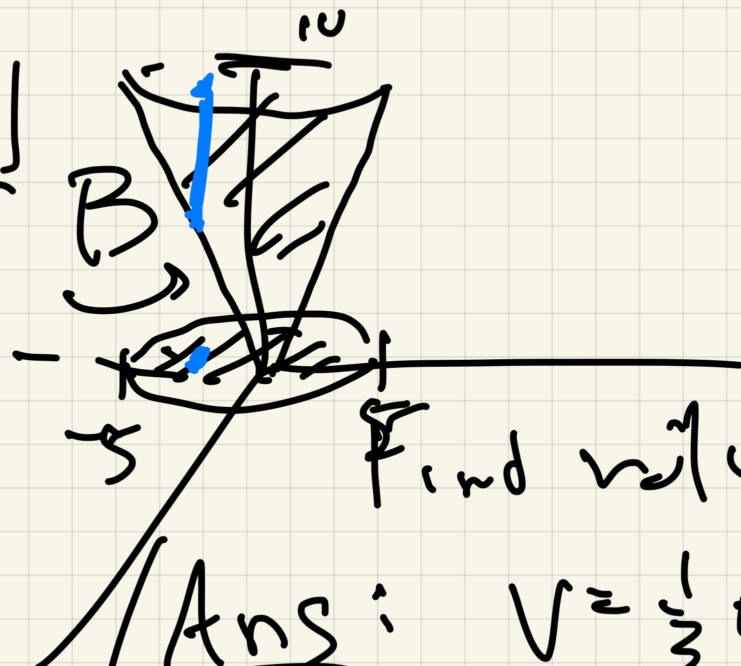
4/5 | Calc3

$$\iiint_B f(x,y,z)\,dV$$

Endpoints / region of integration B

Ex |



$z = 2\sqrt{x^2+y^2}$

Find volume of B

Ans: $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi 5^2 \cdot 10 = \frac{250\pi}{3}$ ✓

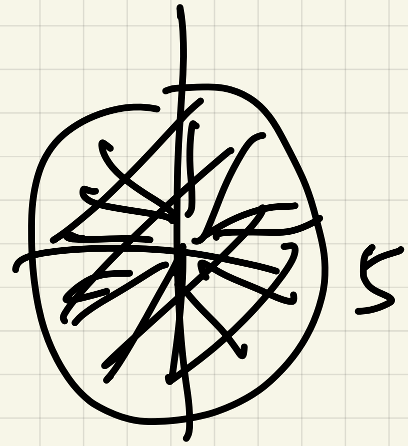
$$V = \int_{-5}^{5}\int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}}\int_{2\sqrt{x^2+y^2}}^{10} dz\,dy\,dx$$

$$\int_{-5}^{5}\int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}} 10 - 2\sqrt{x^2+y^2}\,dy\,dx$$

// Polars

$$\int_0^{2\pi}\int_0^5 (10-2r)\, r \, dr\, d\theta$$

coor factor

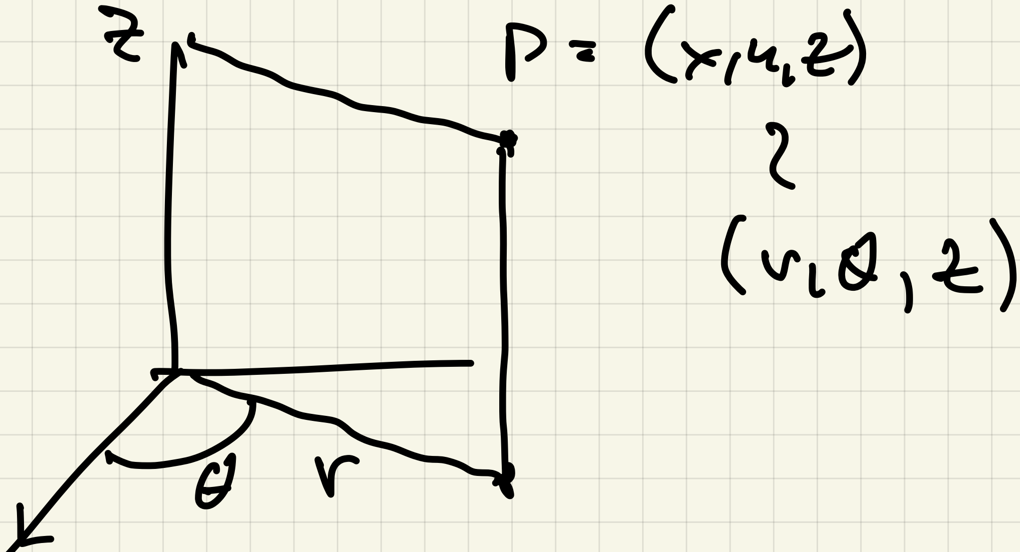


$$5r^2 - \frac{2}{3}r^3 \Big|_0^5 =$$

$$125 - \frac{2}{3}125 = \left(1-\frac{2}{3}\right)125 = \frac{125}{3}$$

$$\int_0^{2\pi} \frac{125}{3}\, d\theta = \frac{125}{3}\cdot 2\pi$$

Cylindrical Coordinates

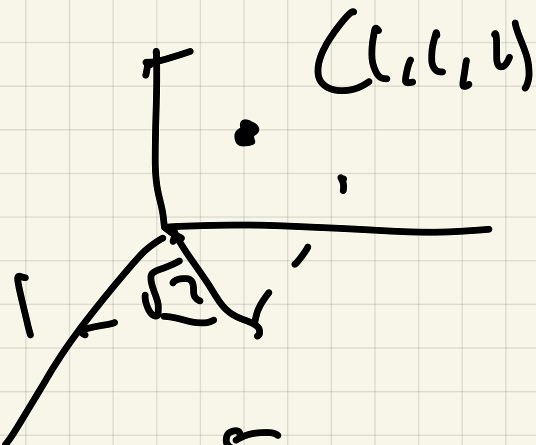


$x = r\cos\theta$ | $r^2 = x^2 + y^2$

$y = r\sin\theta$ | $\tan\theta = \frac{y}{x}$

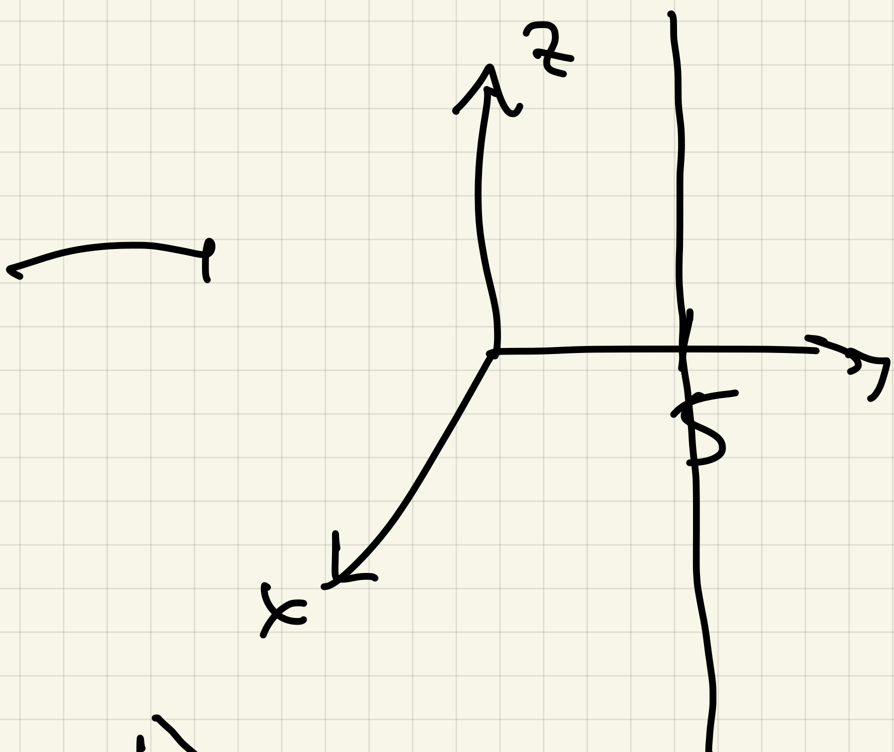
$z = z$

Ex $\overset{x\ y\ z}{(1,1,1)} \approx (\sqrt{2}, \pi/4, 1)$

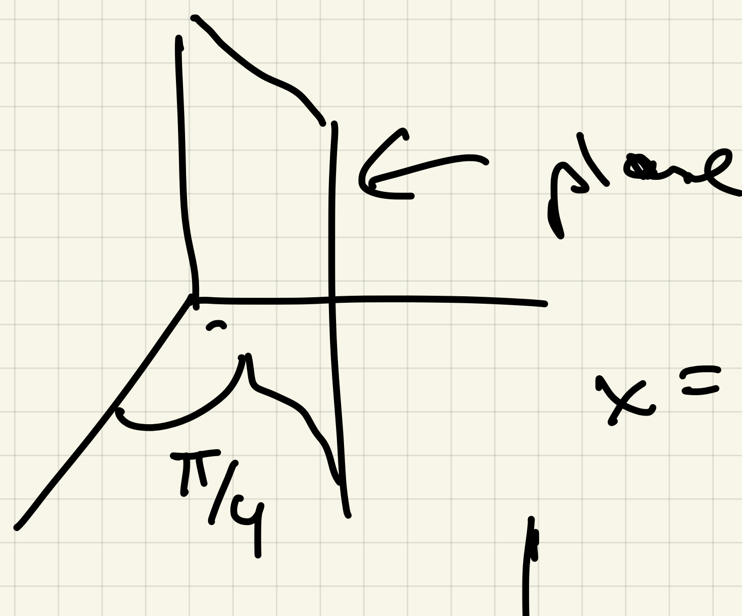


(a) $r = 5$

$\theta = \pi/2$



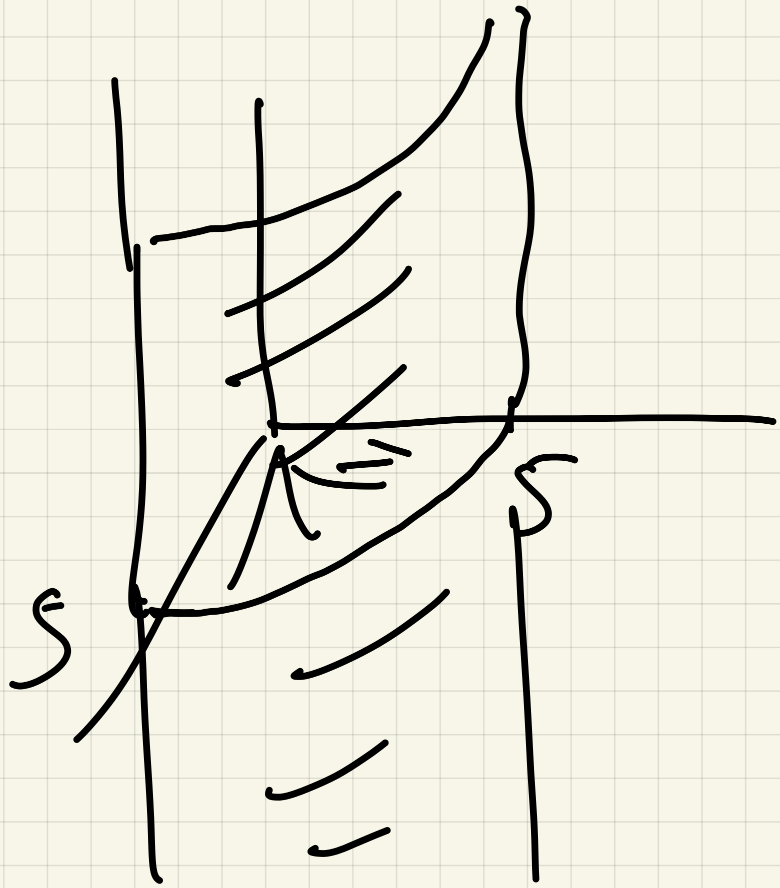
(b) $\theta = \pi/4$



$x = y$

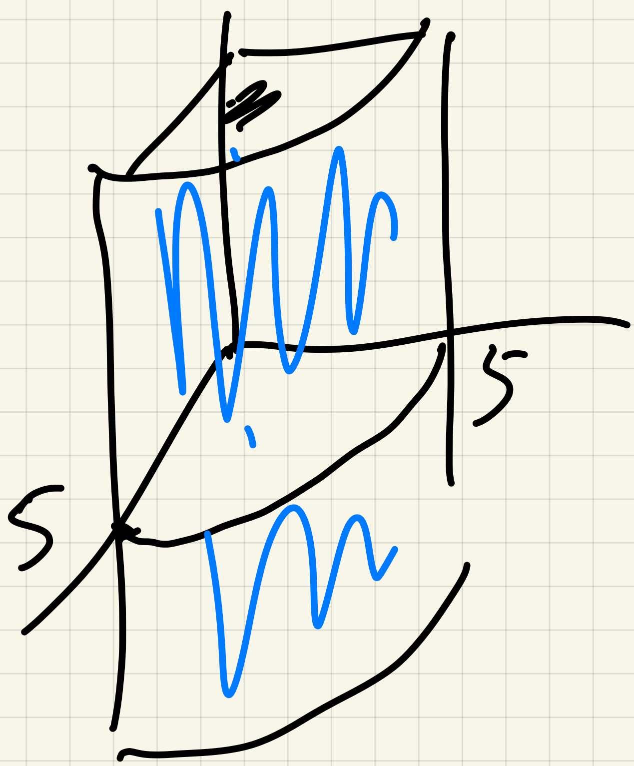
(c) $r = 5$

$0 \le \theta \le \pi/2$

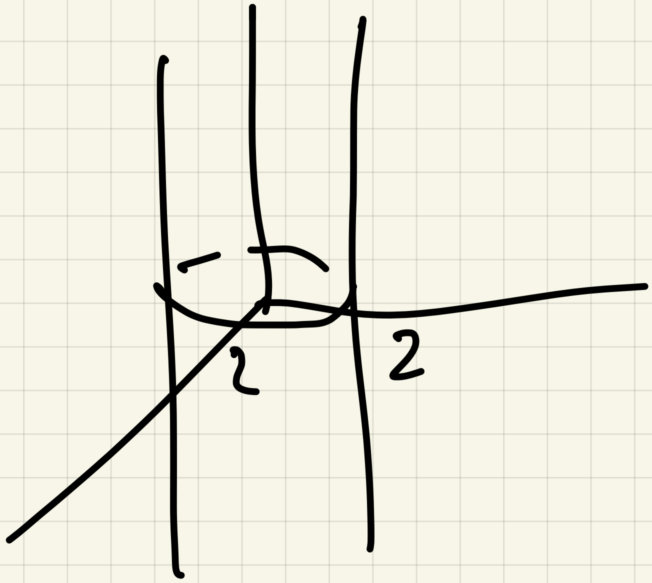


(d) $0 \le r \le 5$

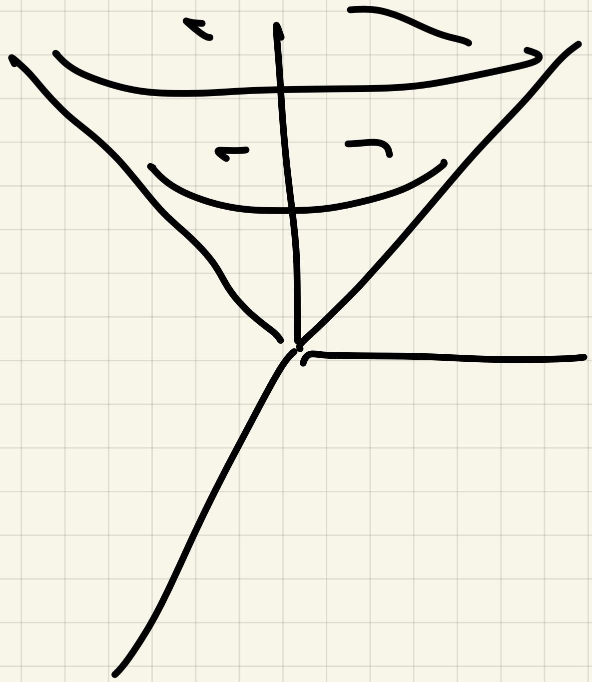
$0 \le \theta \le \pi/2$



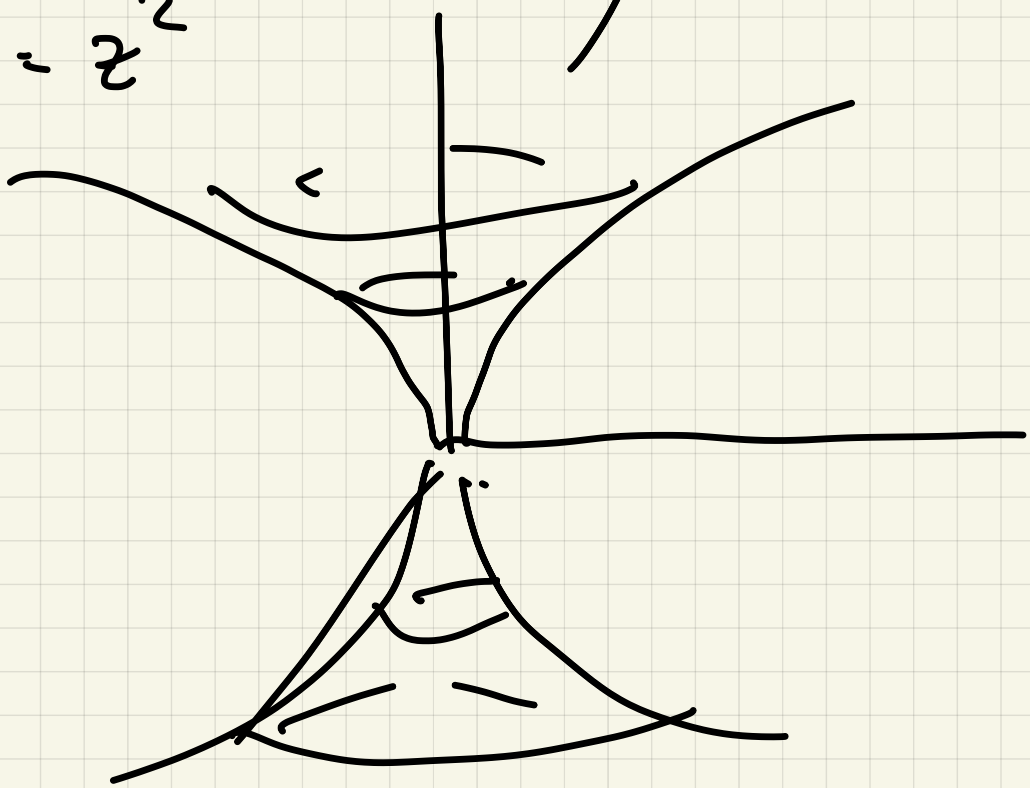
(e) $r = 2 \quad (x^2 + y^2 = 4)$



(f) $r = z$



(g) $r = z^2$

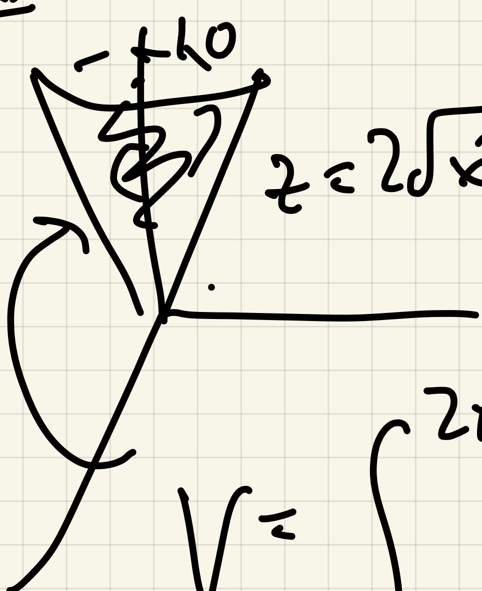


Integrals :

$$\iiint_B f(x,y,z)\,dV = \iiint_B f(r\cos\theta, r\sin\theta, z)\cdot r\,dz\,dr\,d\theta$$

$\uparrow$ conv fact

Ex)



$z = 2\sqrt{x^2+y^2} = 2r$

$z = 2r$

$$V = \int_0^{2\pi}\int_0^{10}\int_0^{z/2} r\,dr\,dz\,d\theta$$

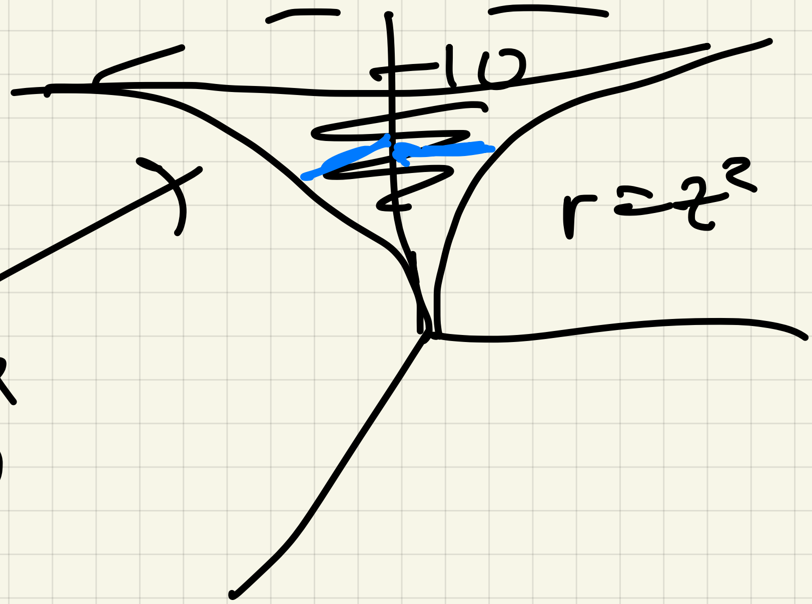
$$\int_0^{2\pi}\int_0^{5}\int_{2r}^{10} r\,dz\,dr\,d\theta$$

$$\frac{1}{2}r^2\Big|_0^{z/2}$$

$$\int_0^{10}\frac{z^2}{8}\,dz = \frac{z^3}{24}\Big|_0^{10} =$$

$$\int_0^{2\pi} \frac{1000}{24}\, d\theta = \frac{2000\pi}{24} = \frac{250\pi}{3} \checkmark$$

Ex2



find volume

$0 \le z \le 10$

$0 \le r \le z^2$

$$V = \int_0^{2\pi} \int_0^{10} \int_0^{z^2} r \, dr \, dz \, d\theta$$

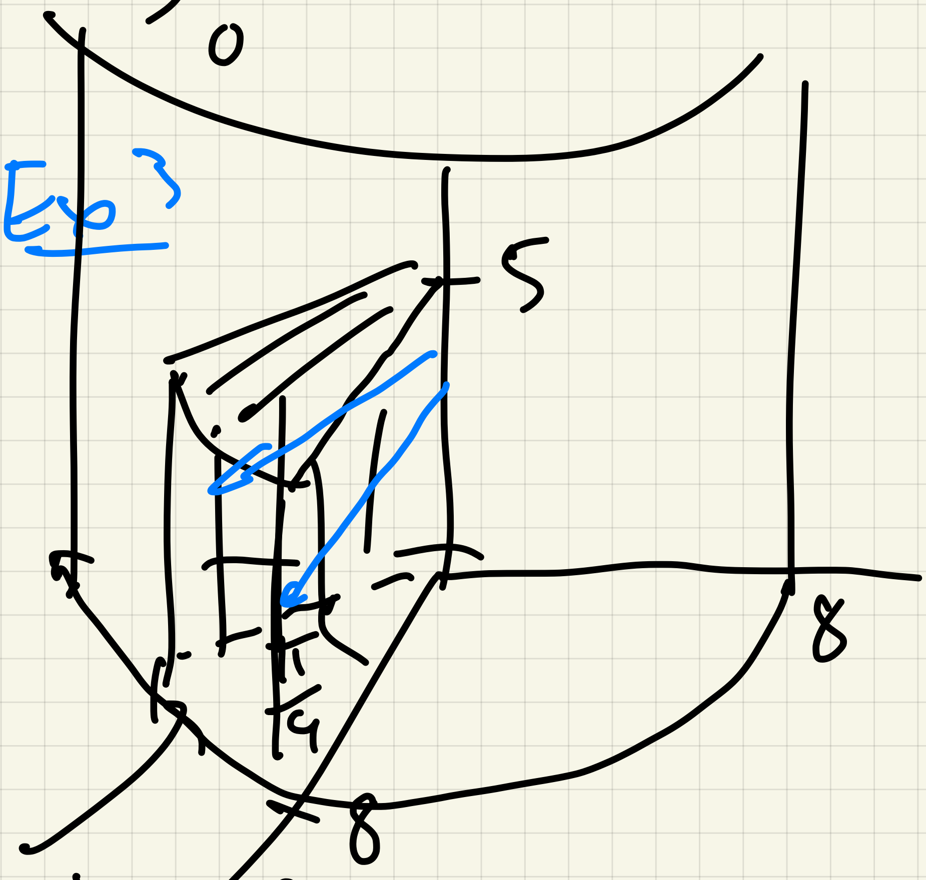
$\theta \quad z \quad r$

$$\frac{1}{2} r^2 \Big|_0^{z^2} = \int_0^{10} \frac{1}{2} z^4 \, dz =$$

$$\frac{1}{10} z^5 \Big|_0^{10} = \frac{10^5}{10} = 10^4$$

$$\int_0^{2\pi} 10000\, d\theta = 20{,}000\pi$$

Ex 3



Find volume

$$V = \int_{-\pi/4}^{0} \int_0^8 \int_0^5 r\, dz\, dr\, d\theta$$

$$\int_0^8 5r = \frac{5}{2} r^2 \Big|_0^8 = \frac{5 \cdot 8^2}{2}$$

$$\int_{-\pi/4}^{0} \frac{5 \cdot 8^2}{2}\, d\theta =$$

$$\frac{5}{2} 8^2 \theta \Big|_{-\pi/4}^{0} =$$

$$160\theta \Big|_{-\pi/4}^{0} =$$

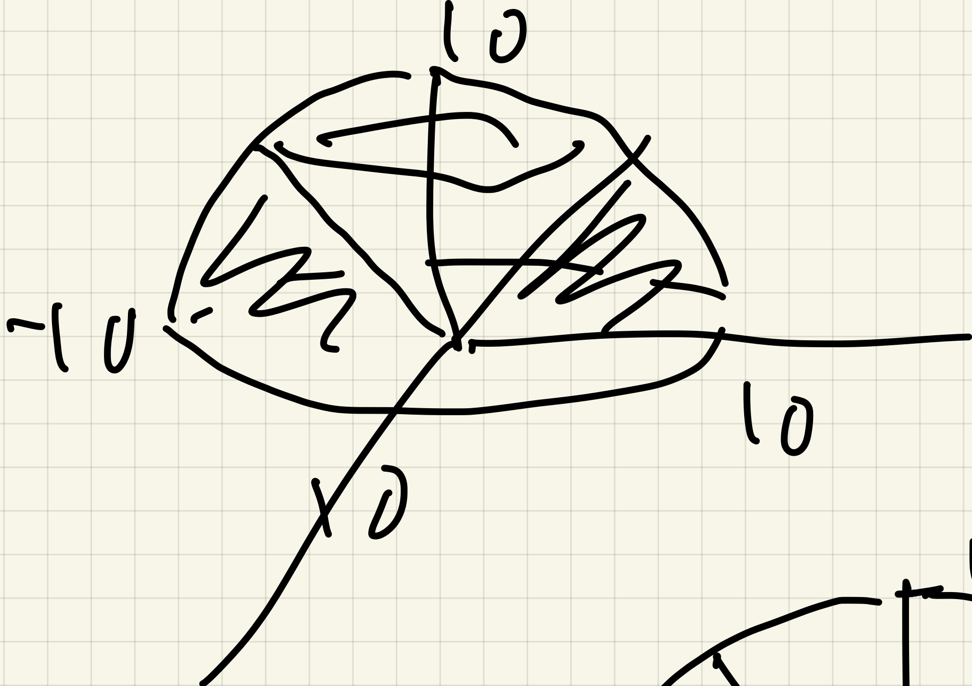
$$0 - (-\pi/4)\, 160 =$$

$$40\pi$$

Ex 4 Find volume of solid inside sphere

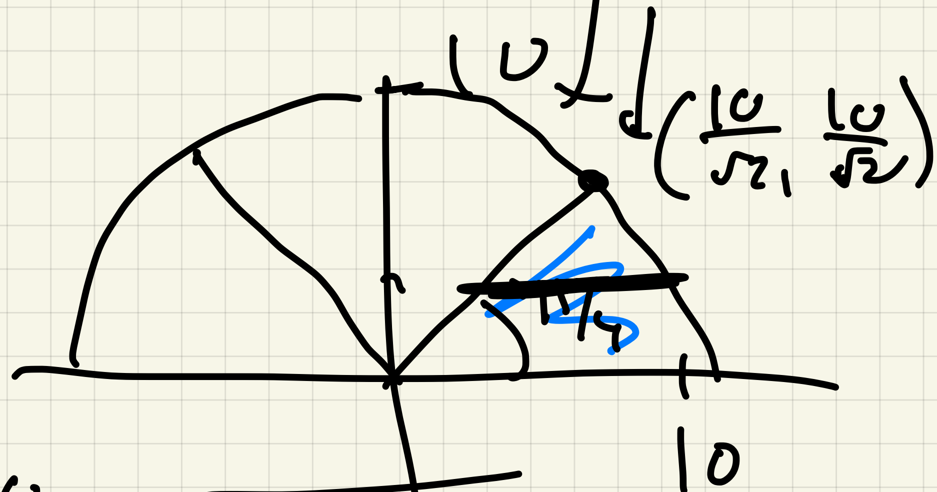
$x^2 + y^2 + z^2 = 100$, $z \geq 0$

outside cone $z = \sqrt{x^2 + y^2}$



$r^2 + z^2 = 100$

$r = \sqrt{100 - z^2}$



$$V = \int_{0}^{2\pi} \int_{0}^{\frac{10}{\sqrt{2}}} \int_{z}^{\sqrt{100 - z^2}} r \, dr \, dz \, d\theta$$

θ z

$$\frac{1}{2} r^2 \Big|_{z}^{\sqrt{100 - z^2}} =$$

$$\frac{1}{2}(100 - z^2) - \frac{1}{2} z^2$$

$$\int_0^{10/\sqrt{2}} 50 - z^2 \, dz =$$

$$10/\sqrt{2} = 5\sqrt{2}$$

$$50z - \frac{z^3}{3} \Big|_0^{5\sqrt{2}}$$

$$250\sqrt{2} - \frac{125 \cdot 2\sqrt{2}}{3}$$

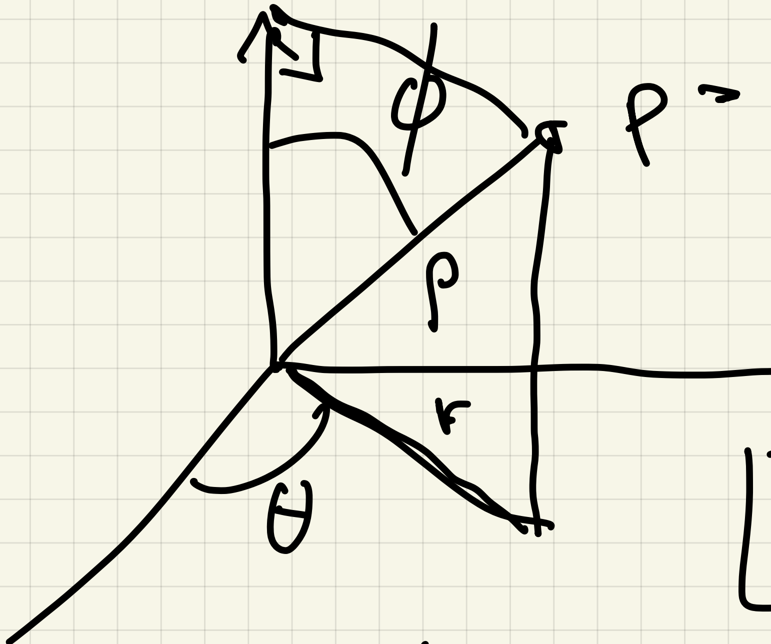
$$250\sqrt{2} - \frac{250\sqrt{2}}{3} =$$

$$250\sqrt{2}\left(1 - \frac{1}{3}\right) = \frac{500\sqrt{2}}{3}$$

$$\int_0^{2\pi} \frac{500\sqrt{2}}{3} \, d\theta = \frac{1000\sqrt{2}\,\pi}{3}$$

But this can be done even more easily!

Spherical Coordinates



$0 \le \phi \le \pi$

θ = polar angle

ρ = dist to origin

ϕ = angle from positive z axis

Connections:

$r = \rho \sin\phi$ $\Rightarrow$ $x = r\cos\theta$

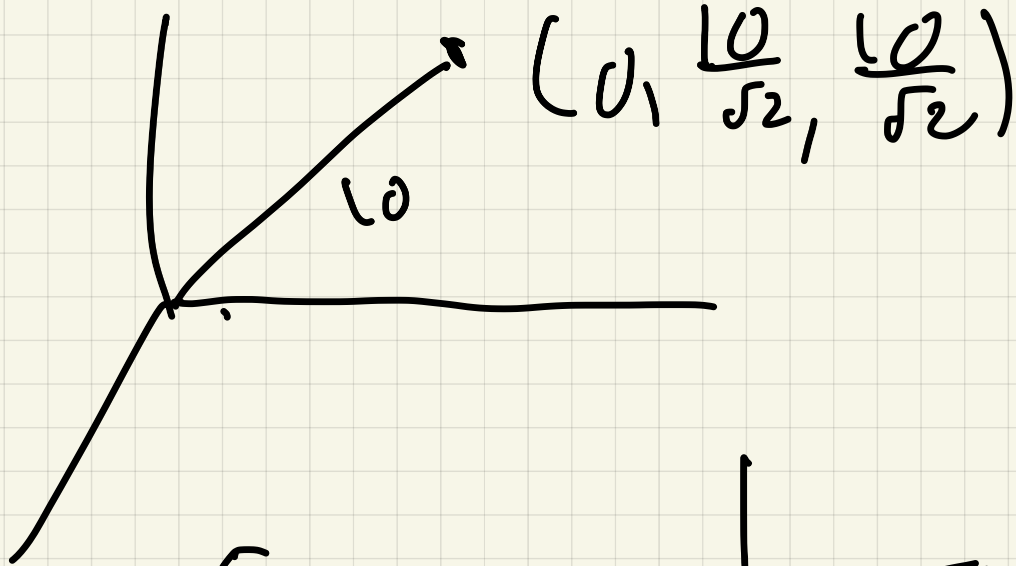
$z = \rho\cos\phi$ $\quad = \rho\sin\phi\cos\theta$

$y = r\sin\theta = \rho\sin\phi\sin\theta$

$\rho = \sqrt{x^2+y^2+z^2} = \sqrt{r^2+z^2}$

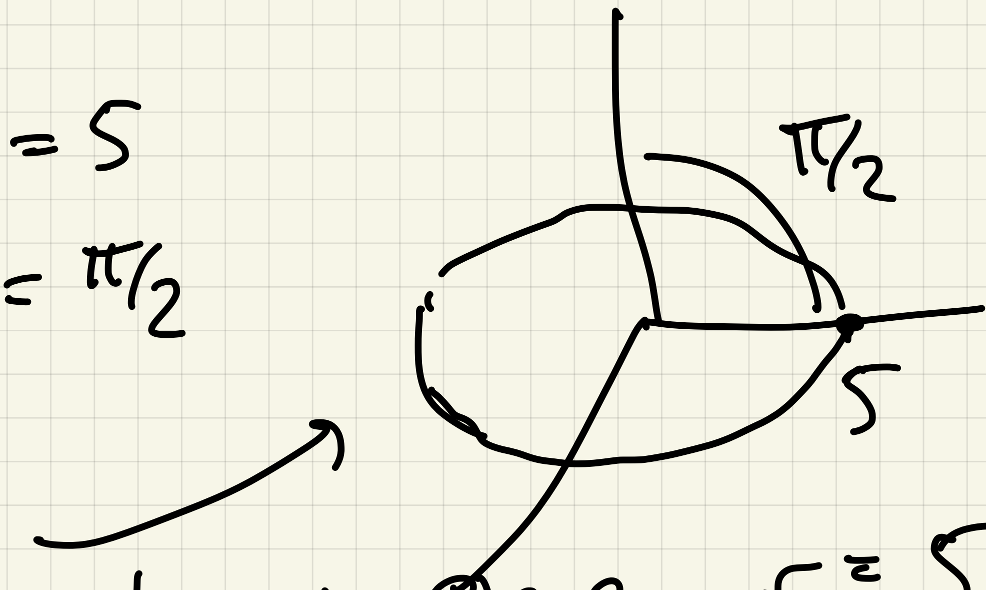
Ex5 $\rho = 10,\ \theta = \pi/2,\ \phi = \pi/4$

(a)



(b) $\rho = 5$

$\phi = \pi/2$

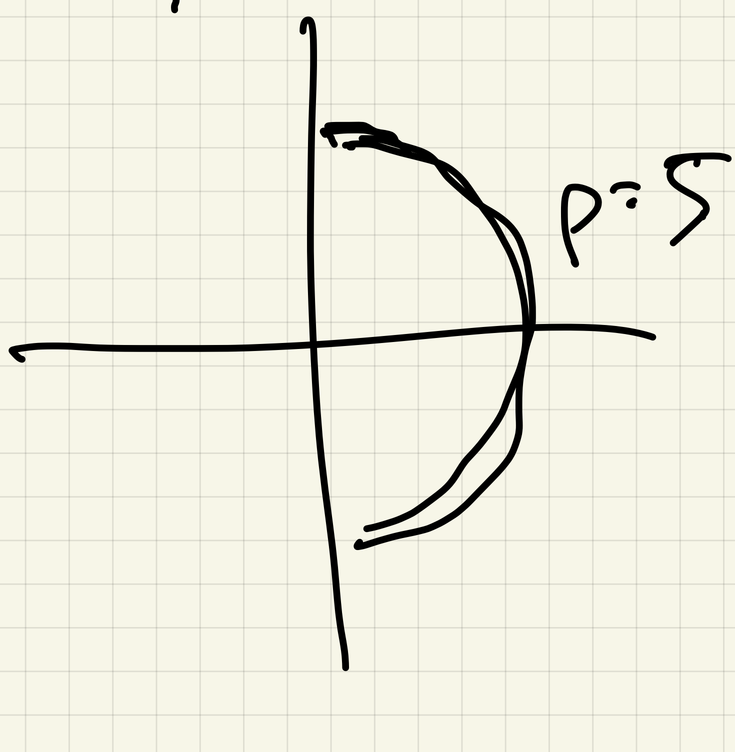


circle in plane $r = 5$

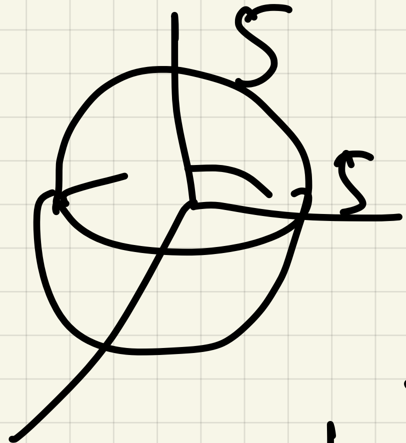
(c) $\rho = 5$

$\theta = \pi/2$

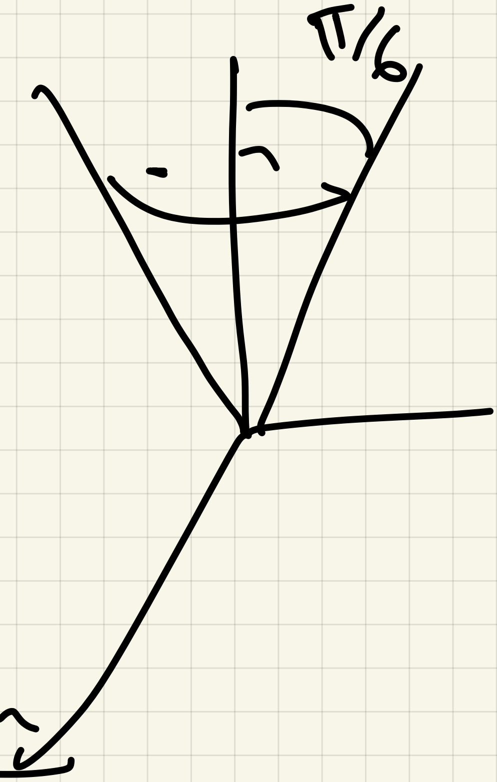
$(0 \leq \phi \leq \pi)$



(d) $\rho = 5$



(e) $\phi = \pi/6$



Integration

$$\iiint_B f(x,y,z)\,dV =$$

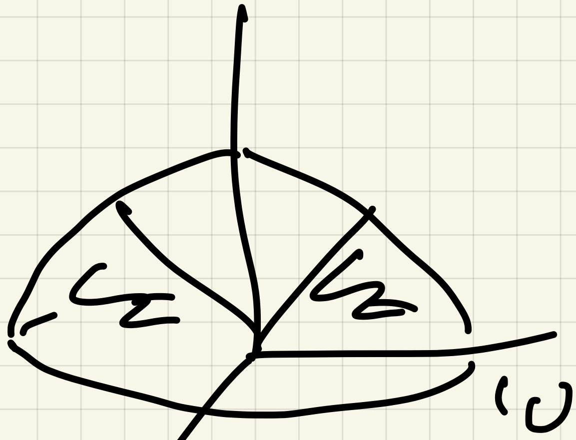
$$\iiint_B f(\underset{x}{\rho \sin\phi\cos\theta},\ \underset{y}{\rho\sin\phi\sin\phi},\ \underset{z}{\rho\cos\phi})\,\rho^2\sin\phi\ d\rho\, d\phi\, d\theta$$

y

coord factor

$\rho^2 \sin\phi$

[x y] revisited



$$V = \int_0^{2\pi} \int_{\pi/4}^{\pi/2} \int_0^{10} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$\theta \quad \phi \quad \rho$