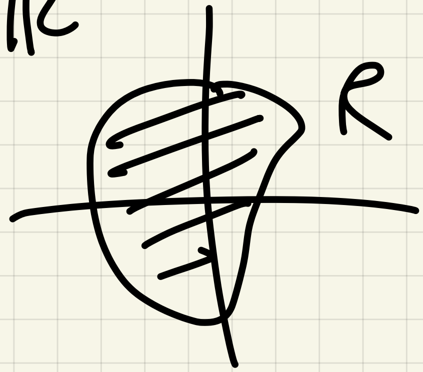
10/22 Calc 3 $z=f(x,y)$

last time $\quad R \subset \mathbb{R}^2$

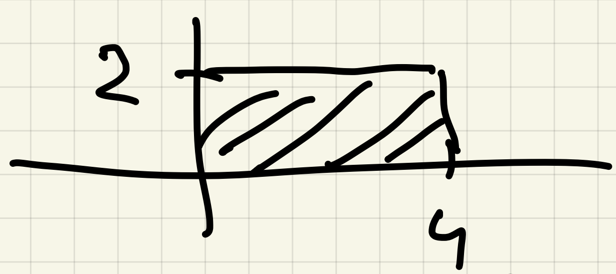


$z=f(x,y)$ on

$$\iint_R f(x,y)\,dA = \text{signed volume under surface (graph) of } z=f(x,y)$$

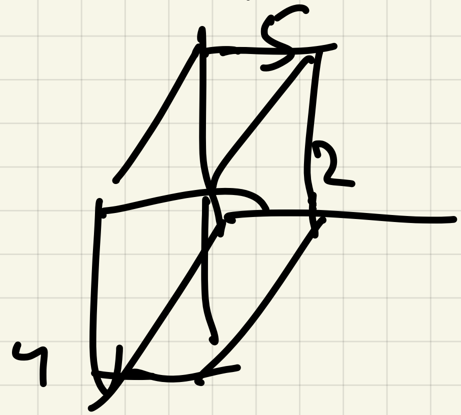
$$\left(= \lim_{|\Delta|\to 0} \sum_{i=1}^{n} f(x_i,y_i)\,\Delta A_i \right)$$

Ex) $\quad R =$

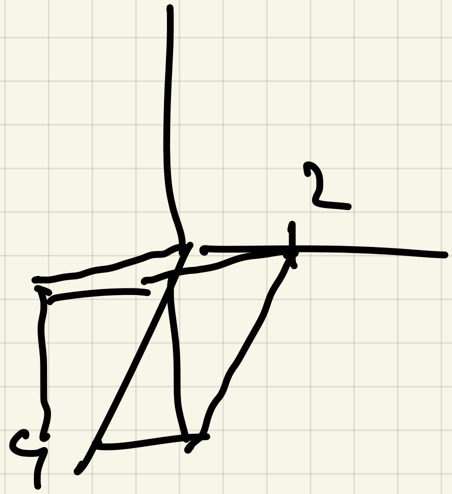


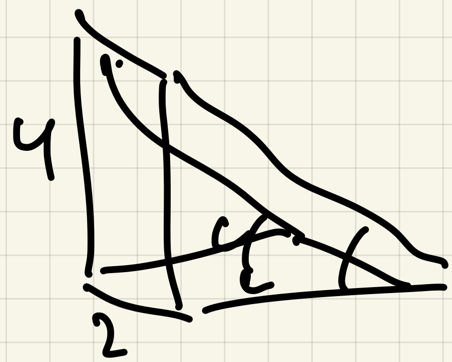
(a)

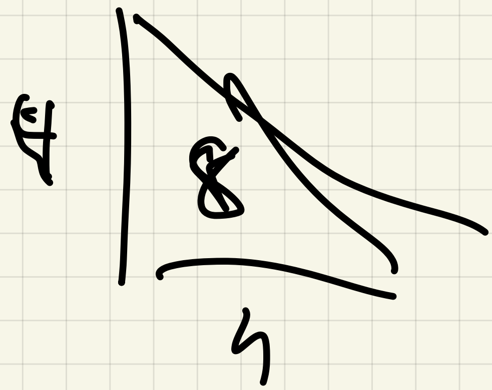
$$\iint_R 5\,dA = 40$$



(b) $\iint_R x \, dA$



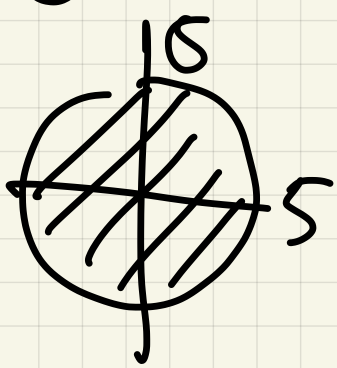




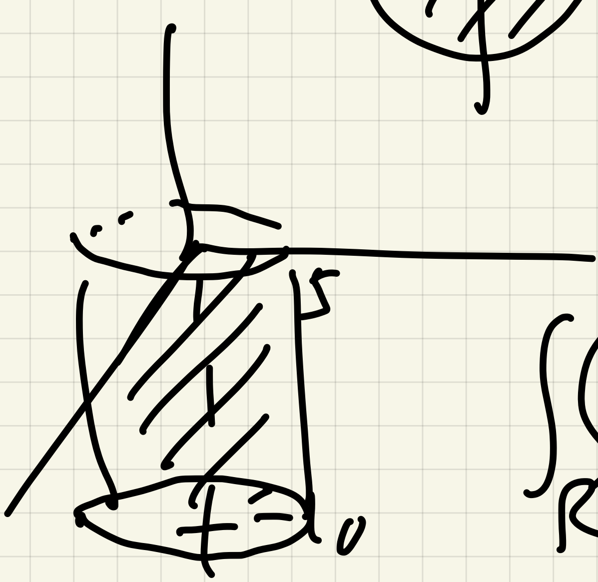
$8 \cdot 2 = 16$

(c) $z = -10$

$R =$



$x^2+y^2 \le 25$



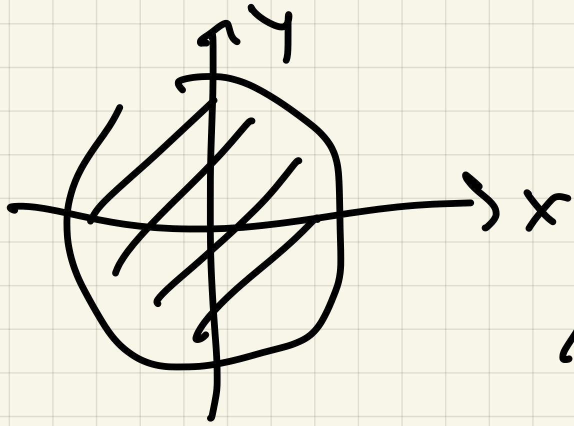
$\iint_R -10 \, dA =$

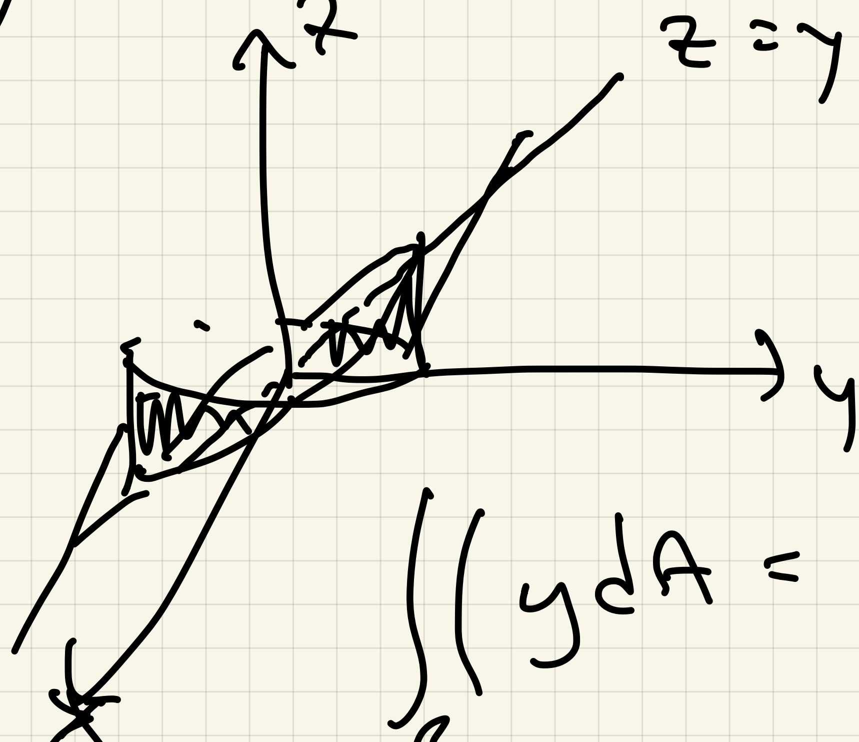
$-$ (volume) $=$

$-\pi \cdot 5^2 \cdot 10 = -250\pi$

$\pi r^2 h$

(d) $z = y$ $\quad R$ same





$$\iint_R y \, dA = 0$$

How to compute $\iint_R f \, dA$??

Answer: Iterated integrals

(A) Ex2

$$\int (\underline{x} + y^3)\, dx$$

$$\frac{1}{2}x^2 + xy^3 + C(y)$$

function of y

$$\int (x + y^3)\, dy =$$

$$xy + \frac{1}{4}y^4 + C(x)$$

function of x

(B) Endpoints: (also similar)

Ex3

(a)
$$\int_{x=0}^{x=2} (x+y^3)\, dx = \frac{1}{2}x^2 + xy^3 \Big|_{x=0}^{x=2}$$

$$= (2 + 2y^3) - 0 = 2 + 2y^3$$

(b) $\displaystyle\int_{x=y}^{x=y^2} = \underline{\tfrac{1}{2}x^2 + xy^3}\Big|_{x=y}^{x=y^2} =$

$$= \left(\tfrac{1}{2}y^4 + y^5\right) - \left(\tfrac{1}{2}y^2 + y^4\right)$$

(c) $\displaystyle\int_{y=0}^{y=x} (x+y^3)\,dy =$

$$xy + \tfrac{1}{4}y^4 \Big|_0^x = x^2 + \tfrac{1}{4}x^4$$

Can <u>iterate</u> this:

<u>Ex 4</u>

(a) $$\int_{y=0}^{y=1} \underbrace{\left[\int_{x=0}^{x=2} (x+y^3)\,dx\right]}_{\text{Ex 3}} dy$$

$$= \int_{y=0}^{y=1} \left(\tfrac{1}{2}x^2 + xy^3 \Big|_0^2\right) dy =$$

$$\int_{y=0}^{y=1} 2 + 2y^3\,dy$$

$$= 2y + \frac{1}{2}y^4 \Big|_0^1 = 2 + \frac{1}{2}$$

$$= 5/2$$

(b) $$\int_{x=0}^{x=2} \underbrace{\int_{y=0}^{y=1} (\underline{x+y^3})\,dy}\,dx$$

$$xy + \frac{1}{4}y^4 \Big|_{y=0}^{y=1} = x + \frac{1}{4}$$

$$\int_{x=0}^{x=2} \left(x + \frac{1}{4}\right) dx =$$

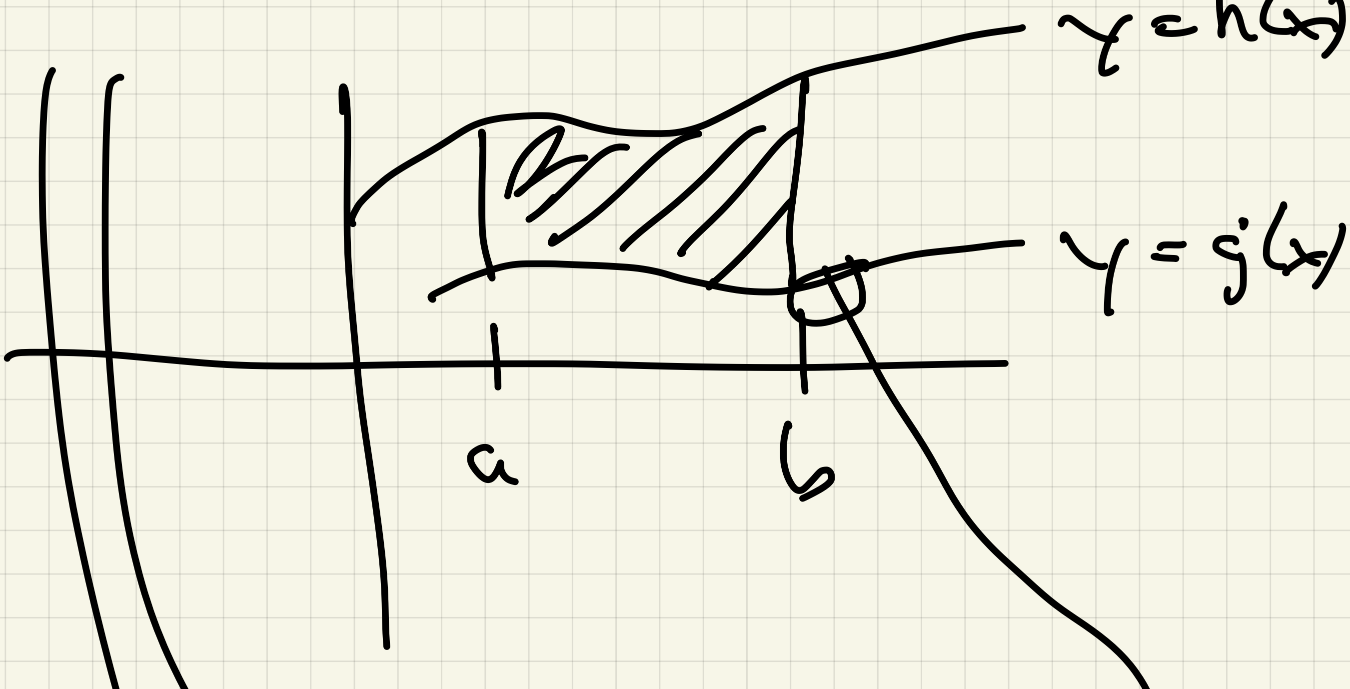
$$\frac{1}{2}x^2 + \frac{1}{4}x \Big|_0^2 =$$

$$2 + \frac{1}{2} = 5/2$$

<u>Fubini's theorem</u>:

$$\int_a^b \underbrace{\int_{g(x)}^{h(x)} f(x,y)\,dy}\,dx$$

$$a \le b, \quad g(x) \le h(x)$$

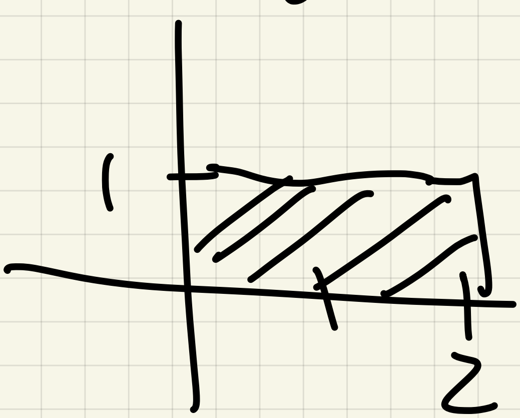


$$\iint_R f(x,y)\,dA$$

R is region

<u>In Ex 4</u>: For both integrals, region is

$0 \le x \le 2$
$0 \le y \le 1$

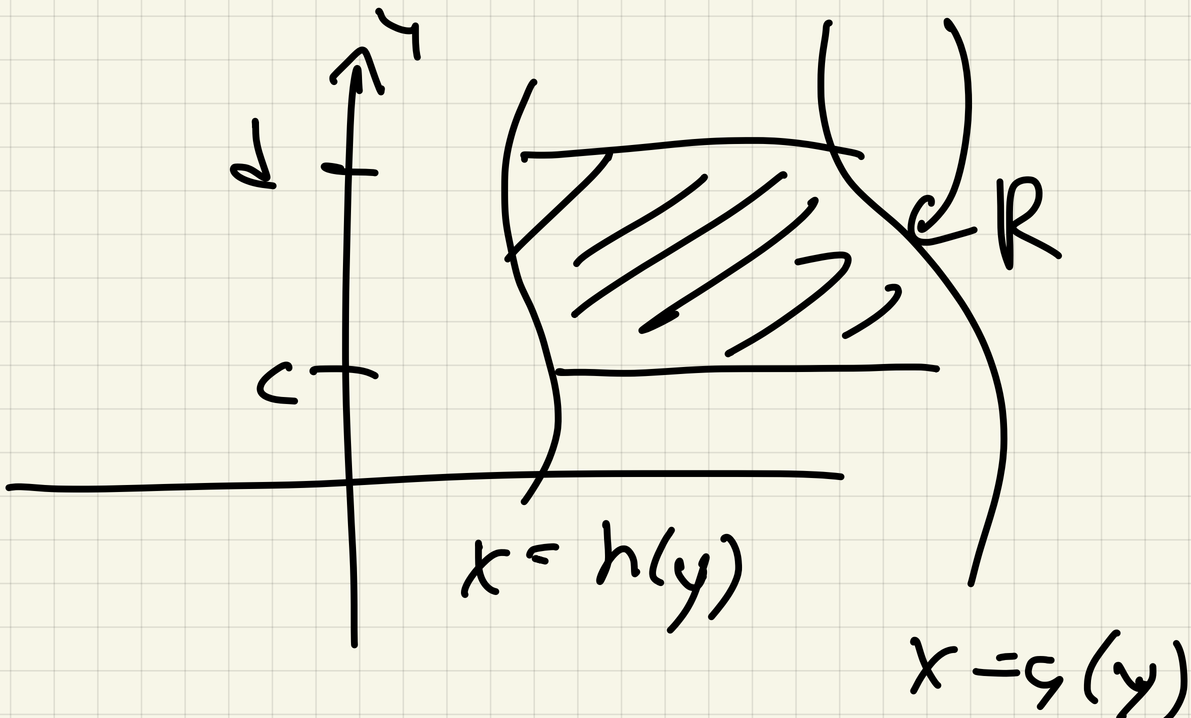


Also: If $c \le d$

& $h(y) \le g(y)$

then

$$\int_c^d \int_{h(y)}^{g(y)} f(x,y)\,dA = \iint_R f(x,y)\,dA$$

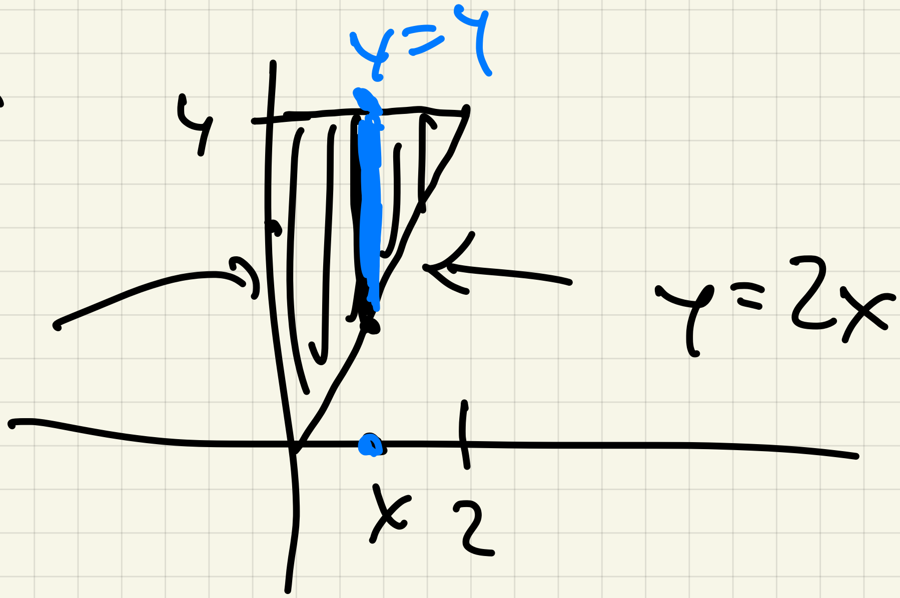


For the Ex 1(c)

$$\iint_R -10\,dA = \int_{-5}^{5} \int_{\sqrt{25-x^2}}^{\sqrt{25-x^2}} -10\,dy\,dx$$

Ex 5

R



$$\iint_R 2xy\,dA = ??$$

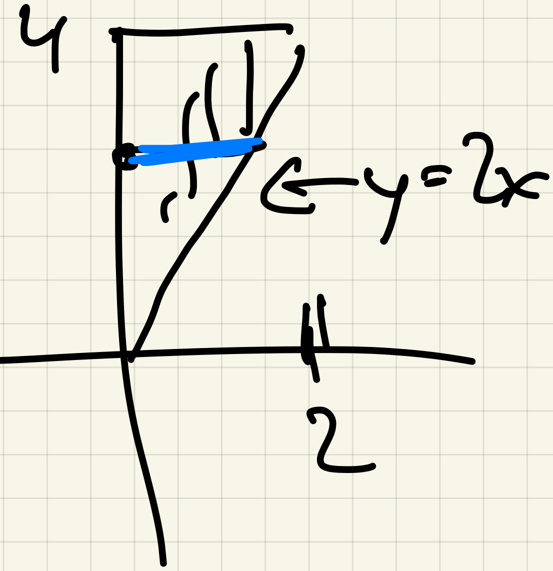
$$\int_{x=0}^{x=2}\int_{y=2x}^{y=4} 2xy\,dy\,dx =$$

$$xy^2\Big|_{y=2x}^{y=4} = 16x-4x^3$$

$$\int_0^2 16x-4x^3\,dx = 8x^2-x^4\Big|_0^2 =$$

$$32-16=16$$

$$\underbrace{\int_{y=0}^{y=4} \int_{x=0}^{x=y/2} 2xy \, dx \, dy}$$

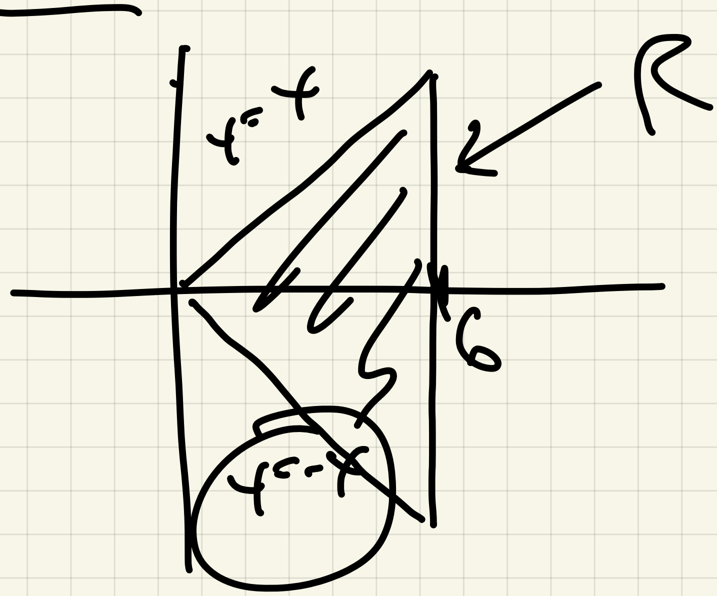


$$x^2 y \Big|_{x=0}^{x=y/2} = \frac{y^3}{4}$$

$$\int_0^4 \frac{y^3}{4} = \frac{y^4}{16}\Big|_0^4 = \frac{4^4}{16} = \frac{256}{16} =$$

16 ✓

Ex 6



$$\iint_R 1 \, dA$$

Rmk: $\int_a^b 1\,dx = x\Big|_a^b = b-a$

$= \text{width of interval}$

$\iint_R 1\,dA = \text{Area of } R$

dxdy

dy dx

dy dx:

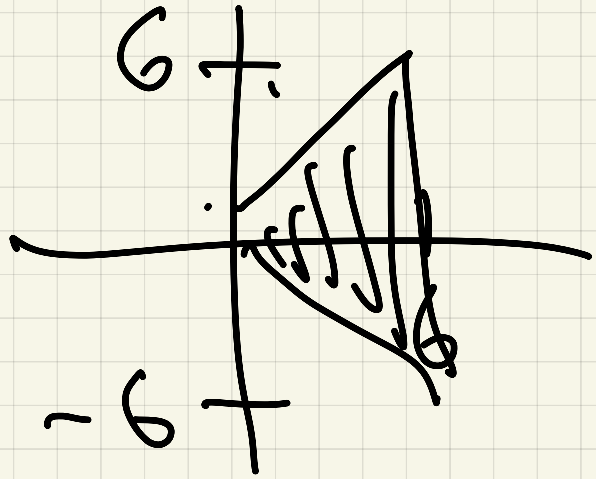
$$\int_0^6 \int_{-x}^{x} 1\,dy\,dx =$$

$$y\Big|_{-x}^{x} = x-(-x) = 2x$$

$$\int_0^6 2x\,dx = x^2\Big|_0^6 = 36 \checkmark$$

dxdy

$$\int_{-6}^{6}\int_{??}^{6} dx\,dy$$



problem

split up integral

$$\| $$

<u>top</u> <u>bottom</u>

$$\underbrace{\int_0^6\int_y^6 1\,dx\,dy}_{\text{LHS}} + \underbrace{\int_{-6}^0\int_{-y}^6 1\,dx\,dy}_{\text{RHS}}$$

<u>LHS</u> $x\Big|_y^6$

$$\int_0^6 6-y\,dy = 6y-\frac{y^2}{2}\Big|_0^6 =$$

$$36-18=18$$

<u>RHS</u> $\displaystyle \int_{-6}^0 \underbrace{\int_{-y}^6 1\,dx\,dy} =$

$$\int_{-6}^{0}$$

$$\underbrace{x \Big|_{-y}^{6}}_{\text{inside}} = 6-(-y) = 6+y$$

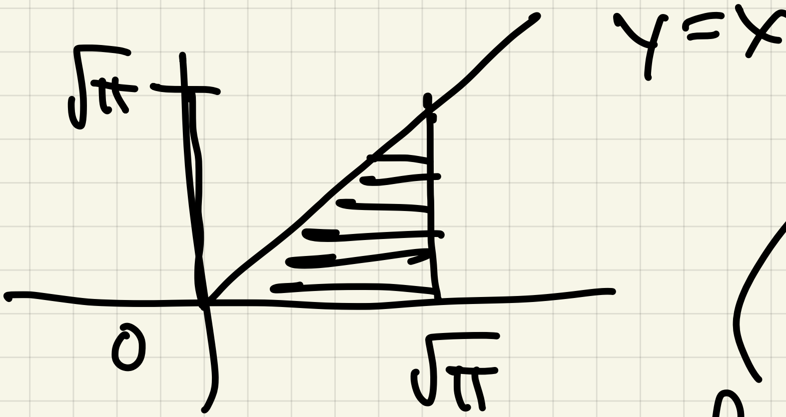
$$\int_{-6}^{0} 6+y\,dy = 6y + \frac{y^2}{2}\Big|_{-6}^{0} =$$

$$0-(-36+18) = 18 \checkmark$$

<u>Ex 7</u> $$\int_{0}^{\sqrt{\pi}}\int_{y}^{\sqrt{\pi}} \sin(x^2)\,dx\,dy = ??$$

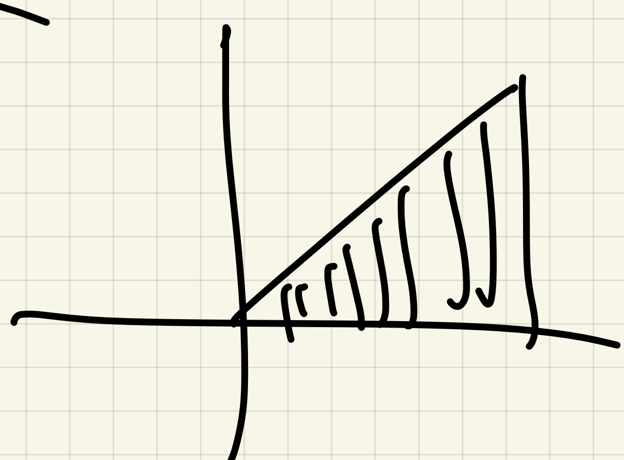
<u>HARD</u>

<u>describe</u> <u>region</u> : R



$$R \begin{cases} 0 \le y \le \sqrt{\pi} \\ y \le x \le \sqrt{\pi} \end{cases}$$

$$0 \le x \le \sqrt{\pi}$$
$$0 \le y \le x$$



$$\underbrace{\int_0^{\sqrt{\pi}} \underbrace{\int_0^x \sin(x^2)\, dy}\, dx}$$

$$y \sin(x^2) \Big|_0^x = x \sin(x^2) - 0$$

$$\int_0^{\sqrt{\pi}} x \sin(x^2)\, dx \qquad \begin{aligned} u &= x^2 \\ du &= 2x\, dx \\ \tfrac{1}{2} du &= x\, dx \end{aligned}$$

$$\|$$

$$\int_{u=0}^{u=\pi} \frac{1}{2}\sin u\,du =$$

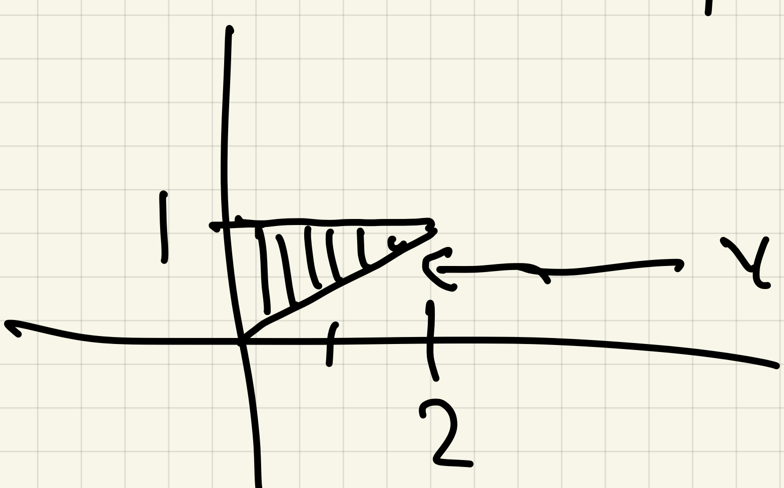
$$-\frac{1}{2}\cos u \Big|_0^{\pi} =$$

$$-\frac{1}{2}(-1) - \left(-\frac{1}{2}\right)(1) = 1.$$

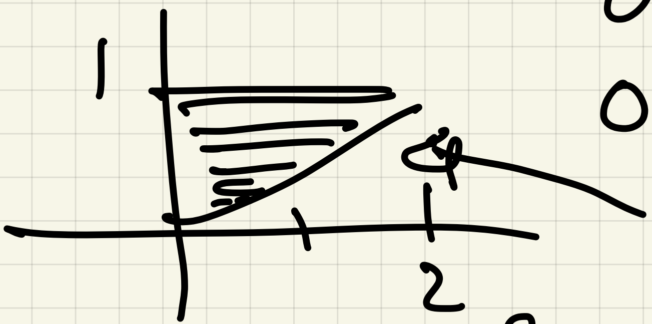
<u>Ex 8</u> $$\int_0^2 \underbrace{\int_{x/2}^1 x\sqrt{1+y^3}\,dy}_{\text{hard}}\,dx$$

<u>region</u>: $0 \le x \le 2$

$x/2 \le y \le 1$



switch order



$0 \le y \le 1$

$0 \le x \le 2y$

$x = 2y$

$$\int_0^1 \int_0^{2y} y\sqrt{1+y^3}\,dx\,dy$$

$$\sqrt{1+y^3} \cdot \frac{x^2}{2}\Big|_0^{2y} =$$

$$\sqrt{1+y^3}\left(\frac{(2y)^2}{2}\right) - 0$$

$$= 2\sqrt{1+y^3}\,y^2$$

$$\int_0^1 2\sqrt{1+y^3}\,y^2\,dy$$

$u = 1+y^3$

$du = 3y^2\,dy$

$$\frac{1}{3}\,du = y^2\,dy$$

$$\int \frac{1}{3} 2\sqrt{u}\,du = \int \frac{2}{3}\sqrt{u}\,du =$$

$$\int \frac{2}{3} u^{\frac{1}{2}}\,du = \frac{2}{3}\,\frac{2}{3}\,u^{3/2} =$$

$$\frac{4}{9}u^{3/2} =$$

$$\frac{4}{9}(1+y^3)^{3/2}\Big|_0^1 =$$

$$\frac{4}{9}2^{3/2} - \frac{4}{9}1 =$$

$$\frac{4}{9}(2\sqrt{2}-1)$$