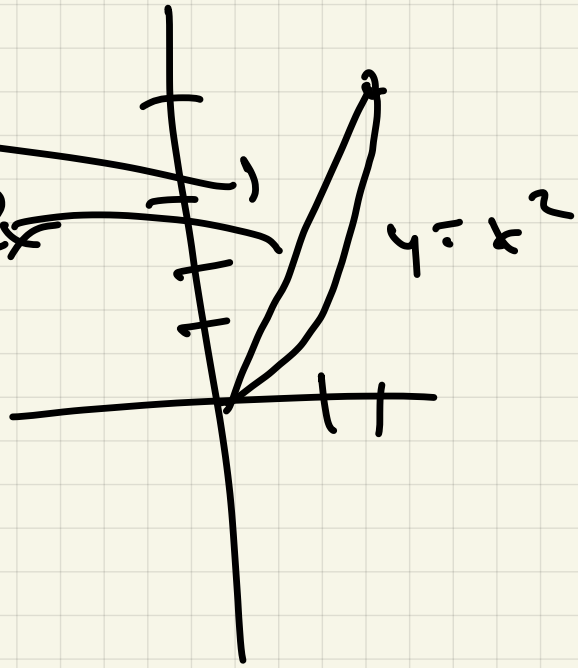
4/18 Calc 3 Quiz 19

1. $r(t) = \langle t, 2t \rangle$ $y = 2x$ $y = x^2$

$0 \le t \le 2$



$$\int_C x^2 + y \, ds = \int_0^2 (t^2 + 2t)\sqrt{5}\, dt$$

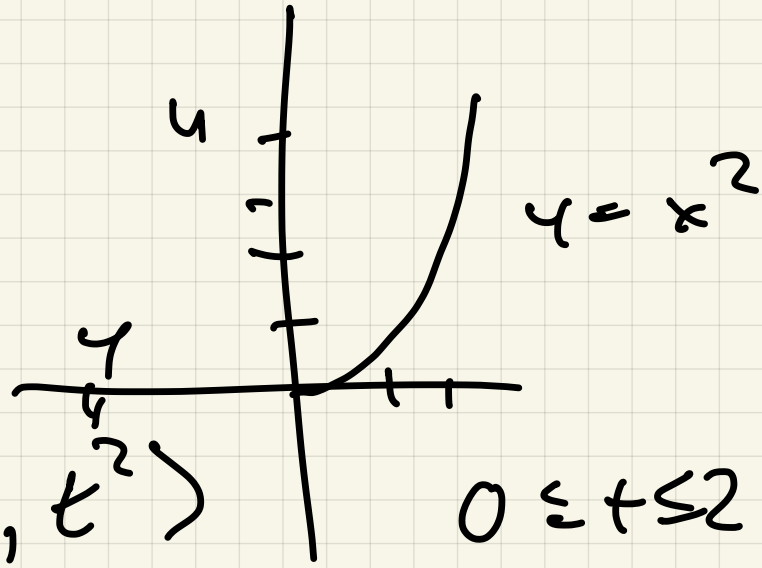
$x^2 + y$

$r'(t) = \langle 1, 2 \rangle$

$|r'| = \sqrt{5}$

$$\sqrt{5}\left(\frac{t^3}{3} + t^2\right)\Big|_0^2 = \frac{20\sqrt{5}}{3}$$

2.



$y = x^2$

| $x$ | $y$ |
|---|---|
| 1 | 1 |

$r(t) = \langle t, t^2 \rangle$ $0 \le t \le 2$

$y = x^2$ ↑

$r' = \langle 1, 2t \rangle$

$|r'| = \sqrt{1 + 4t^2}$

$$\int_0^2 t^2 + t^2 \sqrt{1+4t^2}\, dt$$

$t$: $x^2$ to $1$

$x$: $1$

$$\int_0^2 2t^2 \sqrt{1+4t^2}$$

Last time: vector field

$F(x,y,z) = (M, N, P)$ ($x$, $y$, $z$)

$$\int_C F \cdot dr = \int_C M\,dx + N\,dy + P\,dz$$

F conservative if $F = \nabla f$

(f potential function)

($\Leftrightarrow$ $M\,dx + N\,dy + P\,dz$ is exact)

Test: In 2D $\quad \vec{F}(x,y) = (M, N)$

$\vec{F}$ conservative $\Leftarrow$ $M_y = N_x$

In 3D

$\vec{F}$ conservative $\Leftrightarrow$ $M_y = N_x$, $M_z = P_x$, $N_z = P_y$

$\Downarrow$

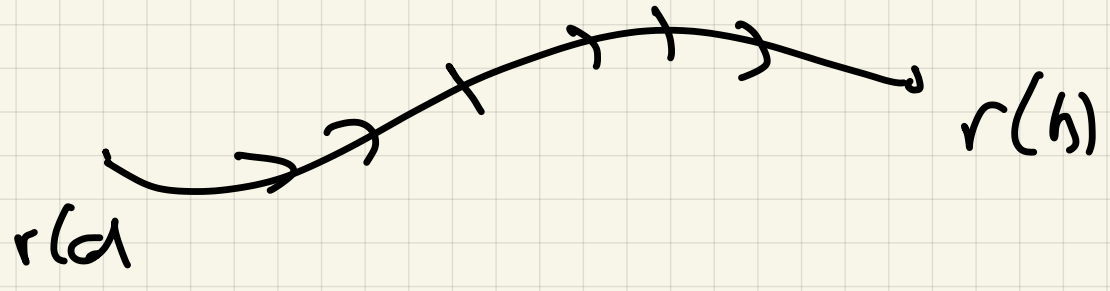
$\text{Curl}\ \vec{F} = \nabla \times \vec{F} = \vec{0}$

Theorem (FTCL I):

If $\vec{r}(t)$ gives $C: a \le t \le b$

and $\vec{F} = \nabla f$, then

$$\int_C \vec{F} \cdot d\vec{r} = f(r(b)) - f(r(a))$$

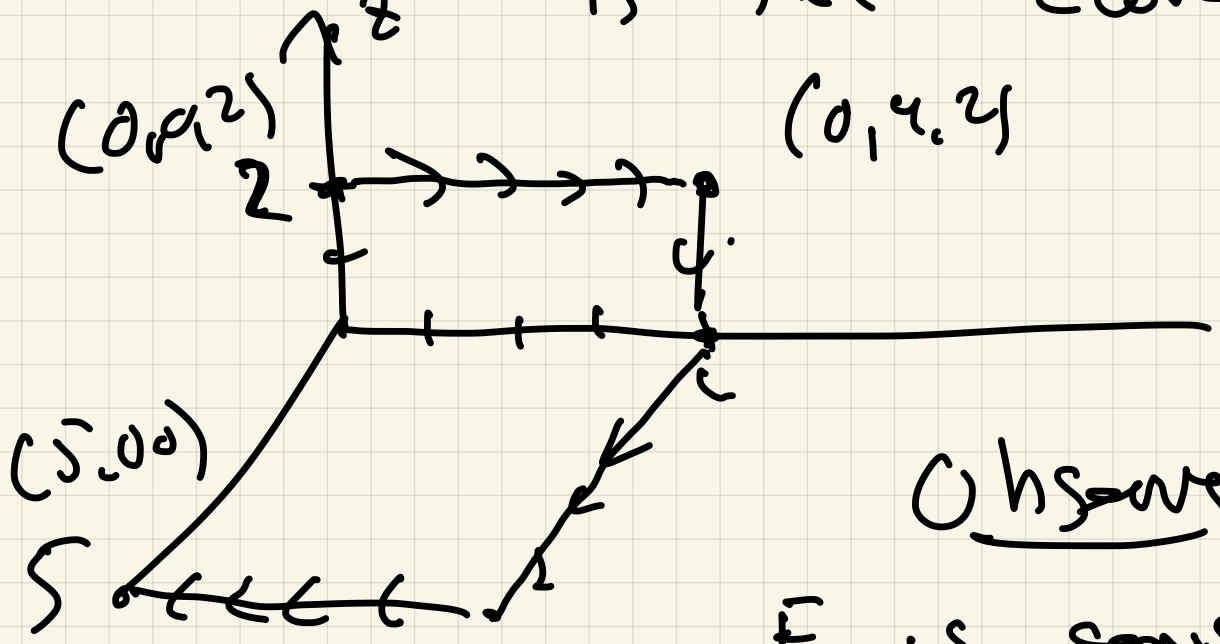


Ex 1

Find $\int_C (x+z^2+3y)\,dx + (3x-\sin y+z)\,dy + (xz^2+yz+z^3)\,dz$

M N

where C is the cone




Observe:

F is conservative

$$\text{Curl } F = \begin{vmatrix} \hat{i} & \hat{j} & k \\ \partial_x & \partial_y & \partial_z \\ x+z^2+3y & 3x-\sin y+z & 2xz+yz+z^3 \end{vmatrix}$$

$i(1-1) - j(2x-2x) + k(3-3) = \vec{0}$

F Conservative:

$F = \nabla f =$ ① $f_x = x + z^2 + 3y$

② $f_y = 3x - \sin y + z$

③ $f_z = 2xz + y + z^3$

① $f = \frac{1}{2}x^2 + xz^2 + 3xy + G(y,z)$

② $f_y = 0 + 0 + 3x + G_y$

$3x - \sin y + z$ $=$

$G = \cos y + yz + H(z)$

$f = \frac{1}{2}x^2 + xz^2 + 3xy + \cos y + yz + H(z)$

③ $f_z = 0 + 2xz + 0 + 0 + y + H_z$

$= 2xz + y + z^3$

so

$$f = \frac{1}{2}x^2 + xz^2 + 3xy + \cos y + yz + \frac{1}{4}z^4$$

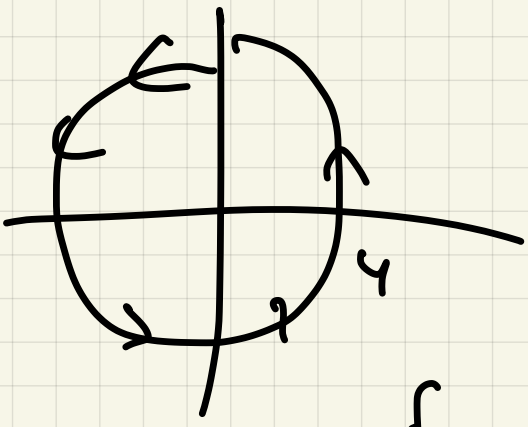
$$\int_C M\,dx + N\,dy + P\,dz =$$

$$f(5,0,0) - f(0,0,2)$$

$$\left(\frac{25}{2} + \cancel{x}\right) - \left(\cancel{x} + 4\right) = \frac{25}{2} - 4$$

$$= \frac{17}{2}$$

Ex 2 $$\int_C (x+3y)\,dx + (3x - \sin y)\,dy$$




$$F = (x+3y,\ 3x - \sin y)$$
$$M \qquad N$$

$$M_y = 3 = N_x$$

$$f = \frac{1}{2}x^2 + 3xy + \cos y$$

$$r(t) = (4\cos t,\ 4\sin t)$$

$$0 \le t \le 2\pi$$

$$r(0) = r(2\pi) = (4,0)$$

$$\int_C F \cdot dr = f(4,0) - f(1,0) = 0$$

**Rmks:** $F$ conservative,

$C$ closed curve

$\Downarrow$

$$\int_C F \cdot dr = 0$$

**Thm:** $F$ conservative

$\Updownarrow$

$\int_C F \cdot dr = 0$ for every closed curve $C$