10/29 | Calc 3

$$\iint_R f(x,y)\, dA$$

Last time

$$\int_a^b \int_{g(x)}^{h(x)} f(x,y)\, dy\, dx$$

Fubini's thm

Endpoints (similar calc 2)

Ex 1 Set up endpoints for

$\iint_R 1\, dA$ for regions R

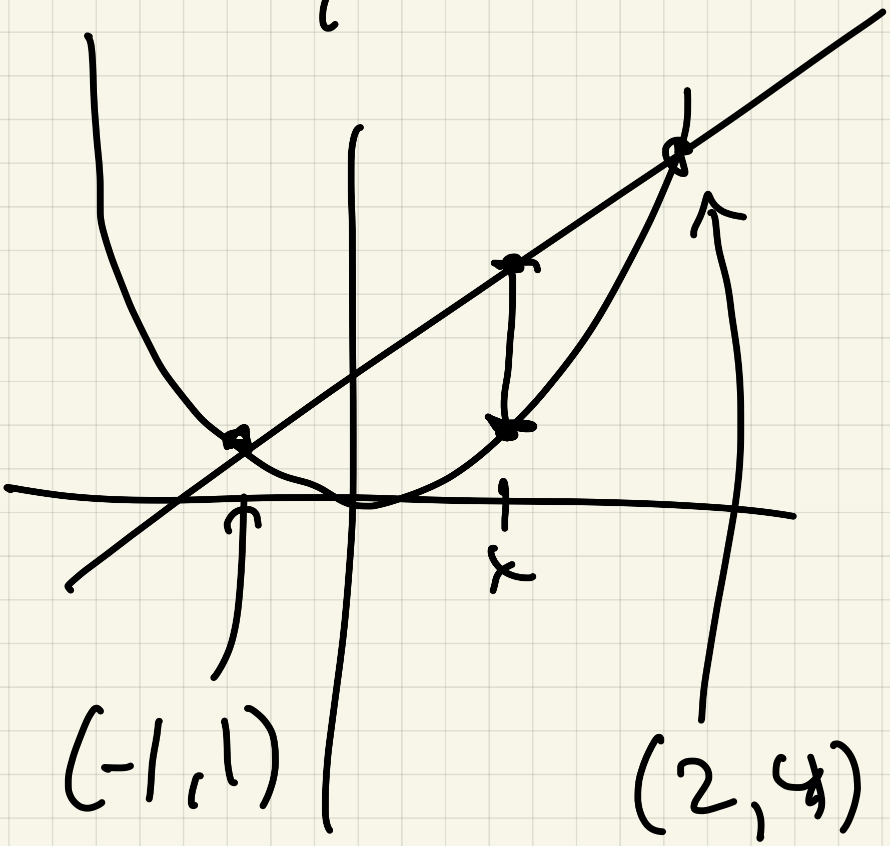
$dx\,dy$ / $dy\,dx$

Remark 1: §14.3

$$\iint_R 1\, dA = \text{Area of } R$$

(a) R bounded by

$y = x^2$ and $y = x+2$



$x^2 = x+2$

$\Downarrow$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$x = 2, -1$

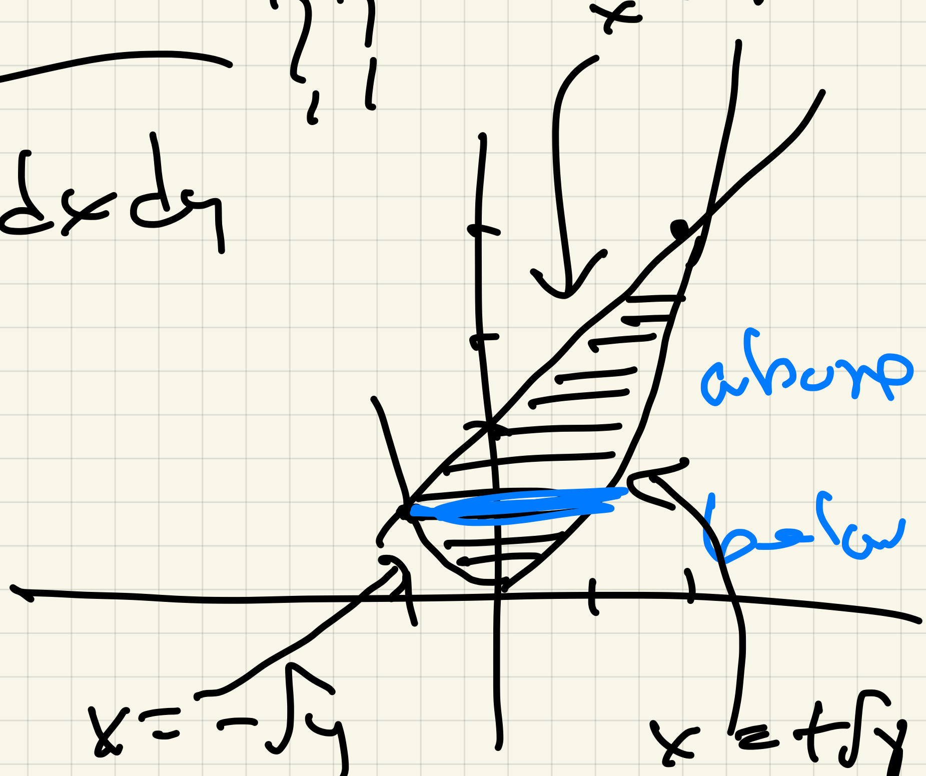
$$\iint_R 1\, dA = \int_{-1}^{2} \int_{x^2}^{x+2} 1\, dy\, dx$$

$$\int_0^4 \int_{?}^{?} 1\, dx\, dy \quad ??$$

$y = x^2$

$\Downarrow$

$x = \pm\sqrt{y}$

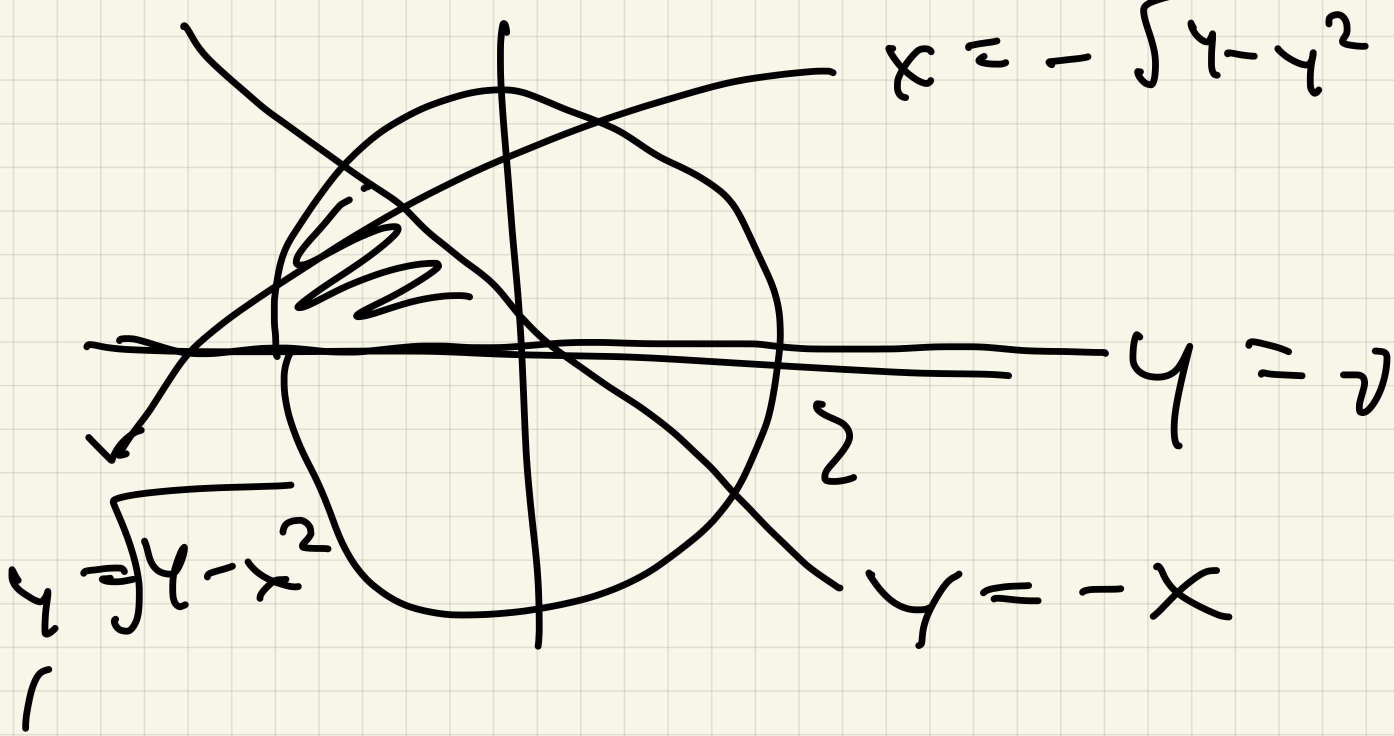


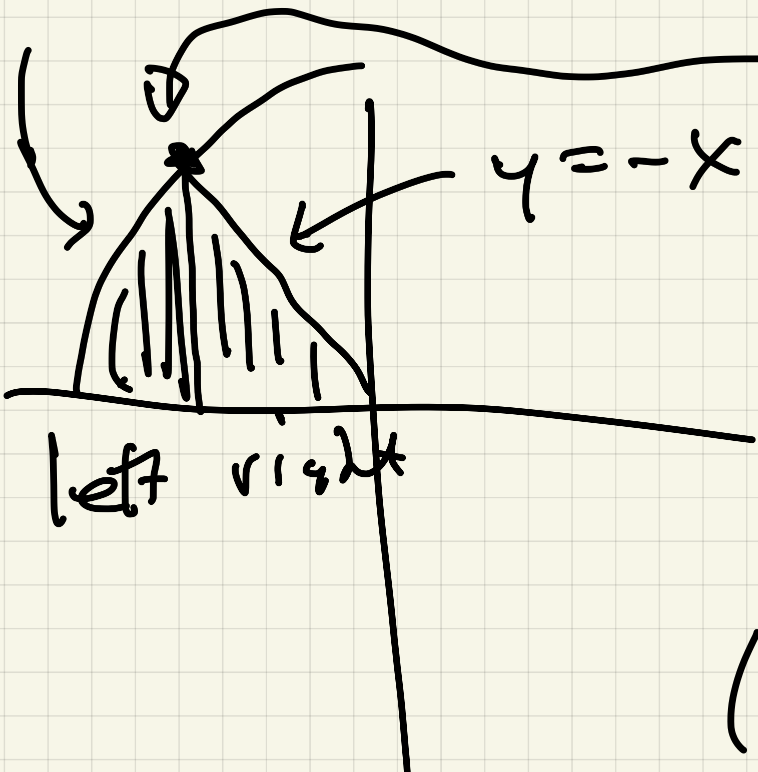
$$\int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} 1\,dx\,dy + \int_1^4 \int_{y-2}^{\sqrt{y}} 1\,dx\,dy$$

bottom top

(b) Region bounded by

$$\begin{cases} \boxed{x^2+y^2=4} \\ y=-x \\ y=0 \\ \text{quadrant II} \end{cases} \qquad y=\sqrt{4-x^2}$$





$y=-x$

$x^2+y^2=4$

$x^2+(-x)^2=4$

$2x^2=4$

$x^2=2$

$x=\pm\sqrt{2}$

$(-\sqrt{2},\sqrt{2})$

$$\iint_R 1\,dA = \int_{-2}^{-\sqrt{2}}\int_{0}^{\sqrt{4-x^2}} 1\,dy\,dx \;+$$

left

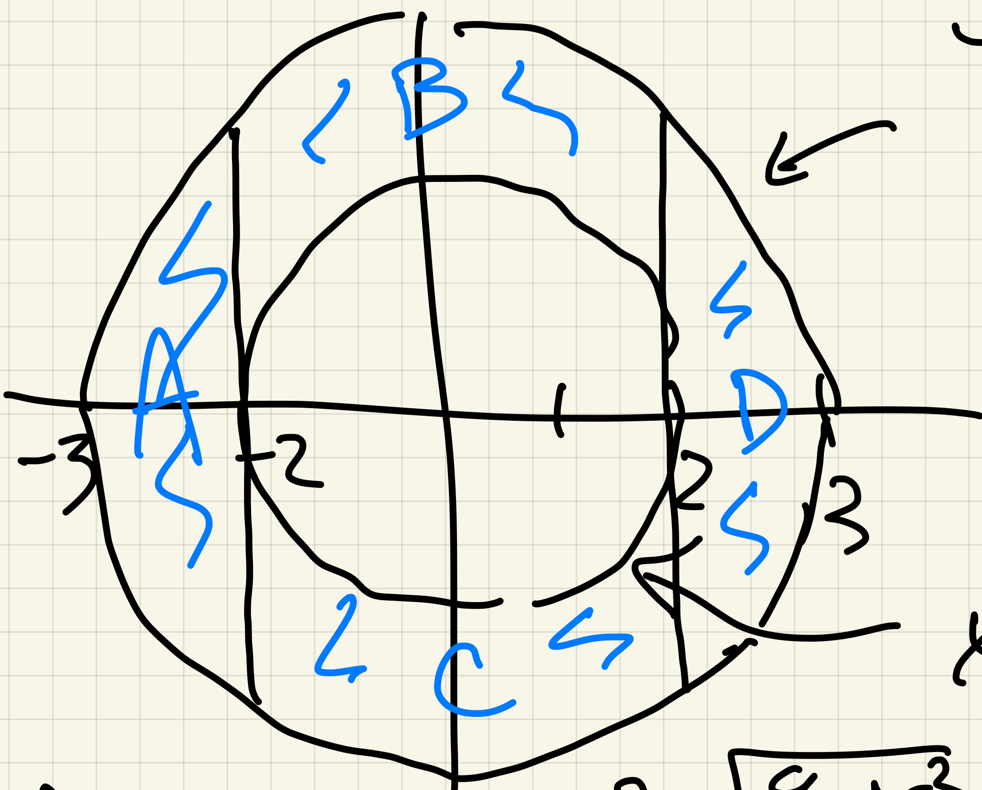
$$\int_{-\sqrt{2}}^{0}\int_{0}^{-x} 1\,dy\,dx$$

right

$$\iint_R 1\,dA = \int_{0}^{\sqrt{2}}\int_{-\sqrt{4-y^2}}^{-y} 1\,dx\,dy$$

6) Region between circles

$x^2+y^2=9$, $x^2+y^2=4$



$$\iint_R 1\,dA = \underbrace{\int_{-3}^{-2}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} 1\,dy\,dx}_{A} +$$

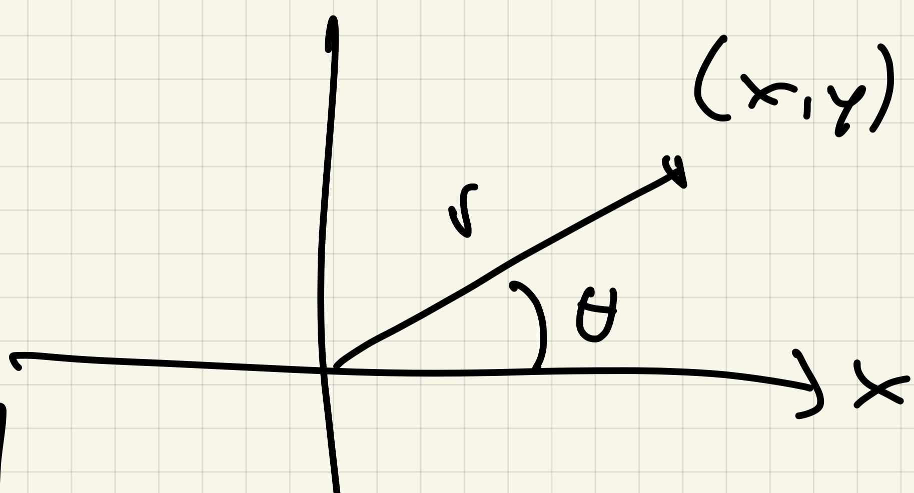
$$\underbrace{\int_{-2}^{2}\int_{\sqrt{4-x^2}}^{\sqrt{9-x^2}} 1\,dy\,dx}_{B} +$$

C $\displaystyle\int_{-2}^{2}\int_{-\sqrt{9-x^2}}^{-\sqrt{4-x^2}} 1\,dy\,dx \; +$

D $\displaystyle\int_{2}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} 1\,dy\,dx$

(b) & (c) inconvenients

§14.4 Polar coordinates



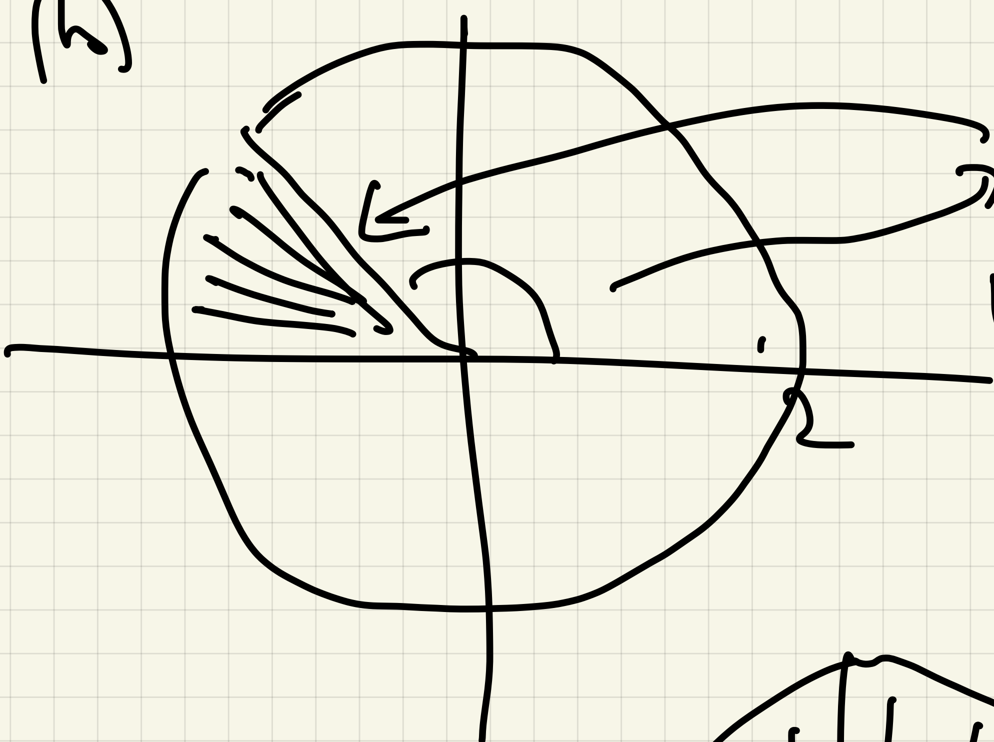
$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}}$$

$r^2 = x^2 + y^2$

$\theta = \arctan \frac{y}{x}$

Ex2 Describe region R in (b) + (c) with polar coords r, θ

(b)

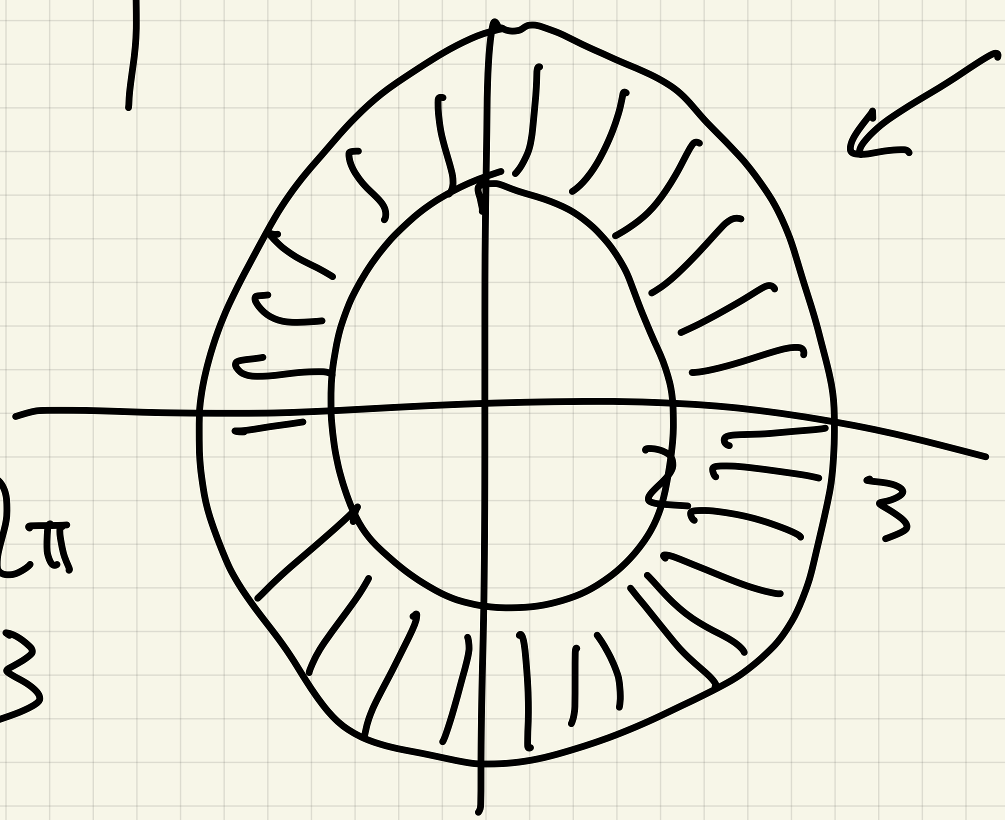


$3\pi/4 \le \theta \le \pi$

$135^\circ \le \theta \le 180^\circ$

$0 \le r \le 2$

(c)



$0 \le \theta \le 2\pi$

$2 \le r \le 3$

Theorem 1:

If R = region in (x,y) coords

G = region in polar coords

Then

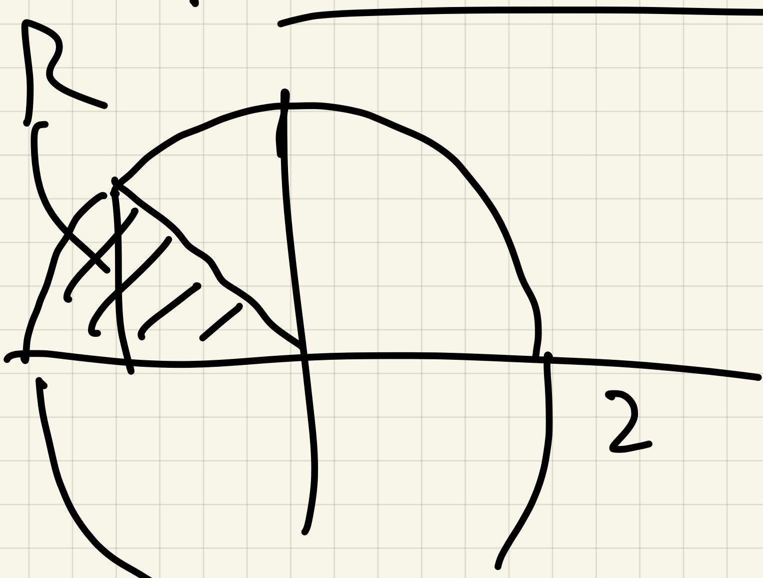
$$\iint_R f(x,y)\,dx\,dy = \iint_G f(r\cos\theta, r\sin\theta)\, r \,dr\,d\theta$$

conversion factor

Plausible check:

(b)

radius? circle



$$\text{Area of } R = \iint_R 1\,dA = \int_0^{\sqrt{2}} \int_{-\sqrt{4-y^2}}^{-y} 1\,dx\,dy$$

$$= \frac{1}{8}(\pi \cdot 2^2) = \frac{4\pi}{8} = \frac{\pi}{2}$$

Thm 1

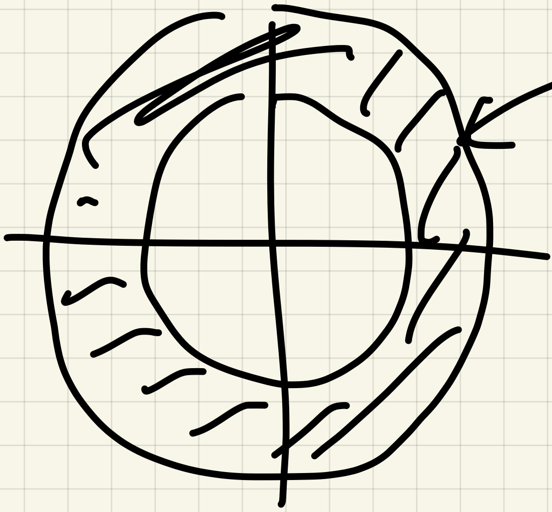
$$= \int_{3\pi/4}^{\pi} \int_0^2 1\,r\,dr\,d\theta$$

$$\frac{1}{2} r^2 \Big|_0^2 = \frac{1}{2} \cdot 2^2 - 0$$

$$= 2$$

$$\int_{3\pi/4}^{\pi} 2\,d\theta = 2\theta \Big|_{3\pi/4}^{\pi} =$$

$$2\left(\pi - 3\pi/4\right) = 2\left(\frac{\pi}{4}\right) = \frac{\pi}{2} \checkmark$$



$9\pi - 4\pi = 5\pi$

Area

$$\int_0^{2\pi}\int_2^3 r\,dr\,d\theta =$$

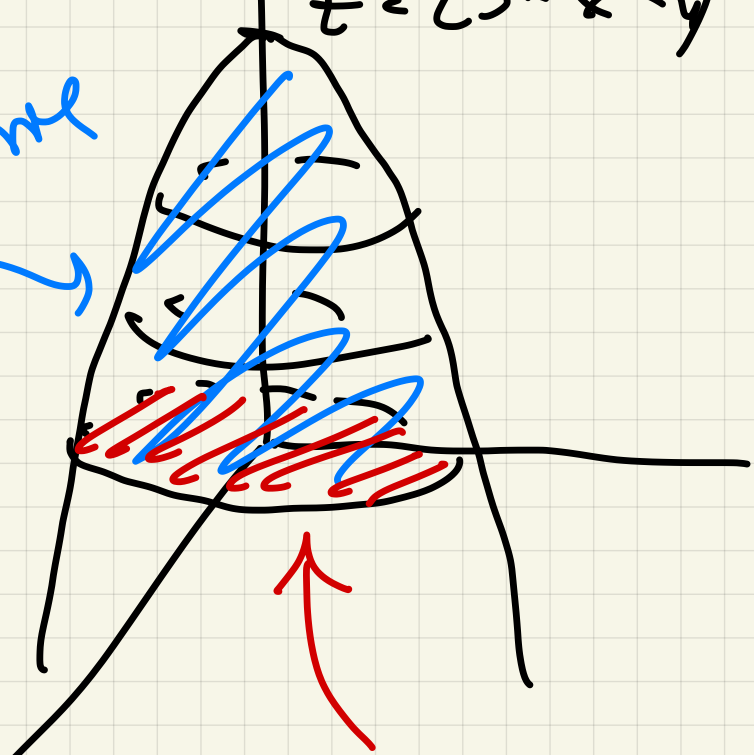
$$\int_0^{2\pi} \frac{r^2}{2}\Big|_2^3 \, d\theta =$$

$$\int_0^{2\pi} \frac{9-4}{2}\, d\theta = \frac{5}{2}\theta\Big|_0^{2\pi} = 5\pi \quad ✓$$

Ex 4

$z = 25 - x^2 - y^2$

find volume of b/w region ($z \geq 0$)

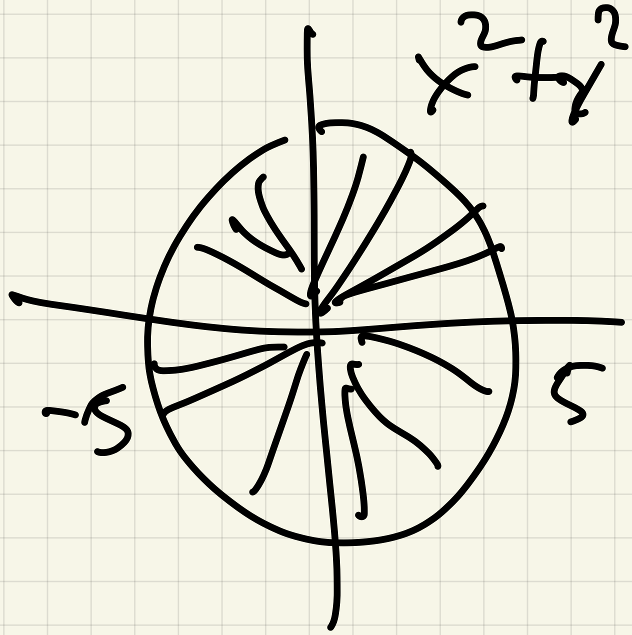


disk
$x^2+y^2 \le 25$
R

<u>Volume</u> $\iint_R 25 - x^2 - y^2 \, dA =$

$$\int_{-5}^{5} \int_{-\sqrt{25-x^2}}^{+\sqrt{25-x^2}} 25 - x^2 - y^2 \, dy \, dx$$

$x^2+y^2=25$



Can do it!, but easier to use polar!

$$\int_0^{2\pi} \int_0^5 \underbrace{25 - (r\cos\theta)^2 - (r\sin\theta)^2}_{\parallel} \, r \, dr \, d\theta$$

$$\int_0^{2\pi}\int_0^5 (25-r^2)r\,dr\,d\theta$$

$$\int 25r - r^3\,dr$$

$$= 25\frac{r^2}{2} - \frac{r^4}{4}\Big|_{r=0}^{r=5}$$

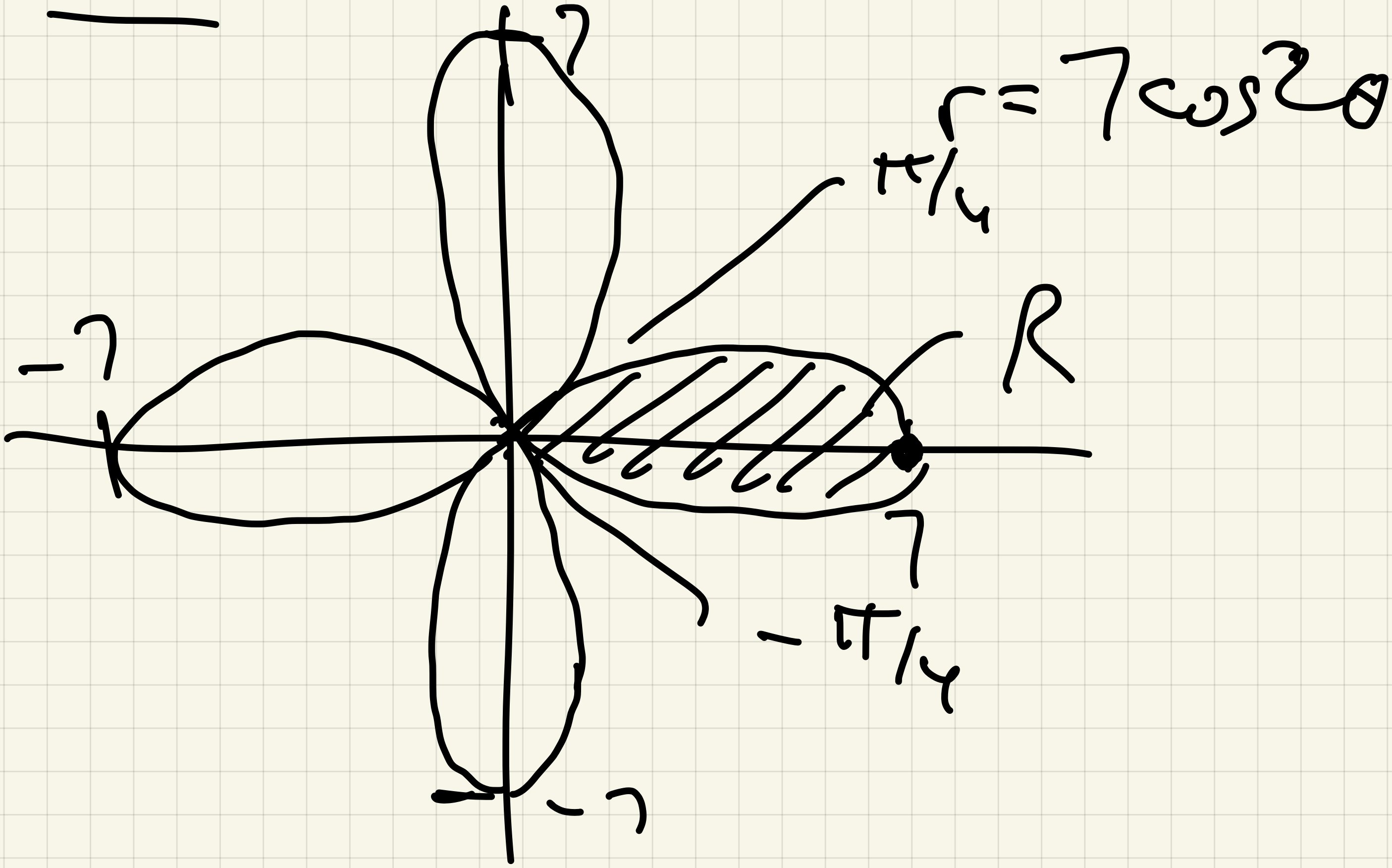
$$25\cdot\frac{25}{2} - \frac{625}{4} =$$

$$\frac{625}{2} - \frac{625}{4} = \frac{625}{4}$$

$$\int_0^{2\pi} \frac{625}{4}\,d\theta =$$

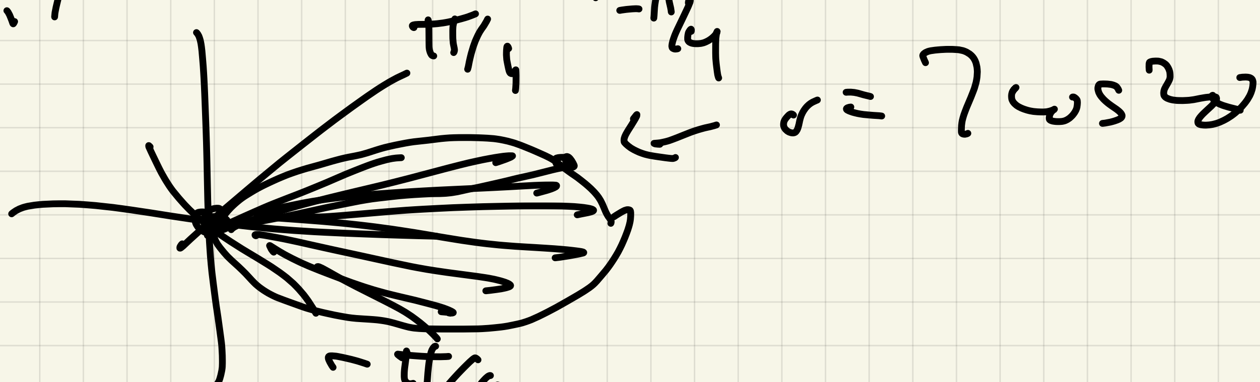
$$\frac{625}{4}\theta\Big|_0^{2\pi} = \boxed{\frac{625\pi}{2}}$$

Ex5 Find area of R



$$\text{Area} = \iint_R 1\, dA =$$

$$\int_{?}^{?}\int_{?}^{?} 1\, dx\, dy = \int_{-\pi/4}^{\pi/4}\int_{0}^{7\cos 2\theta} r\, dr\, d\theta$$



$$\int_0^{7\cos 2\theta} r\,dr = \frac{1}{2}r^2\Big|_0^{7\cos 2\theta} =$$

$$\frac{1}{2}(7\cos 2\theta)^2 = \frac{49}{2}\cos^2 2\theta$$

$$\text{Area} = \frac{49}{2}\int_{-\pi/4}^{\pi/4} \boxed{\cos^2 2\theta}\,d\theta$$

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

$$\frac{49}{2}\int_{-\pi/4}^{\pi/4} \frac{1+\cos 4\theta}{2}\,d\theta =$$

$$\frac{49}{4}\int \underline{1} + \cos 4\theta \; d\theta =$$

$$= \qquad 0$$

$$\frac{49}{4}\left(\theta + \frac{1}{4}\cancel{\sin 4\theta}\right)\Bigg|_{-\pi/4}^{\pi/4}$$

$$\frac{49}{4}\left(\pi/4 - \left(-\pi/4\right)\right) =$$

$$\frac{49}{4}\left(\frac{\pi}{2}\right) = \frac{49\pi}{8}$$