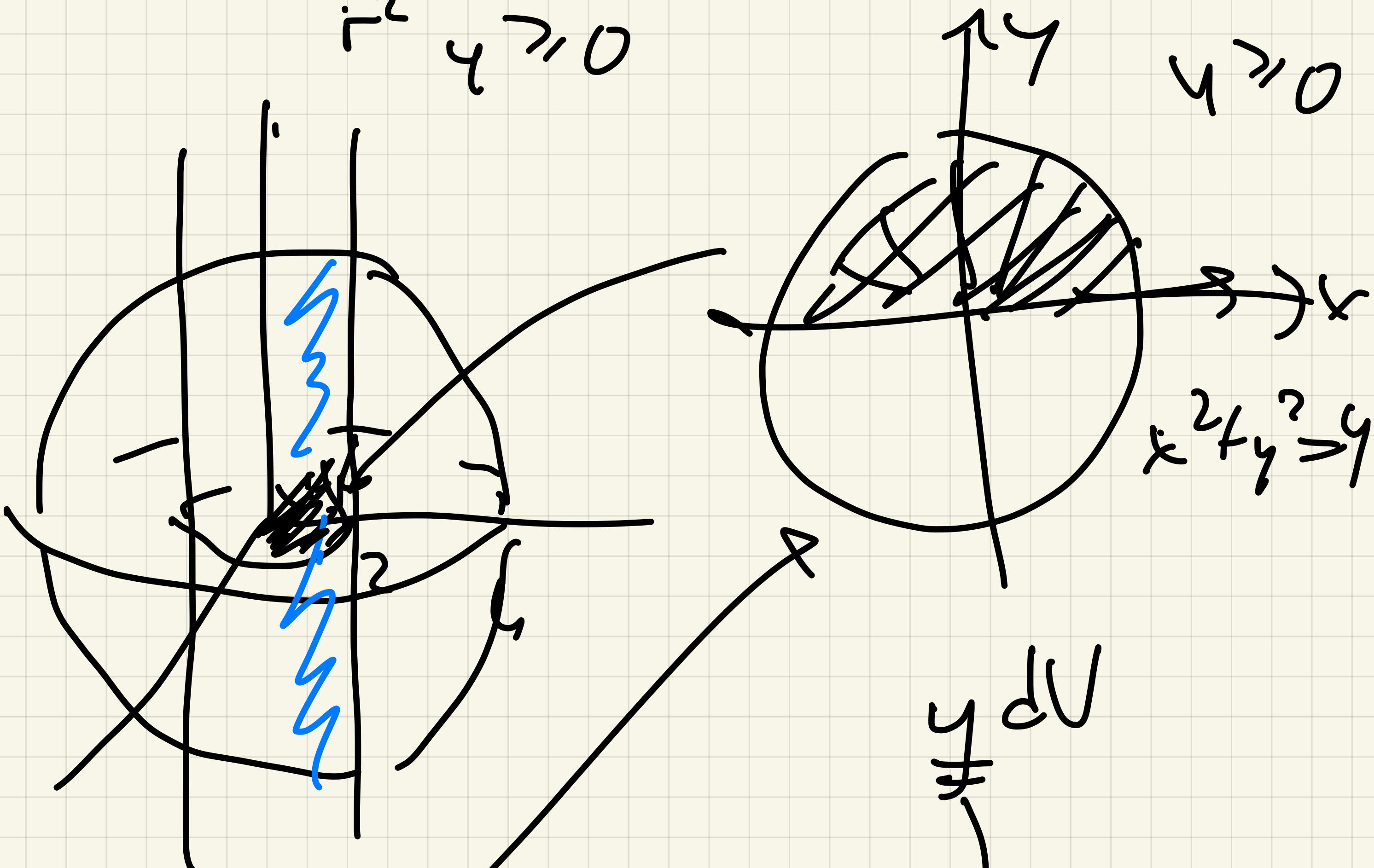
11/14 Calc 3 Quiz 18

1. $x^2+y^2=4 \rightarrow z = \pm\sqrt{16-r^2}$

$\underbrace{x^2+y^2}_{r^2}+z^2=16$

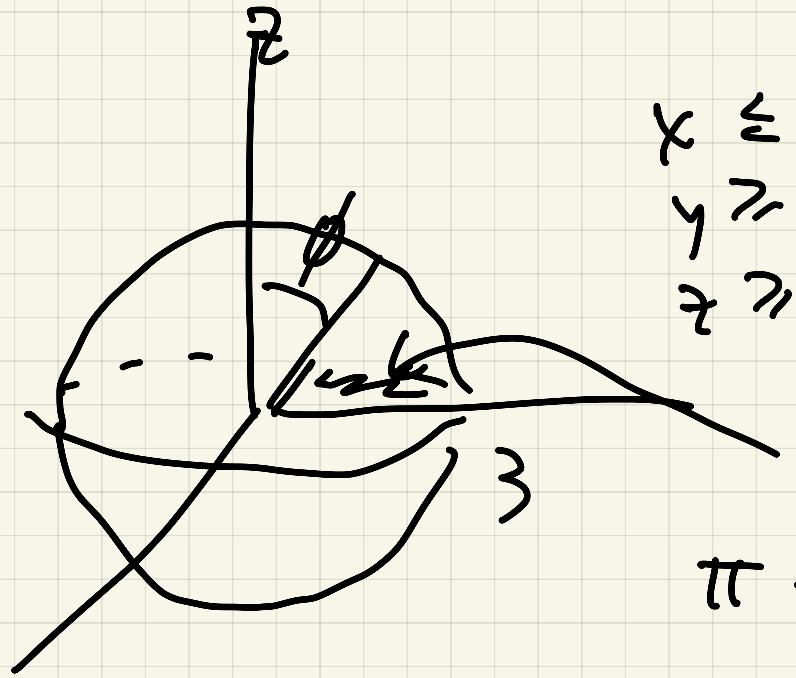
$y \geq 0$



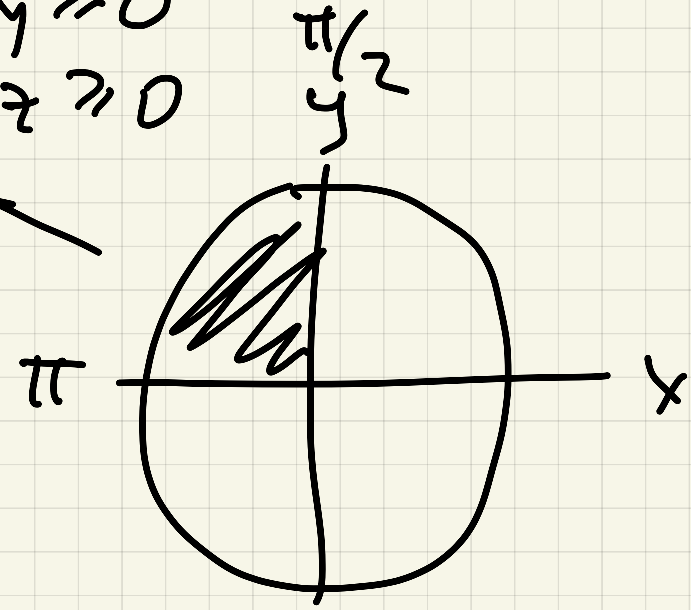
$y \, dV$

$$\int_0^{\pi} \int_0^{2} \int_{-\sqrt{16-r^2}}^{\sqrt{16-r^2}} r\sin\theta \; r \, dz \, dr \, d\theta$$

$\theta \quad r$



$x \le 0$
$y \ge 0$
$z \ge 0$



$$\int_{\pi/2}^{\pi}\int_{0}^{\pi/2}\int_{0}^{3} \underbrace{\rho\sin\phi\sin\theta}_{y}\,\underbrace{\rho^2\sin\phi}\, d\rho\, d\phi\, d\theta$$

$\theta \quad \phi$

$$y = \underline{r}\sin\theta$$
$$= \rho\sin\phi\sin\theta$$

Exam 3 → Thursday

13.7
Ch 14
15.1 – 15.2

§ 15.1 $\int_C f \cdot ds$

§ 15.2 F = vector fields

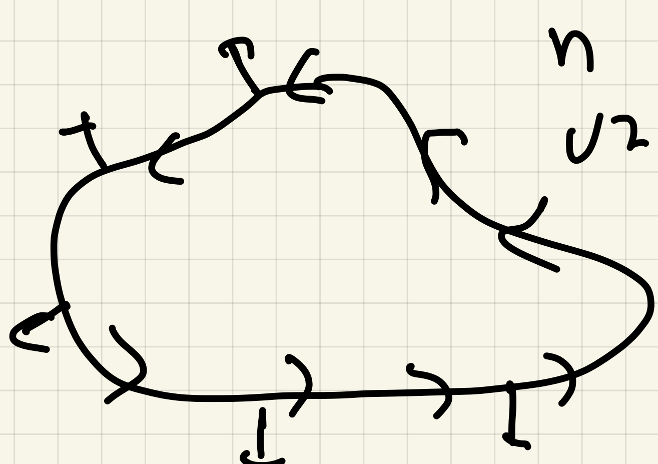
$$F(x,y,z) = \langle \underset{x}{M(x,y,z)}, \underset{y}{N(\ )}, \underset{z}{P} \rangle$$

Work flow $\int_C \bar{F} \cdot \bar{T}\, ds = \int_C \bar{F} \cdot dr =$

$$\boxed{\int_C M\,dx + N\,dy + P\,dz}$$

(parametrize C, direction matters)

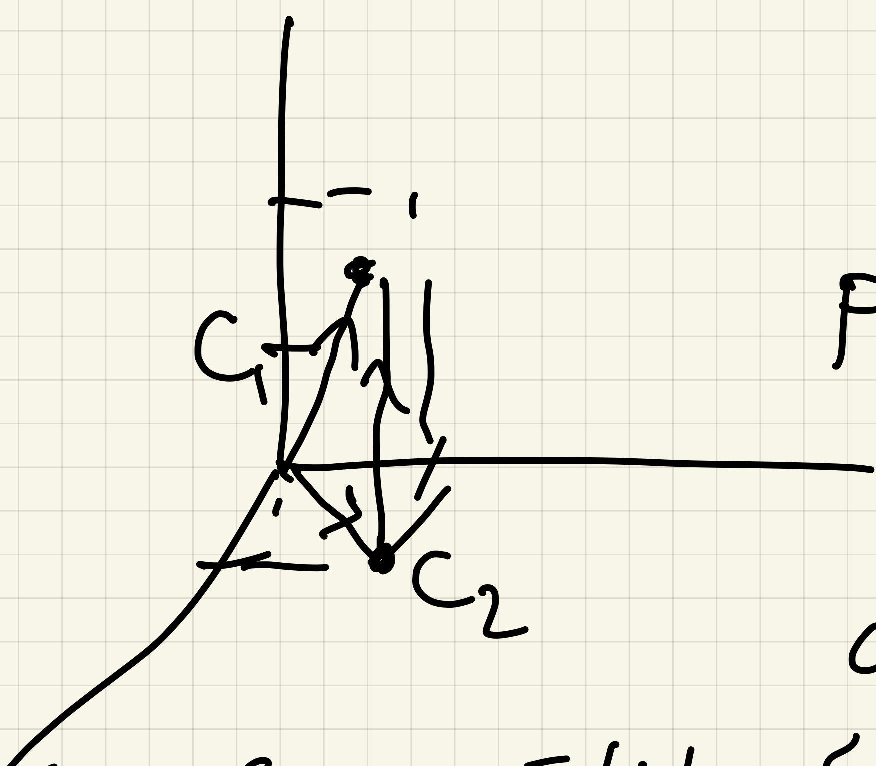
Flux integral $\int_C F \cdot n\, ds = \int M\,dy - N\,dx$



Ex① Find $\int_C \bar{F}\cdot d\bar{r}$ $\quad \bar{F} = \langle y, x+1, 2z \rangle$

(a) $C = C_1$ line segm. from $(0\,0\,0)$ to $(1\,1\,2)$

(b) C_2 is line segm from $(0\,0\,0)$ to $(1\,1\,0)$ followed by line segm. from $(1,1\,0)$ to $(1\,1\,2)$



$F = \langle y, x+1, 2z \rangle$

$0 \le t \le 1$

(a) C_1 : $\bar{r}(t) = \langle t, t, 2t \rangle$

$$r' = \langle 1, 1, 2 \rangle$$

$$\int F \cdot dr = \int_0^1 F(r(t)) \cdot \langle 1,1,2 \rangle \, dt$$

$$2z$$

$$\int \langle t, t+1, 4t \rangle \cdot \langle 1, 1, 2 \rangle \, dt$$

$$= \int_0^1 t + t + 1 + 8t \, dt =$$

$$\int_0^1 10t + 1 \, dt = 5t^2 + t \Big|_0^1 = 6$$

$$\int y \frac{dx}{dt} + (x+1) \frac{dy}{dt} + 2z \frac{dz}{dt} \, dt$$

$$r(t) = \langle t, t, 2t \rangle$$

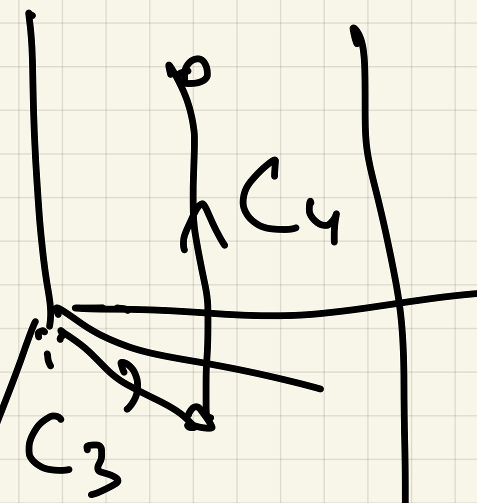
$$x \quad y \quad z$$

$$\int_0^1 t \cdot 1 + (t+1) \cdot 1 + 2(2t) \cdot 2 \, dt$$

(b)

$$C_2 = C_3 \cup C_4$$

$$0\ 0\ 0 \qquad 1\ 1\ 0$$



C_3:

$$\overset{x}{\langle t},\overset{y}{t},\overset{z}{0}\rangle \quad 0 \le t \le 1$$

$$\int_0^1 t \cdot 1 + (t+1)\cdot 1 + 0 \, dt$$

$$= \int_0^1 2t+1 \, dt = t^2+t \Big|_0^1 = 2$$

C_4

$$\vec{r}(t) = \langle \overset{x}{1}, \underset{y}{1}, \underset{z}{t} \rangle \quad 0 \le t \le 2$$

$$\int_0^2 1\cdot 0 + (1+1)\cdot 0 + 2t \cdot 1 \, dt$$

$$= \int_0^2 2t \, dt = t^2 \Big|_0^2 = 4$$

So $\displaystyle\int_{C_3} F\cdot dr + \int_{C_4} F \cdot dr = 2 + 4 = 6$

<u>Defn</u>: A vector field $\vec{F}$ is

<u>conservative</u> if

$$F = \nabla f, \quad \text{some scalar function } f,$$

f is called a <u>potential function for $\vec{F}$</u>.

<u>Ex</u>) If $\vec{F}$ conservative, find potential function

(a) $\vec{F}(x,y) = \langle 2x, 2y \rangle$

$$\text{want } f: \langle f_x, f_y \rangle$$

$$f = x^2 + y^2$$

(b) $\vec{F}(x,y) = \langle 2y, 2x \rangle$

$$f = 2xy$$

(c) $F(x,y) = \langle 2x, 1 \rangle$

$$f = \cancel{2}\, x^2 + y$$

(d) $F(x,y) = \langle 1, 2x \rangle$

$$f = ???$$

If $F = \nabla f, = \langle f_x, f_y \rangle$

then $f_{xy} = f_{yx}$

$$\frac{\partial}{\partial y}(1) = \frac{\partial}{\partial x}(2x)$$

$$\begin{array}{ccc} \| & & \| \\ 0 & \neq & 2 \end{array}$$

<u>Test</u>: $F(x,y) = (M, N)$

F conservative $\Leftrightarrow \partial M/\partial y = \partial N/\partial x$

$$M_y = N_x$$

<u>Test</u>: $F(x,y,z) = (M, N, P)$

$\langle f_x, f_y, f_z \rangle$

$\bar{F}$ conservative $\Leftrightarrow$

$M_y = N_x,\ M_z = P_x,\ N_z = P_y$

The <u>curl</u> of $F(x,y,z) = (M,N,P)$ is

$$\text{Curl } F = \nabla \times \bar{F} =$$

$$\nabla = \langle \partial/\partial x, \partial/\partial y, \partial/\partial z \rangle$$

$$\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ M & N & P \end{vmatrix} =$$

$$i\left(\partial P/\partial y - \partial N/\partial z\right) + j(\quad) + k(\quad)$$

$$=$$

$$\langle P_y - N_z,\ M_z - P_x,\ N_x - M_y \rangle$$

$\Downarrow$

So $\quad$ Curl $F = \langle 0, 0, 0 \rangle \Leftrightarrow$

$\vec{F}$ conservative

<u>Ex 3</u> $\quad$ Compute Curl F,

$$F = \left\langle -\frac{z}{x^2}, \frac{1}{z}, \frac{-y}{z^2} + \frac{1}{x} \right\rangle$$

Curl $\vec{F} = \nabla \times \vec{F} =$

$$\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -\frac{z}{x^2} & \frac{1}{z} & \frac{-y}{z^2} + \frac{1}{x} \end{vmatrix} =$$

$$\langle 0, 0, 0 \rangle$$

So $\vec{F}$ is conservative

$$f = \frac{y}{z} + \frac{z}{x}$$

Ex 4 $\quad F = \langle 2xyz, x^2z, x^2y + 3z^2 \rangle$

$$\text{Curl } F = \begin{vmatrix} \hat{i} & \hat{j} & k \\ \partial_x & \partial_y & \partial_z \\ 2xyz & x^2z & x^2y + 3z^2 \end{vmatrix}$$

$$\langle 0, 0, 0 \rangle$$

$F =$ conser.

$$F = \nabla f =$$

$$f = x^2yz + x^2zy$$

Fundamental Theorem of line integrals

If F is conservative, and $F = \nabla f$, then

$C: \; r(t) = \; , \quad a \le t \le b$

Then

$$\int_C \bar{F} \cdot d\bar{r} = \int_C \underline{\nabla f} \cdot dr =$$

$$f(r(b)) - f(r(a))$$

Why?

$$\bar{r}(t) = \langle x(t), y(t), z(t) \rangle$$

$$a \le t \le b$$

$$\int_a^b \nabla f(r(t)) \cdot \bar{r}'(t) \, dt$$

$$\parallel$$

$$\int_a^b f_x(r(t)) \cdot \frac{dx}{dt} + f_y(\quad) \frac{dy}{dt} +$$

$$f_z(\quad) \frac{dz}{dt}$$

$$\parallel$$

$$\int_a^b \nabla f(r(t)) \cdot \bar{r}'(t)$$

$$\parallel$$

$$\int_a^b \frac{d}{dt}\left(\underline{\underline{f(r(t))}} \right)$$

$$f(r(t))\Big|_a^b = f(r(b)) - f(r(a))$$

<u>Ex 1</u> $\quad F = \langle \overset{M}{y}, \overset{N}{x+1}, \overset{P}{2z} \rangle$

$C: (0\,0\,0)$ to $\langle 1\,1, 2\rangle$

<u>Check</u>:
$$\left.\begin{aligned} M_y &= N_x = 1 \\ M_z &= N_x = 0 \\ N_z &= P_y = 0 \end{aligned}\right\}$$

So F conservative:

$F = \nabla f$, $\quad f = ??$

$$f = xy + y + z^2$$

so $\displaystyle\int_C F \cdot dr = f(1,1,2) - f(0,0,0)$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \shortparallel$

$6 - 0 = 6$ ✓

~~Note~~ Thm:

F conservative $\Rightarrow \int_C F \cdot dr$ depends only on endpoints (not depend on path)