6/11 Calc3

Last time Partial derivative

Can compute higher partials

Ex) $z = g(x,y) = x^3 + 3x^6y^2 - y^{10}$

$g_x = 3x^2 + 18x^5y^2$

$g_y = 6x^6y - 10y^9$

Second partial derivatives,

$g_{xx} = (g_x)_x = 6x + 90x^4y^2$

$g_{xy} = (g_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial g}{\partial x}\right) = 36x^5y$

$g_{yy} = (g_y)_y = 6x^6 - 90y^8$

$g_{yx} = (g_y)_x = 36x^5y$ ← same

Theorem: If f_{xy} and f_{yx} ($z = f(x,y)$) are continuous on an open disk around $(x,y) = (a,b)$,

Then $f_{xy}(a,b) = f_{yx}(a,b)$

[and similarly with other mixed partial higher derivatives]

Ex2 $w = h(x,y,z) =$

$$3x^2y - 5y\cos z + ze^{xy} + \ln xy$$

$$h_x = 6xy + yze^{xy} + \frac{1}{x}$$

$$h_y = 3x^2 - 5\cos z + xze^{xy} + \frac{1}{y}$$

$\to h_z = +5y\sin z + e^{xy}$

~~Un~~ Find 2^{nd} partials:

$$h_{xx} = 6y + y^2 z e^{xy} + \frac{-1}{x^2}$$

$$h_{xy} = h_{yx} = 6x + ze^{xy} + xyze^{xy}$$

$$h_{xz} = h_{zx} = ye^{xy}$$

$$h_{yy} = x^2 z e^{xy} - \frac{1}{y^2}$$

$$h_{zy} = h_{yz} = 5\sin z + xe^{xy}$$

$$h_{zz} = 5y\cos z$$

<u>Ex 3</u> $g(x,y,z) = \sin(xy^2z)$

<u>Compare</u> $\boxed{g_{zzx}}$ and $\boxed{g_{zxz}}$

$$g_z = +\underbrace{\cos(xy^2z)}\cdot\underbrace{xy^2}$$

$$g_{zx} = \underbrace{-y^2 z}\,\underbrace{\sin(xy^2 z)}\cdot\underbrace{xy^2} + y^2\cos(xy^2 z)$$

$$g_{zxz} = -y^2\sin(xy^2 z)\,xy^2 - y^2 z\cos(xy^2 z)(xy^2)\cdot(xy^2) - y^2\sin(xy^2 z)\cdot(xy^2)$$

$$g_{zz} = -\sin(xy^2 z)\cdot(xy^2)(xy^2)$$

$$\underbrace{\boxed{-x^2y^4}\sin(xy^2 z)}$$

$$\boxed{g_{zzx}} = -2xy^4\sin(xy^2 z) - x^2y^4\cos(xy^2 z)\cdot(y^2 z)$$

§13.4 Chain Rule:

If $z = f(x,y)$ and $x = g(t)$, $y = h(t)$ are functions of t,

then $z = f(x,y) = f(g(t), h(t))$

and $$\frac{dz}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

<u>Ex!</u> $z = \underbrace{x^2 + xy + y^3}_{} = f(x,y)$

$\underline{x = t^2}$, $\underline{y = t^3}$

(a) Find $\frac{dz}{dt}$ with chain rule

(b) Check by substitution.

(a) <u>Chain Rule</u>:

$$\frac{dz}{dt} = \underbrace{\frac{\partial z}{\partial x} \frac{dx}{dt}}_{} + \underbrace{\frac{\partial z}{\partial y} \cdot \frac{dy}{dt}}_{}$$

$$= (2x+y)(2t) + (x+3y^2)(3t^2)$$

$$= (2t^2+t^3)(2t) + (t^2+3t^6)(3t^2)$$

$$= 4t^3 + 2t^4 + 3t^4 + 9t^8$$

$$= 4t^3 + 5t^4 + 9t^8$$

(b) $z(t) = x^2 + xy + y^3$ $\quad x = t^2$, $y = t^3$

$$= t^4 + t^5 + t^9$$

$$\boxed{\frac{dz}{dt}} = 4t^3 + 5t^4 + 9t^8$$

<u>More sophisticated:</u>

$$z = f(x,y), \quad x = g(s,t), \quad y = h(s,t)$$

$$z = f(x,y) = \underline{f(g(s,t), h(s,t))}$$

$$\Downarrow$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$$

<u>Ex 2</u> $z = \arctan\left(\frac{y}{x}\right)$ ←

$\boxed{x = r\cos\theta}$ ←

$y = r\sin\theta$

Find $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \theta}$

with chain rule

$$\frac{\partial z}{\partial r} = \boxed{\frac{\partial z}{\partial x}} \cdot \boxed{\frac{\partial x}{\partial r}} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$= \boxed{\frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right)} \cos\theta +$$

$$\frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x} \cdot \sin\theta =$$

$$\frac{-y}{x^2+y^2}\cos\theta + \frac{x}{x^2+y^2}\,\underline{\sin\theta}$$

$$\underbrace{\frac{-r\sin\theta\ \cos\theta}{r^2\cos^2\theta + r^2\sin^2\theta}}_{= r^2} + \frac{r\cos\theta\ \sin\theta}{r^2}$$

$$= \frac{-r\sin\theta\cos\theta + r\cos\theta\sin\theta}{r^2} = 0$$

$$\frac{\partial z}{\partial\theta} = z_x x_\theta + z_y y_\theta$$

$$= \frac{-y}{x^2+y^2}(-r\sin\theta) + \frac{x}{x^2+y^2}\, r\cos\theta$$

$$+ \frac{\cancel{r^2}\sin^2\theta}{\cancel{r^2}} + \frac{\cancel{r^2}\cos^2\theta}{\cancel{r^2}} = 1.$$

Reason for simple answers:

$$z = \arctan\left(\frac{y}{x}\right) = \arctan\left(\frac{\cancel{r}\sin\theta}{\cancel{r}\cos\theta}\right)$$

$y = r\sin\theta$

$x = r\cos\theta$

$$= \arctan(\tan\theta)$$

$$= \theta$$

$$\boxed{z = \theta}$$

If $\quad w = k(x, y, z) \quad$

$x = f(s, t)$
$y = g(s, t)$
$z = h(s, t)$

$$w = k\left(f(s,t),\, g(s,t),\, h(s,t)\right)$$

$\cancel{\partial w}/\partial$

$$\partial w/\partial s = \frac{\partial k}{\partial x} \cdot \partial x/\partial s + \partial k/\partial y \cdot \partial y/\partial s + \partial k/\partial z \cdot \partial z/\partial s$$

$$\partial w/\partial t = \partial k/\partial x \cdot \partial x/\partial t + \frac{\partial k}{\partial y} \cdot \partial y/\partial t + \partial k/\partial z \cdot \frac{\partial z}{\partial t}$$