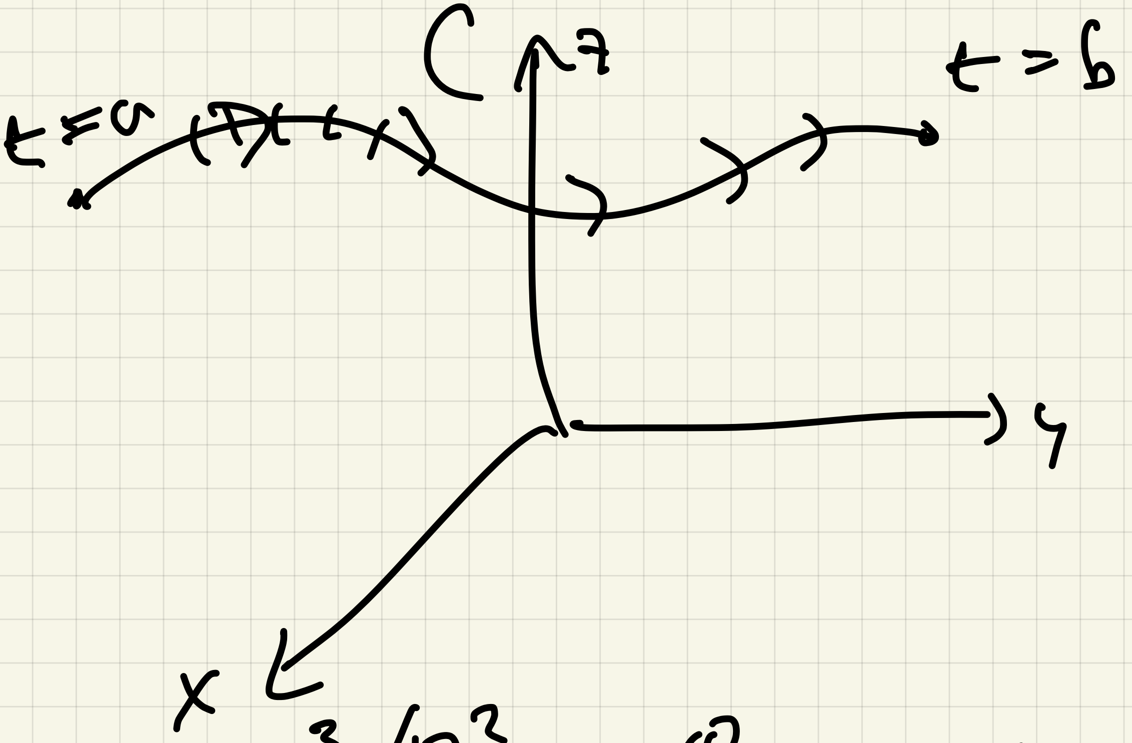
11/3/ Calc3

Exam 3 → Next Thursday

§13.7, Ch 14

Last time Line integrals:

$C : r(t) :$ in $\mathbb{R}^2$ / in $\mathbb{R}^3$ $a \le t \le b$



① $f : \mathbb{R}^3/\mathbb{R}^2 \to \mathbb{R}$ scalar

$$\int_C f\,ds = \int_a^b f(\bar{r}(t))\,|r'(t)|\,dt$$

② $F : \mathbb{R}^3 \to \mathbb{R}^3$ $(\mathbb{R}^2 \to \mathbb{R}^2)$

$$\int_C F \cdot dr \quad \left(= \int F \cdot T \underbrace{ds} \right)$$

$$\|$$

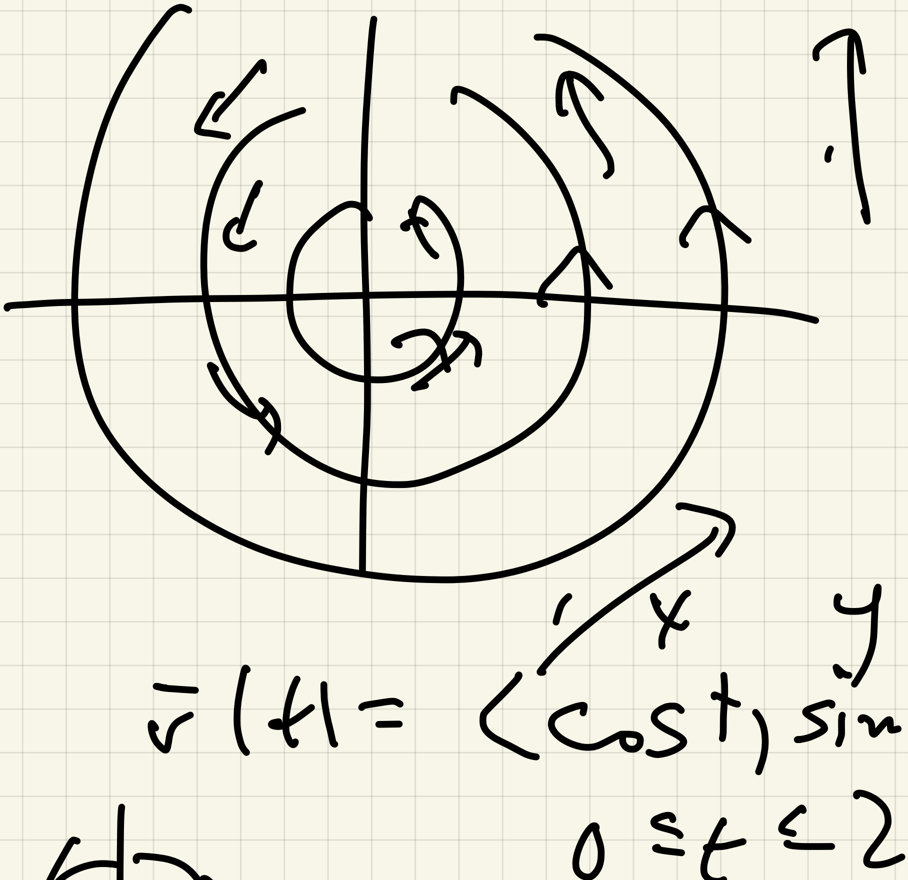
$$\int_a^b F(\bar{r}(t)) \cdot r'(t)\, dt$$

$$\|$$

work integral

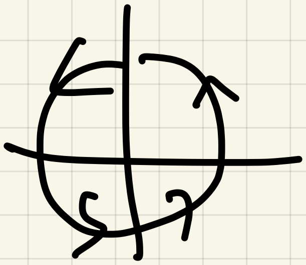
flow integral

Ex) $F(x,y) = \langle -y, x \rangle$



(a) $\bar{r}(t) = \langle \overset{x}{\cos t}, \overset{y}{\sin t} \rangle$

$0 \le t \le 2\pi$



$$\vec{r}\,'(t) = \langle -\sin t, \cos t \rangle$$

$$\int_0^{2\pi} \underset{\cdot}{F}(\underset{||}{\underline{\underline{r(t)}}}) \cdot r'(t)\, dt$$

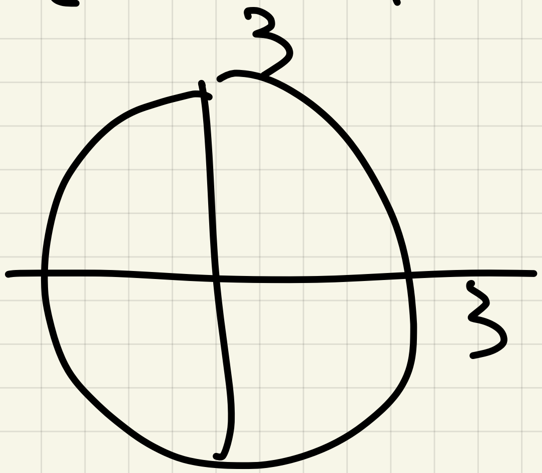
$$\int_0^{2\pi} \underbrace{\langle \underset{-y}{-\sin t}, \underset{x}{\cos t} \rangle \cdot \langle -\sin t, \cos t \rangle}\, dt$$

$$\int_0^{2\pi} 1\, dt = 2\pi$$

(b)

C_2 $\quad r(t) = \langle 3\sin t, 3\cos t \rangle$

$0 \le t \le 2\pi$



$$r' = \langle 3\cos t, -3\sin t \rangle$$

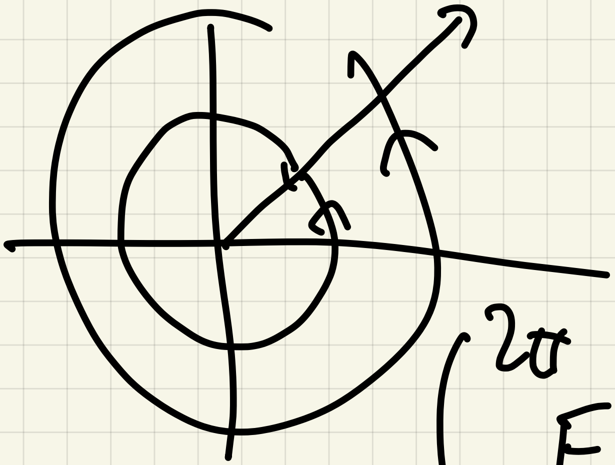
$$\int \underset{-y \quad x}{F}(r(t)) \cdot r'(t)\, dt =$$

$$\int_0^{2\pi} \langle -3\cos t, 3\sin t\rangle \cdot \langle 3\cos t, -3\sin t\rangle \, dt$$

$$= \int_0^{2\pi} -9 \, dt = -18\pi$$

(c) $\overset{x\ \ y}{\vec{r}(t) = \langle t, t\rangle}$, $0 \le t \le 2\pi$

$r'(t) = \langle 1, 1\rangle$



$$\int_0^{2\pi} F(r(t)) \cdot r'(t) \, dt$$

$$\int_0^{2\pi} \underbrace{\langle -t, t\rangle \cdot \langle 1, 1\rangle} \, dt$$

$$\int_0^{2\pi} 0 \, dt = 0$$

Alternate Notation $F = \langle M, N, P \rangle$

$$\int_C F \cdot dr = \int_C M\,dx + N\,dy + P\,dz$$

Ex2 1(a) revisited $\langle -y, x \rangle$

$$\int_C F \cdot dr = \int_C \boxed{-y}\,\overset{\frac{dx}{dt}}{\boxed{dx}} + x\,\underset{\frac{dy}{dt}}{\boxed{dy}}$$

$$\vec{r}(t) = \langle x(t), y(t) \rangle$$

$$= \langle \underset{x}{\cos t}, \underset{y}{\boxed{\sin t}} \rangle$$

$$\frac{dx}{dt} = -\sin t \qquad \boxed{\frac{dy}{dt}} = \cos t$$

$$\int_0^{2\pi} (-\sin t)(-\sin t) + (\cos t)(\cos t)\,dt$$

$$\parallel$$

$$\int_0^{2\pi} 1\,dt = 2\pi$$

Ex 3

$$C_3 : \quad r(t) = \langle \overset{x}{t}, \overset{y}{t} \rangle$$

$$0 \le t \le 2\pi$$

$$\int_C \underset{\shortparallel}{y}\, dx + \underline{\underline{0\, dz}}$$

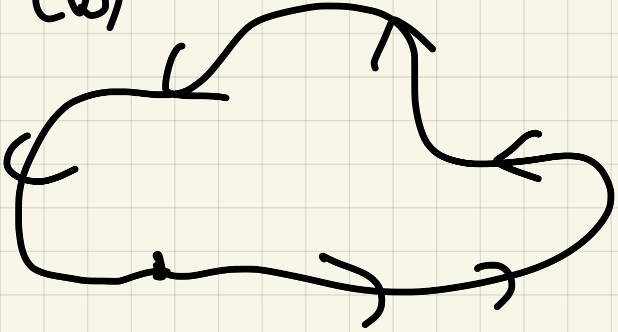
$$\frac{dx}{dt} = 1$$

$$\frac{dy}{dt} = 1$$

$$\int_0^{2\pi} t \cdot 1\, dt = \quad \frac{t^2}{2}\Big|_0^{2\pi} = 2\pi^2$$

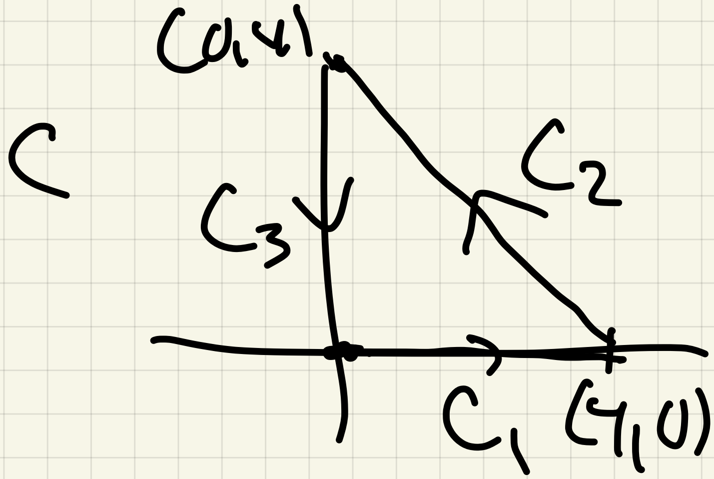
Rmk: If C is closed curve,

$r(a) = r(b)$



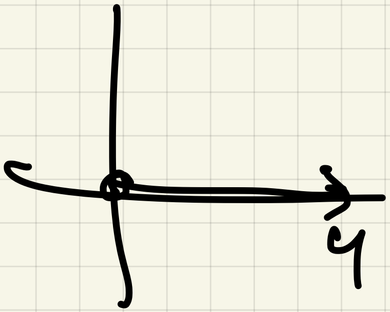
Then $\int_C F \cdot dr$ = circulation integral

Ex 4 $\int_C x\, dx + y\, dy$



$$\int_C x\,dx + y\,dy = \int_{C_1} + \int_{C_2} + \int_{C_3}$$

C_1:



$r = \langle 4t, 0\rangle, \quad 0 \leq t \leq$ [redacted]

$$\int_{C_1} x\,dx + y\,dy = \int_0^1 (4t)\cdot 4 + 0\cdot 0\,dt$$

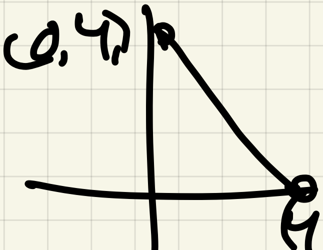
$\|$

$$\int_0^1 16t\,dt$$

$$8t^2\Big|_0^1 = 8$$

$$r' = \langle \underset{\frac{dx}{dt}}{4}, \underset{\frac{dy}{dt}}{0} \rangle$$

C_2



$$r(t) = \langle \underset{x}{4-4t}, \underset{y}{4t} \rangle$$

$$0 \leq t \leq 1$$

$$dx = -4 \quad dy = 4$$

$$\int_{C_2} x\,dx + y\,dy = \int_0^1 \underline{(4-4t)(-4)} + \underline{(4t)}4\, d$$

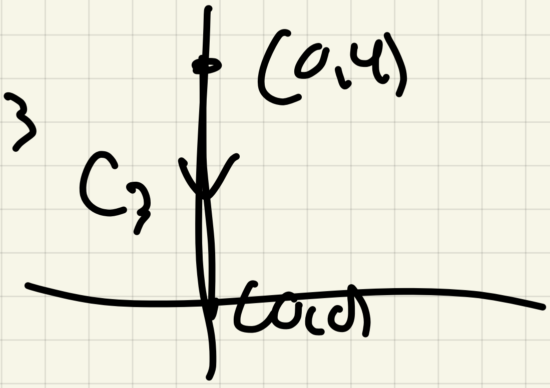
$$=$$

$$\int_0^1 -16 + 16t + 16t =$$

$$\int_0^1 -16 + 32t\, dt$$

$$-16t + 16t^2 \Big|_0^1 = 0$$

C_3



$$r(t) = \langle 0, 4-4t \rangle$$

$$(0,4) + (0,-4)t$$

$$=$$

$$(\underset{x}{0}, \underset{y}{4-4t})$$

$$\frac{dy}{dt} = -4$$

$$\int_{C_3} x\,dx + y\,dy = \int_0^1 0 + (4-4t)(-4)\,dt$$

$$\int_0^1 -16+16t\,dt =$$

$$-16t + 8t^2 \Big|_0^1 = -8$$

$$\int_C = \int_{C_1} + \int_{C_2} + \int_{C_3}$$

$$= 8 + 0 + -8 = 0$$

§ 15.3

<u>Defn</u>: A vector field F is <u>conservative</u> if $F = \nabla f$ for some function f.

f is called <u>potential function</u>

Ex) If $\bar{F}$ is conservative, find a potential function.

(a) $F = \langle 2x, 2y \rangle$

$\langle f_x, f_y \rangle$

$f = x^2 + y^2$

(b) $\bar{F} = \langle 2y, 2x \rangle$

$f_x \quad f_y$

$\nearrow$

$f = 2xy$

(c) $F = \langle 2x, 1 \rangle$

$f = x^2 + y$

(d) $\bar{F} = \langle 1, 2x \rangle$

$f_x \quad f_y$

$\swarrow \qquad \searrow$

$f = x + 2xy$ fails

$$f_x = 1 \qquad\qquad f_y = 2x$$

$$\partial \qquad\qquad \partial$$

$$\left.\begin{array}{ccc} f_{xy} & = & f_{yx} \\ \| & & \| \\ 0 & \neq & 2 \end{array}\right\} \longleftarrow$$

Test: $F = (M, N)$ is conservative

$$\Updownarrow$$

$$\partial M/\partial y = \partial N/\partial x$$

Ex 2

(a) $F = \langle \underbrace{\boxed{x^3} + y + \sin y}_{M}, \underbrace{y + x + x\cos y}_{N} \rangle$

$$\partial M/\partial y = 1 + \cos y$$

$$\| \quad ✓$$

$$\partial N/\partial x = 1 + \cos y$$

$$f = \int M\,dx = \frac{x^4}{4} + xy + x\sin y + C(\underline{\underline{y}})$$

Take $C(y) = \frac{1}{2}y^2$

So

$$f = \frac{x^4}{4} + xy + x\sin y + \frac{1}{2}y^2$$

(b) $F = \langle \underset{M}{\underset{\parallel}{2x+5y}},\ \underset{N}{3x-7y}\rangle$

$M_y = 5 \neq N_x = 3$

Similary, x y z

<u>Test</u> $F = (M, N, P)$ conservative

$\parallel \qquad \Updownarrow$

$M_y = N_x,\ M_z = P_x,\ N_z = P_y$

$\langle f_x, f_y, f_z\rangle$

<u>Ex 3</u>

$F = \langle -\frac{z}{x^2},\ \frac{1}{z},\ -\frac{y}{z^2} + \frac{1}{x}\rangle$

$M \quad N \quad P$

$M_y = 0 \qquad M_z = -\frac{1}{x^2} \; = \checkmark$

$\|$

$N_x = 0 \qquad P_x = (-1)x^{-2} = \frac{-1}{x^2}$

$N_z = -\frac{1}{z^2}$

$\| \checkmark$

$P_y = -\frac{1}{z^2}$

F conservative!

find f:

$f_z = P = -\frac{y}{z^2} + \frac{1}{x} \Rightarrow$

$$f = \int \boxed{-\frac{y}{z^2}} + \frac{1}{x} \, dz =$$

$$f = \frac{y}{z} + \frac{z}{x}$$

Fundamental Theorem of line integrals

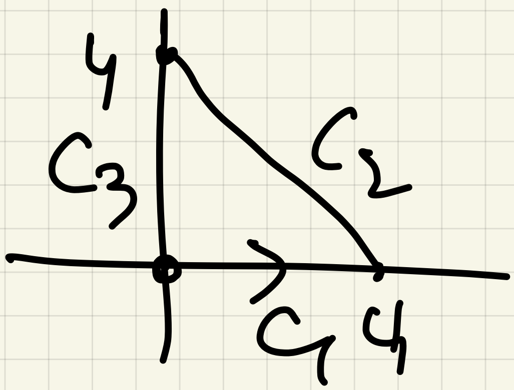
Thm: If $F = \nabla f$ is conservative and $C: r(t)$ $a \le t \le b$, Then

$$\int_C F \cdot dr = f(r(b)) - f(r(a))$$

Why? $$\frac{d}{dt}(f(r(t)) = \underbrace{\nabla f(r(t))}_{F(r(t))} \cdot \underline{\underline{r'(t)}}$$

Ex 4 (Last ex from §15.2)

$$\int_C x\,dx + y\,dy$$



$F = \langle x, y \rangle$
$\quad\ \ M \ \ N$

$M_y = N_x = 0$

$F = \nabla f$, $\quad f = \frac{1}{2}x^2 + \frac{1}{2}y^2$

$$r(a) = (0,0)$$
$$r(b) = (0,0)$$

So $\int_C F\cdot dr = f(0,0) - f(0,0) = 0$

Ex 5 Compute

$$\int_C 2xyz\,dx + x^2z\,dy + (x^2y+3z^2)\,dz$$

$C : r(t) = (t^3+1, \cos\pi t, \sin\pi t)$

$0 \le t \le 1$

Check: $F = (M,N,P) =$

$$(\underset{M}{2xyz}, \underset{N}{x^2z}, \underset{P}{x^2y+3z^2})$$

$M_y = 2xz$ $\quad$ $M_z = 2xy$ ✓

$\cancel{M}\, N_x = 2xz$ $\quad$ $P_x = 2xy$

$N_z = x^2,$ $\quad$ $P_y = x^2$

$\nabla f = F_1$ $\quad f = \int x^2 y + 3z^2 \, dz$

$x^2yz + z^3$