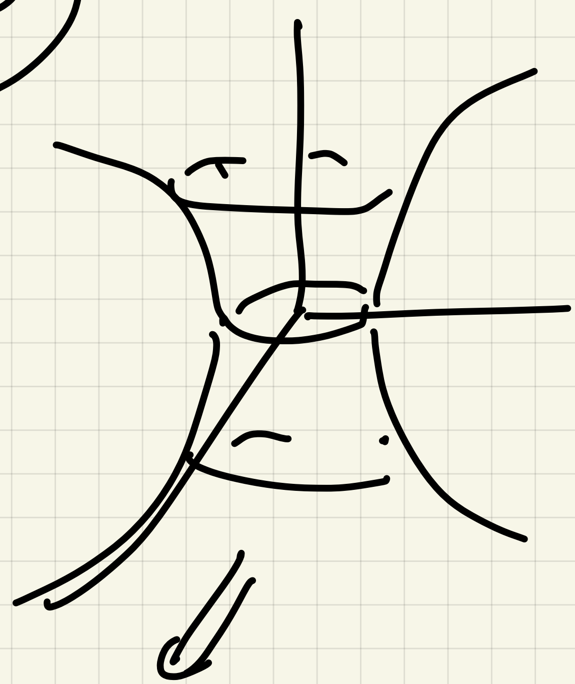
1. Dom $x^2+y^2 \geq 4$

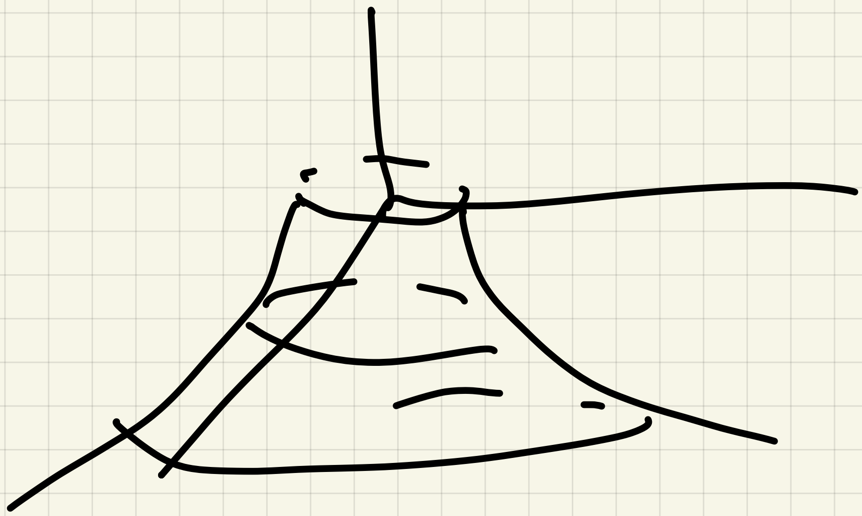


2. Range $z \leq 0$

$$z^2 = x^2+y^2-4$$

$$x^2+y^2 = z^2+4$$





<u>Last time</u>

$$z = f(x, y)$$
$$x = g(s, t)$$
$$y = h(s, t)$$

$$\Downarrow$$

$$z = f(g(s,t), h(s,t))$$

$$\Downarrow \quad \text{and}$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$= z_x \cdot x_s + z_y \cdot y_s$$

$$z_t = z_x \cdot x_t + z_y \cdot y_t$$

Similarly $w = k(x,y,z)$, x,y,z are functions of s,t

$$\boxed{w_s = w_x \cdot x_s + w_y \cdot y_s + w_z \cdot z_s}$$

Ex $w = x^2 + xy + z^2$

$x = s^3 + t^4$

$y = \sin(st)$

$z = e^{(s+3t)}$

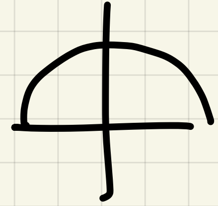
Then

$$\frac{\partial w}{\partial s} = (\underline{2x+y}) \cdot 3s^2 + x \cdot t\cos(st) + 2z \cdot e^{(s+3t)}$$

$$\parallel$$

$$(2(s^3+t^4) + \sin st) \cdot 3s^2 + (s^3+t^4)\, t\cos(st) + 2e^{(s+3t)} \cdot e^{(s+3t)}$$

$\partial w/\partial t$ similar



Implicit Differentiation

Calc: $y = \sqrt{25-x^2} = (25-x^2)^{1/2}$

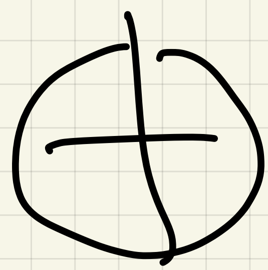
$$\frac{dy}{dx} = \frac{1}{2}(25-x^2)^{-1/2}\cdot(-2x)$$

$$= \frac{-x}{\sqrt{25-x^2}}$$

Explicit

Implicit: $y^2 = 25 - x^2 \Rightarrow$

$$x^2 + y^2 = 25$$



differentiate equation:

$$2x + 2y\cdot\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-2x}{2y} = -\frac{x}{y}$$

Can do same here:

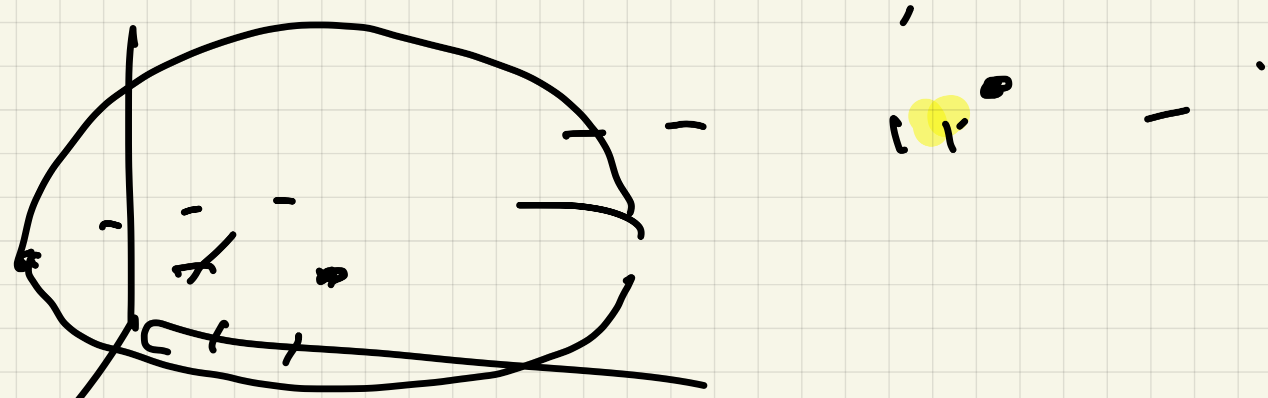
Ex 0 $z = \sqrt{9-(x+1)^2-(y-2)^2}$

Then $\partial z/\partial x(0,0) = -\frac{1}{2}$ $\quad x=0$

$\quad y=0$

$\quad \Downarrow$

$\partial z/\partial y(0,0) = 1$ $\quad z=2$



Ex 1 Can compute implicitly

$$z^2 = 9-(x+1)^2-(y-2)^2$$

$$(x+1)^2+(y-2)^2+z^2 = 3^2$$

Find $\partial z/\partial x$

take $\partial/\partial x$

$$2(x+1)' + 0 + 2z \cdot \frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x} = \frac{-2(x+1)}{2z} = -\frac{(x+1)}{z}$$

at $(0,0,2)$, $\frac{\partial z}{\partial x} = \frac{-1}{2}$ ✓

take $\partial/\partial y$

$$0 + 2(y-2)' + 2z\frac{\partial z}{\partial y} = 0$$

$$\frac{\partial z}{\partial y} = -\frac{(y-2)}{z}$$

<u>Ex 2</u> If $x^3 + xy + z^2 + yz^3 - 4 = 0$

defines a function $z = f(x,y)$

near $\underset{x\ y\ z}{(1,1,1)}$

Find $\partial z/\partial x\,(1,1,1)$ *

$$\partial z/\partial y\,(1,1,1)$$

$$x^3 + \underline{xy} + z^2 + \underline{yz^3} - 4 = 0$$

take $\partial/\partial x$

$$\underline{3x^2 + y} + \underbrace{2z\cdot\frac{\partial z}{\partial x} + y\cdot 3z^2\cdot\frac{\partial z}{\partial x}} = 0$$

$$\frac{\partial z}{\partial x}\left(2z + 3yz^2\right) = -3x^2 - y$$

$$\frac{\partial z}{\partial x} = \frac{-3x^2 - y}{(2z + 3yz^2)}$$

so $\partial z/\partial x\,(1,1,1) = -4/5$

$$\partial/\partial y:\quad x + 2z\frac{\partial z}{\partial y} + 1\cdot z^3 + y\cdot 3z^2\frac{\partial z}{\partial y} = 0$$

$$\frac{\partial z}{\partial y}\left(2z + y\cdot 3z^2\right) = -x - z^3$$

$$\partial z/\partial y = \frac{-x - z^3}{(2z + y^3 \cdot z^2)}$$

$$\partial z/\partial y\,(1,1,1) = -\frac{2}{5}$$

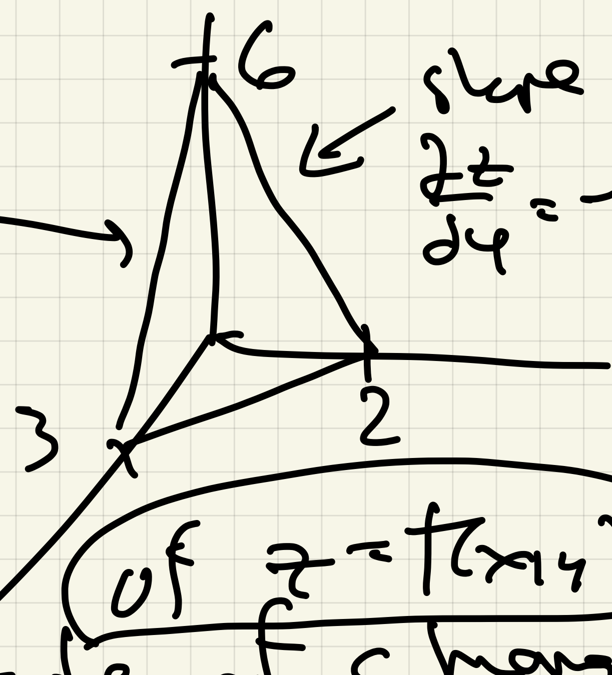
§ 13.5 If $z = f(x,y)$, know

$\partial f/\partial x$ = slope of tangent in $\boxed{\text{pos } x\text{-direction}}$

$\partial f/\partial y$ = slope of tangent line $\boxed{\text{pos } y\text{-direction}}$

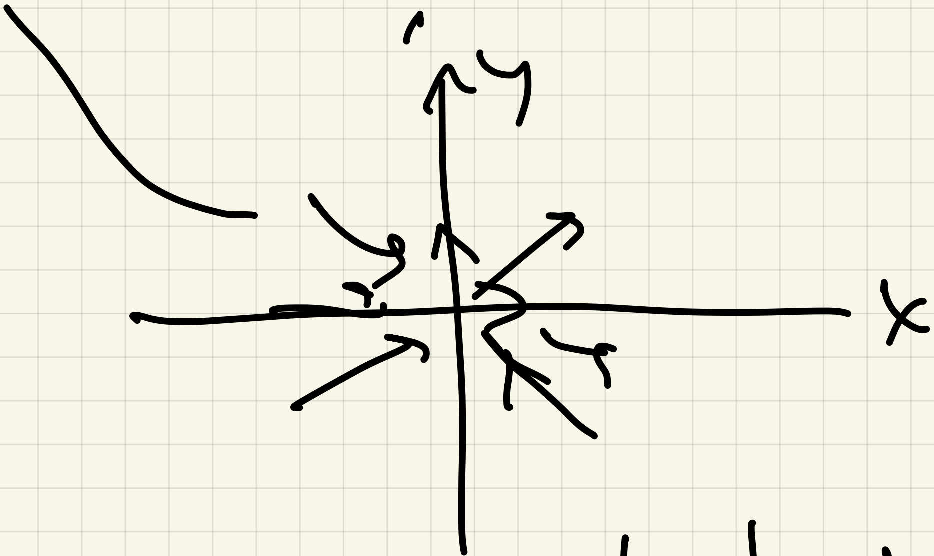
$z = 6 - 2x - 3y$

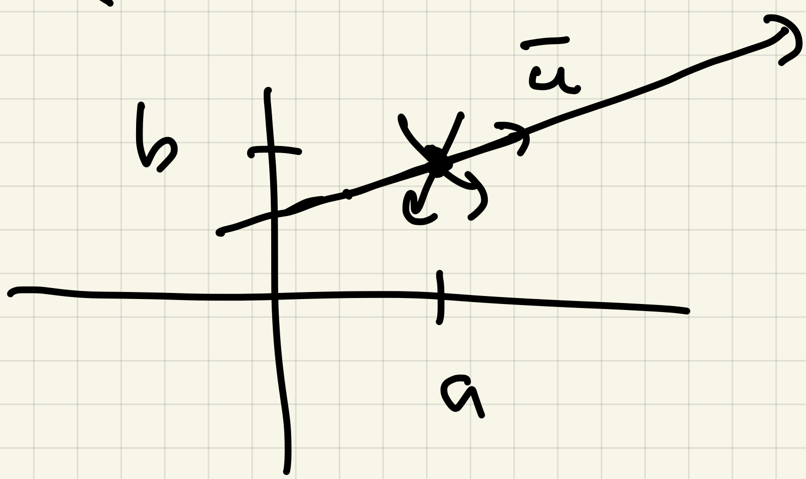
slope is $\partial z/\partial x = -2$



slope $\frac{\partial z}{\partial y} = -3$

What about rate of change of $z = f(x,y)$ in other directions??





A direction is given by a unit vector $u = (u_1, u_2)$

Notice that

$$\bar{r}(t) = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a + u_1 t \\ b + u_2 t \end{pmatrix}$$

<u>direction</u>: $\bar{u}$

<u>speed</u> 1

$$\bar{r}(0) = \begin{pmatrix} a \\ b \end{pmatrix}$$

Definition The directional derivative of $z = f(x,y)$ at (a,b) in direction u

is

$$D_{\vec{u}} f(a,b) := \frac{d}{dt}\left(f(\overset{x}{a+u_1 t}, \overset{y}{b+u_2 t})\right)\Big|_{t=0}$$

$\Vert$ chain rule

$$\frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}$$

$$\frac{\partial f}{\partial x}(a,b)\cdot u_1 + \frac{\partial f}{\partial y}(a,b)\cdot u_2$$

$$\underbrace{\left\langle \frac{\partial f}{\partial x}(a,b), \frac{\partial f}{\partial y}(a,b)\right\rangle}\cdot\langle u_1, u_2\rangle$$

$\nabla f(a,b)$

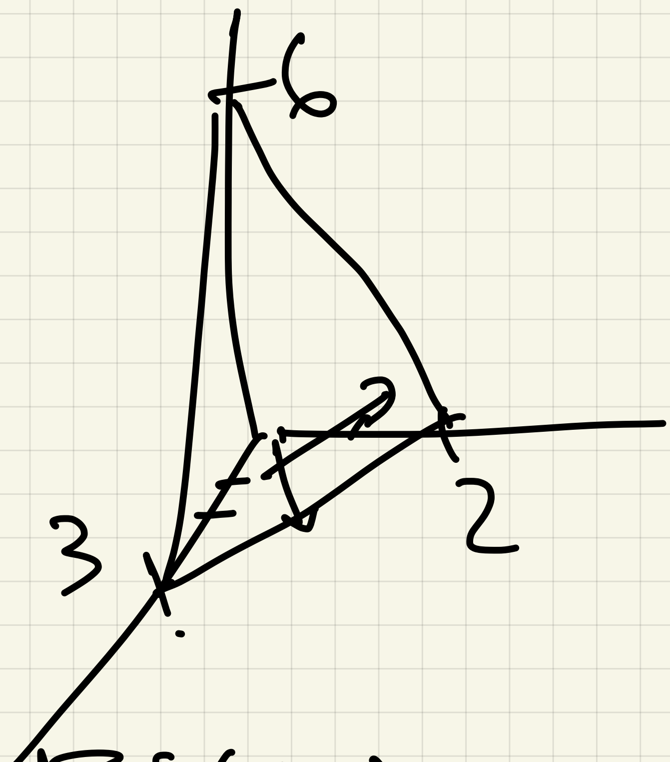
<u>Defn</u> The <u>gradient</u> of $z=f(x,y)$ at (a,b) is

$$\nabla f(a,b) = \langle f_x(a,b), f_y(a,b)\rangle$$

<u>Theorem</u>:

$$D_{\bar{u}} f(a,b) = \nabla f(a,b)\cdot \bar{u}$$

<u>Ex 2</u>



$z = f(x,y)$

$= \underline{\underline{6-2x-3y}}$

(a) Find $\nabla f(0,0)$

(b) Find directional derivatives in $(1,0)$, $(0,1)$.

$(-1,0)$, $(2,1)$ $(-3,2)$

$(2,3)$

(a) $\nabla f = \langle f_x(0,0), f_y(0,0) \rangle$

$= \langle -2, -3 \rangle$

direction	$D_u(f(0,0))$
$(1,0)$	$\langle -2,-3 \rangle \cdot \langle 1,0 \rangle = -2$
$(0,1)$	$\langle -2,-3 \rangle \cdot \langle 0,1 \rangle = -3$
$\langle -1,0 \rangle$	$\langle -2,-3 \rangle \cdot \langle -1,0 \rangle = 2$
$\langle 2,1 \rangle$	$\langle -2,-3 \rangle \cdot \frac{\langle 2,1 \rangle}{\sqrt{5}} = \frac{-7}{\sqrt{5}}$
$\langle -3,2 \rangle$	$\langle -2,-3 \rangle \cdot \frac{\langle -3,2 \rangle}{\sqrt{13}} = 0$
$\langle -2,-3 \rangle$	$\langle -2,-3 \rangle \frac{\langle -2,-3 \rangle}{\sqrt{13}} = \frac{13}{\sqrt{13}} =$

direction of steepest ascent $\longrightarrow$ $\sqrt{13}$