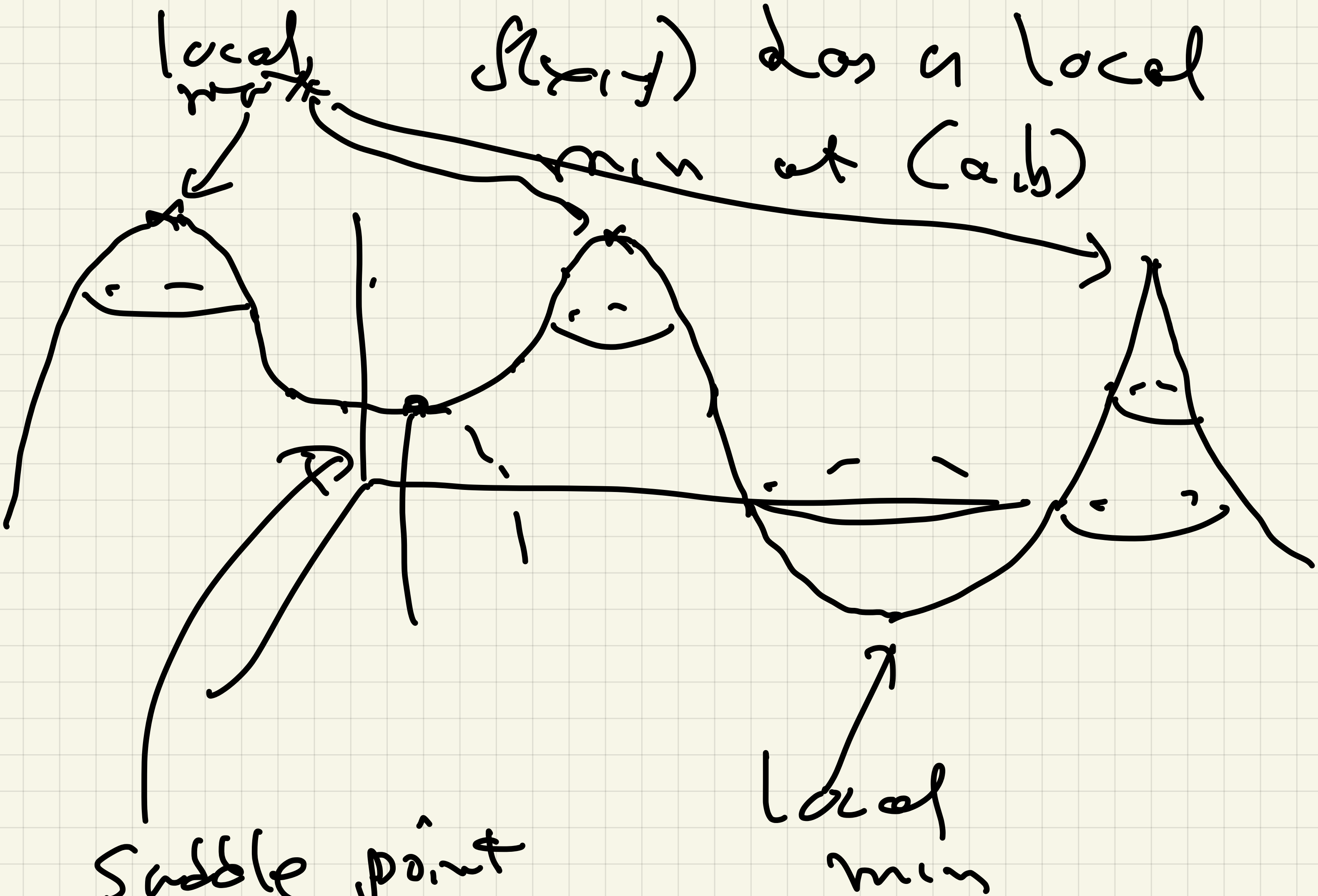
10/15 | Calc3

Last time

$f(x,y)$ has a local maximum at (a,b)

$f(x,y)$ has a local min at (a,b)



Thm If $f(x,y)$ has a local max/min at (a,b), then

$$\nabla f(a,b) = 0 \quad \text{or} \quad \nabla f(a,b) \text{ DNE}$$

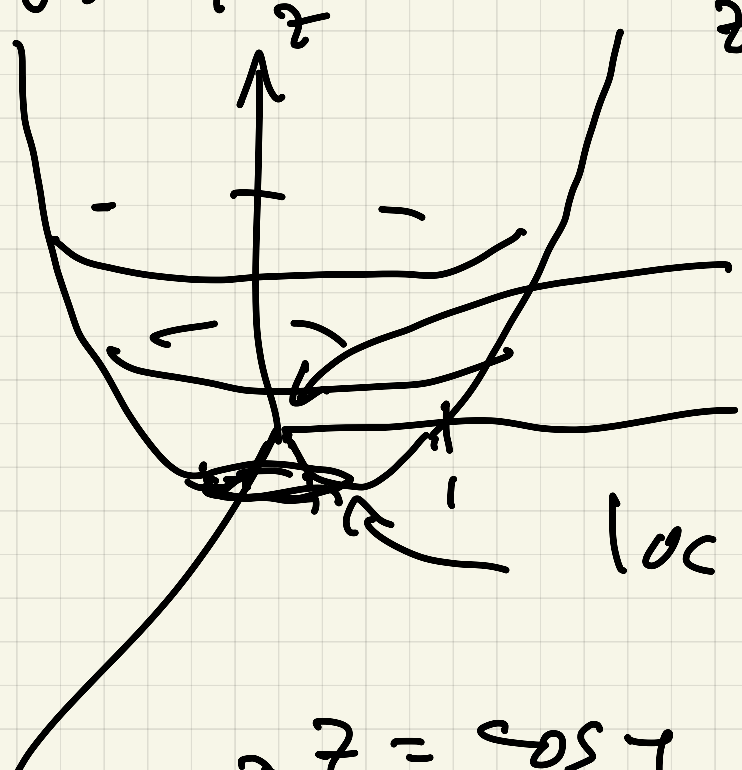
Ex 2 $\quad z = (x^2+y^2) - \sqrt{x^2+y^2}$

$$= \quad z = r^2 - r$$

$r = \sqrt{x^2+y^2}$

$z = r^2 - r$

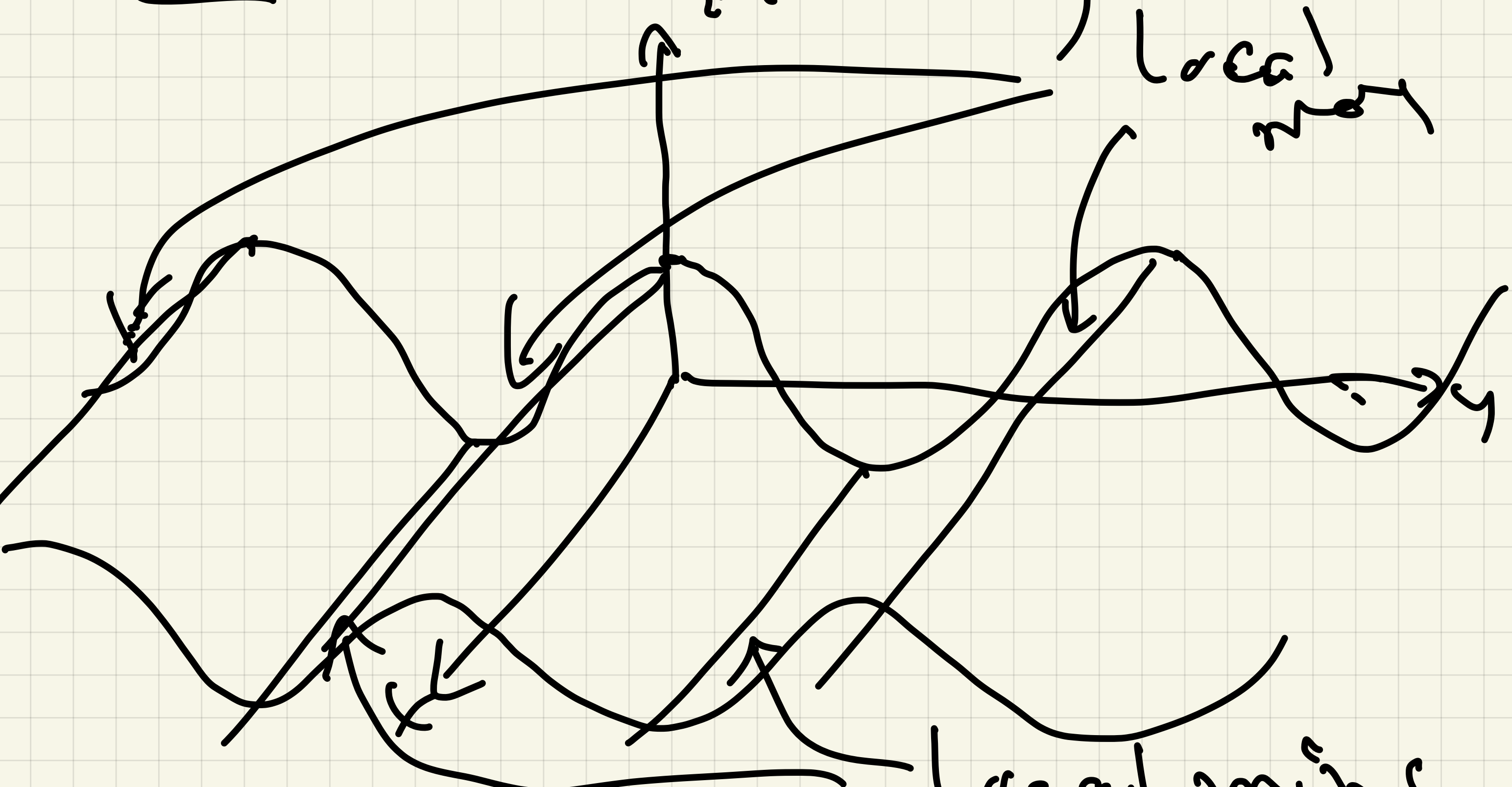
$y^2 - y$



local max

local min.

Ex 3 $\quad z = \cos y$



local max

local mins

$\nabla z = \langle 0, -\sin y \rangle$

Ex4 $z = f(x,y) = x^2 + y^3 - 3y$

find critical points, and sketch

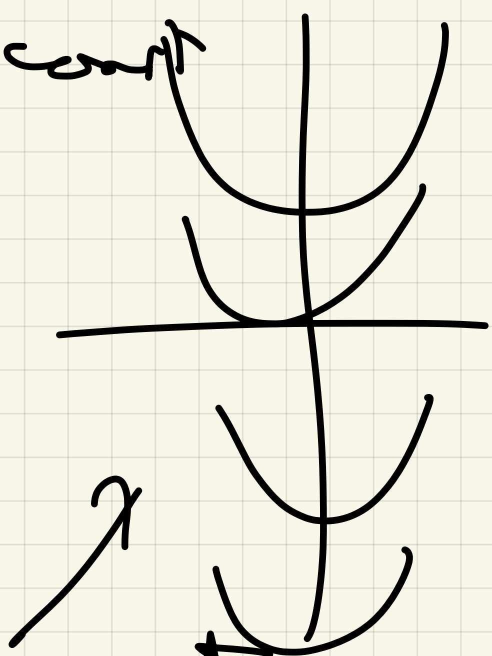
$$\nabla f = \langle 2x, 3y^2 - 3 \rangle = \langle 0, 0 \rangle$$

$$\Downarrow$$

$$x = 0, \quad y = \pm 1$$

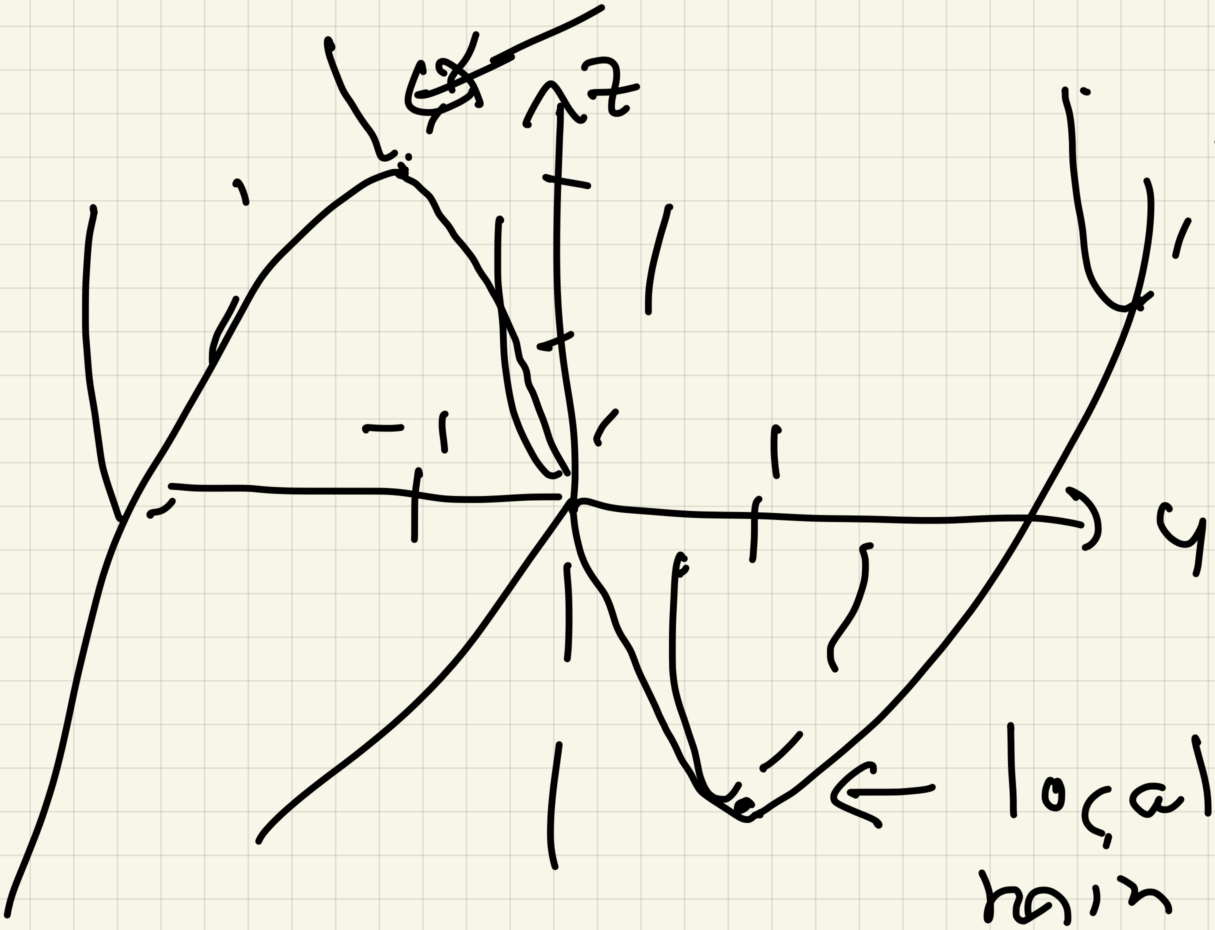
y const: $z = x^2 + \text{const}$

x = 0: $z = y^3 - 3y$



incorporate

neither max/min



<u>IncCalc</u>: 2^{nd} derivative test

If $f'(a)=0$ (crit pt)

$f''(a)>0$ ⟹ local min

$f''(a)<0$ local max

$f''(a)=0$ fails

- <u>Calc3</u>:

<u>Second partials test</u>:

If $z=f(x,y)$ has f_{xx}, f_{xy},

f_{yy} on an open disk around (a,b) and $\nabla f(a,b) = \langle 0, 0 \rangle$

Set $d = f_{xx}(a,b) f_{yy}(a,b) - f_{xy}(a,b)^2$

$$= \det \begin{pmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{xy}(a,b) & f_{yy}(a,b) \end{pmatrix}$$

① $d > 0$, $f_{xx}(a,b) > 0 \rightarrow (a,b)$ is a local min.

② $\boxed{d > 0,}$ $\boxed{f_{xx}(a,b) < 0} \rightarrow (a,b)$ is local max

③ $d < 0$, f has a saddle point at (a,b)

i.e. (a,b) is neither local max nor min

④ $d = 0$, test fails

In Ex 4 $f = x^2 + y^3 - 3y$

$(0, \pm 1)$ crit pts

$f_x = 2x \qquad f_y = 3y^2 - 3$

$f_{xx} = 2, \; f_{yy} = 6y$

$f_{xy} = 0$

$$d = \det \begin{pmatrix} 2 & 0 \\ 0 & 6y \end{pmatrix}$$

$\underline{x = 0, y = 1}$ $\quad \left.\begin{array}{l} \det \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix} = 12 > 0 \\ f_{xx} = 2 > 0 \end{array}\right)$

1

$\Rightarrow (0,1)$ local min.

$x = 0, y = -1 \qquad \det \begin{pmatrix} 2 & 0 \\ 0 & -6 \end{pmatrix} = -12 < 0$

saddle point.

Ex 5 $\quad z = f(x,y) =$

$$\boxed{2xy - \frac{1}{2}(x^4 + y^4) + 1}$$

find crit points and apply 2^{nd} partial test.

$$\nabla f = \langle \underline{2y - 2x^3}, \underline{\underline{2x - 2y^3}} \rangle$$

$$\nabla f = \langle 0, 0 \rangle$$

$$y - x^3 = 0, \quad x - y^3 = 0$$

$$\boxed{y = x^3, \quad x = y^3}$$

$$x = y^3 = (x^3)^3$$

$$x = (x^3)^3 = x^9$$

$$x - x^9 = 0$$

$$\underline{\underline{x}}(1 - x^8) = 0 \Rightarrow x = 0,$$

$$\text{or } x^8 = 1$$

$$x = \pm 1$$

$$\boxed{\begin{matrix} (0,0) \\ (1,1) \\ (-1,-1) \end{matrix}}$$

crit pts

2nd partials

$$f_{xx} = -6x^2, \quad f_{yy} = -6y^2, \quad f_{xy} = 2$$

$$d = \det\begin{pmatrix} -\underline{6x^2} & 2 \\ 2 & -\underline{6y^2} \end{pmatrix}$$

$(0,0) \Rightarrow \quad \det\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} = -4 < 0$

saddle point,

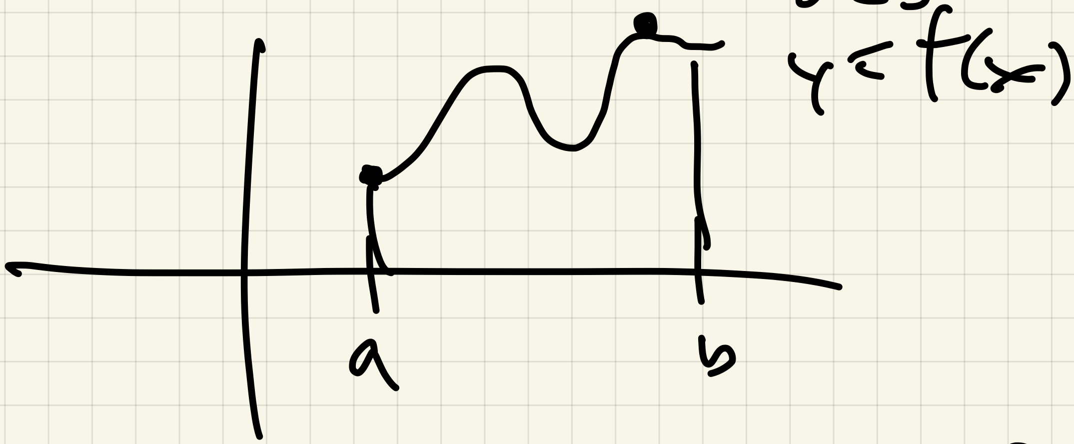
$(1,1)$: $\det\begin{pmatrix} -6 & 2 \\ 2 & -6 \end{pmatrix} = 36 - 4 = 32 > 0$

$f_{xx}(1,1) = -6 < 0$ (2)

local max,

$(-1,-1)$ also local max

Calc 1 Thm If $y = f(x)$ is continuous on $[a,b]$, then f achieves it's max/min values on $[a,b]$



<u>Thm</u> If $f(x,y)$ is continuous on a <u>closed</u> <u>bounded</u> region R, then f achieves its max/min values on R

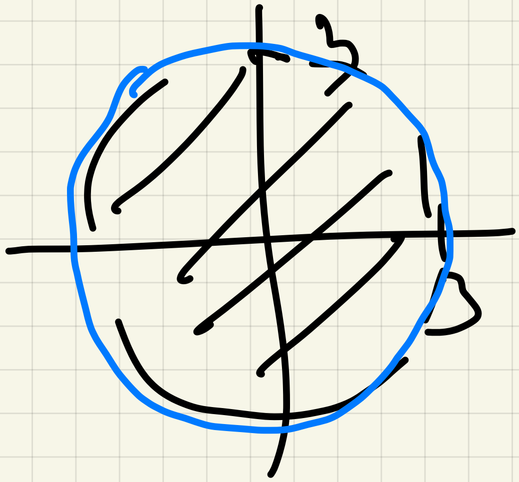
Further: the extreme values occur
① at critical points inside R
or
② on <u>boundary</u> of R

<u>R bounded</u> means

R is contained inside $[a,b] \times [c,d]$

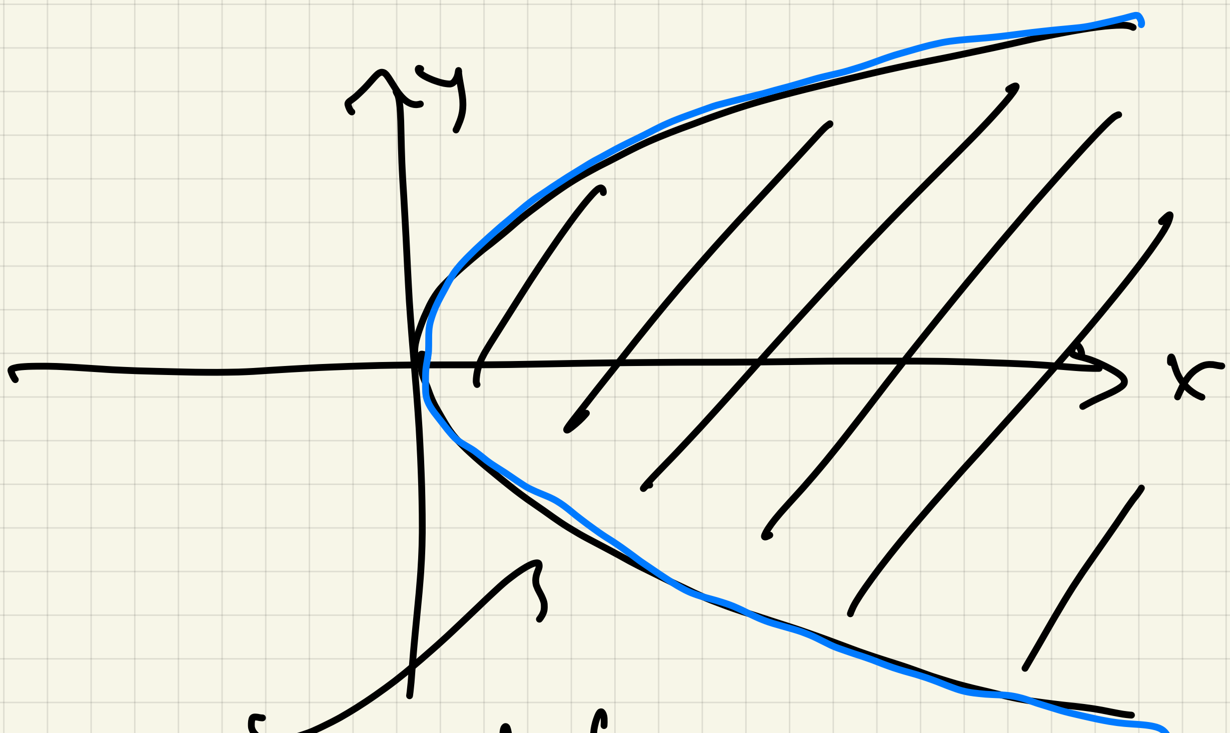
Ex 6 $R = \{(x,y) : x^2+y^2 < 9\}$

(a)



bounded

(b) $R = \{(x,y) : x \geq y^2\}$



unbounded

The boundary of R separates

R from the points not in R

In (a) boundary $x^2+y^2=9$

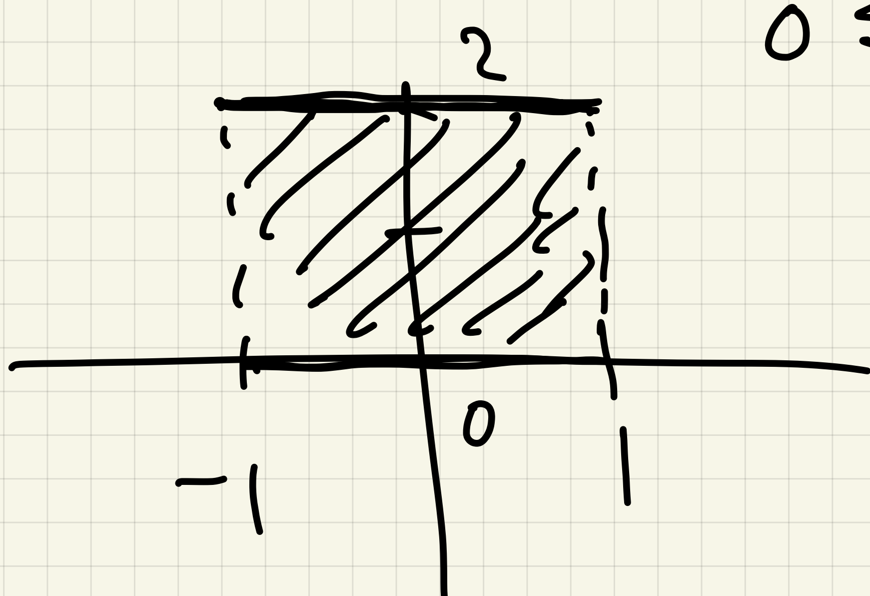
(b) parabola $x=y^2$

R is <u>closed</u> if R <u>contains</u> its <u>boundary</u>

In 6(a) <u>not</u> closed

6(b) is closed

(c) $R = \{(x,y): -1 < x < 1, \ 0 \le y \le 2\}$

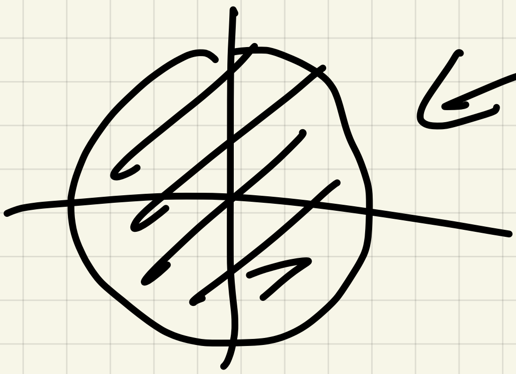


bounded
not
closed

Ex7 Find max/min values

of $g(x,y) = \underline{2x+y+1}$

on $R = \{(x,y) : x^2+y^2 \leq 4\}$



closed ✓, bounded ✓

$\underline{\text{Thm}}$ g achieves max/min values

① crit pts inside

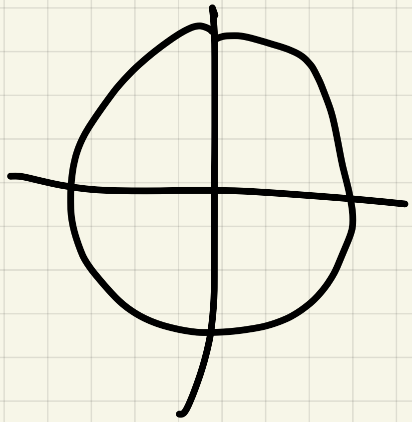
② pt on boundary

$\nabla g = \langle 2, 1 \rangle \neq \langle 0, 0 \rangle$

$\underline{\text{No crit pts}}$!

② $\underline{\text{Check boundary}}$:

$$\boxed{\underline{\text{boundary}}: \quad x^2+y^2=4 \quad \text{circle?}}$$



$$\boxed{\vec{r}(t) = \langle 2\cos t, 2\sin t \rangle \quad 0 \le t \le 2\pi}$$

parametric description

$$g(r(t)) = 2(2\cos t) + 2\sin t + 1$$

$$= 4\cos t + 2\sin t + 1$$

to find max/min

$$\frac{d}{dt}(g(t)) = \frac{d}{dt}(4\cos t + 2\sin t + 1)$$

$$= -4\sin t + 2\cos t = 0$$

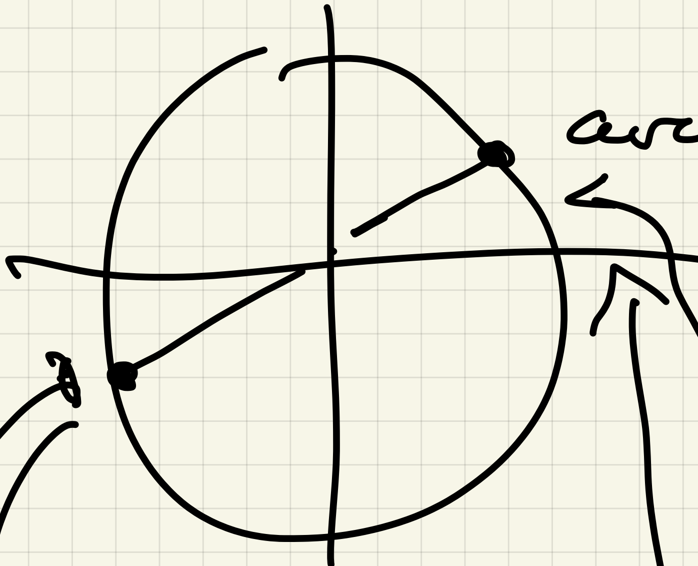
$$\Downarrow$$

$$4\sin t = 2\cos t$$

$$\tan t = \frac{\sin t}{\cos t} = \frac{2}{4} = \frac{1}{2}$$

at $t = \arctan \frac{1}{2}$

$\arctan \frac{1}{2} + \pi$



$\arctan\frac{1}{2}$

$\frac{y}{x}=2$

$2y=x$

$x^2+y^2=4$

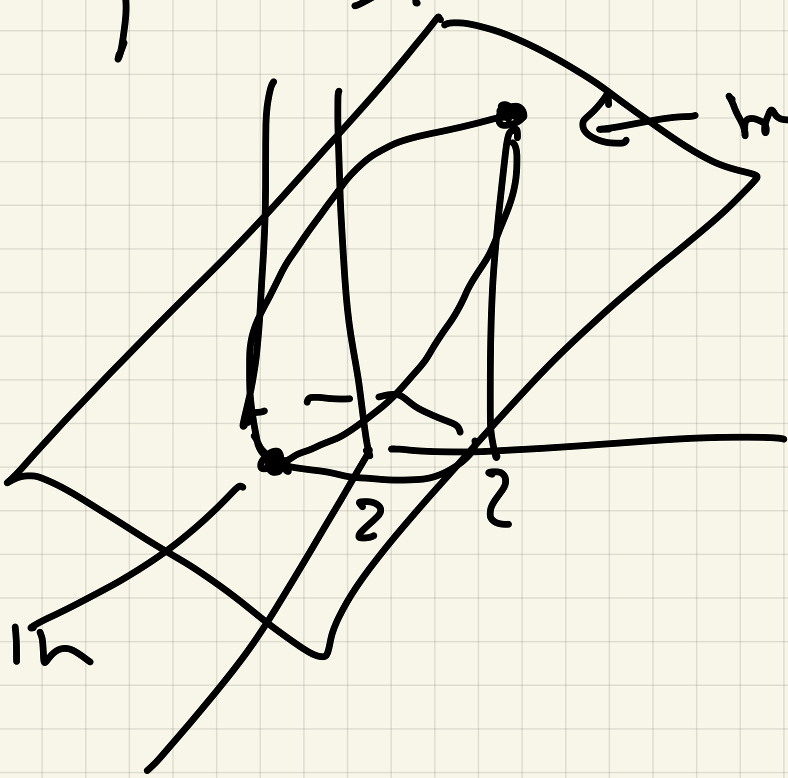
$\left(-\frac{4}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\right)$

$\left(\frac{4}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$

$t=\arctan\frac{1}{2}+\pi$

$g=-3.41$

$g=5.47$



max

min