11/12/ Calc 3

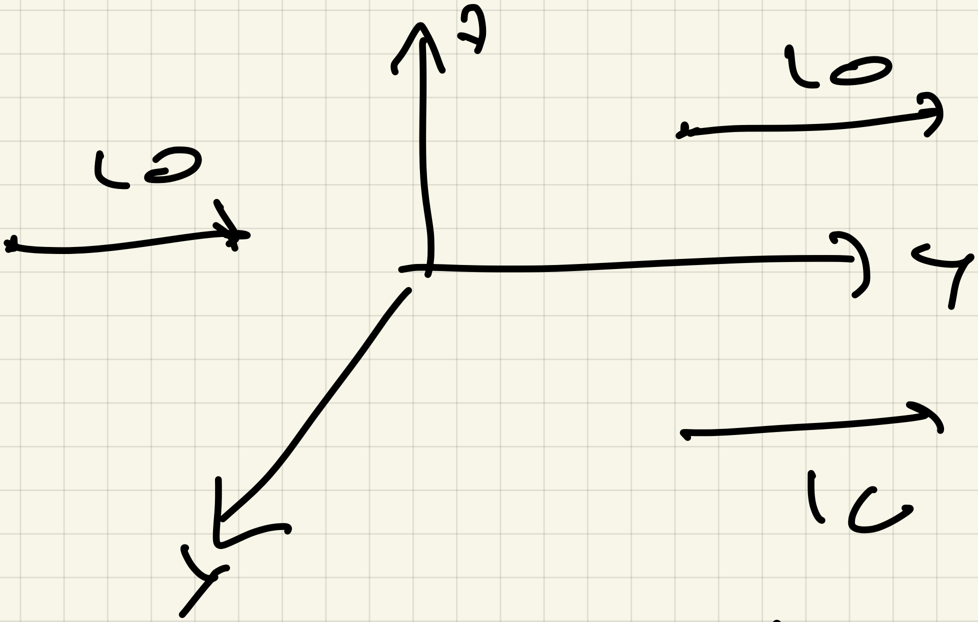
Last time

Vector field :

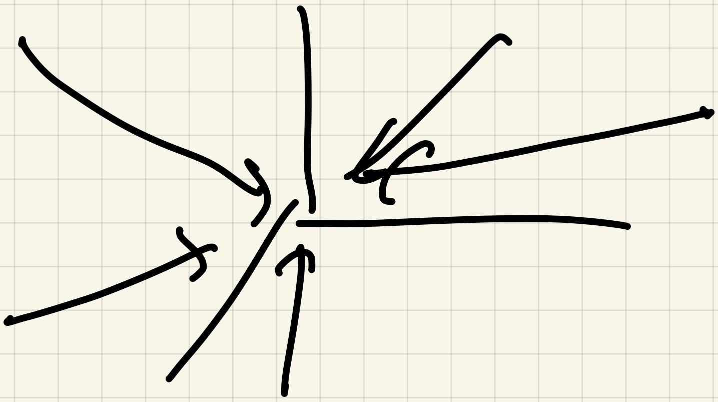
function $F : \mathbb{R}^3 \to \mathbb{R}^3$

$(F : \mathbb{R}^2 \to \mathbb{R}^2)$

Ex! (a) $F(x,y,z) = \langle 0, 10, 0 \rangle$



(b) $G(x,y,z) = G \dfrac{\langle -x, -y, -z \rangle}{|\langle -x, -y, -z \rangle|^3}$

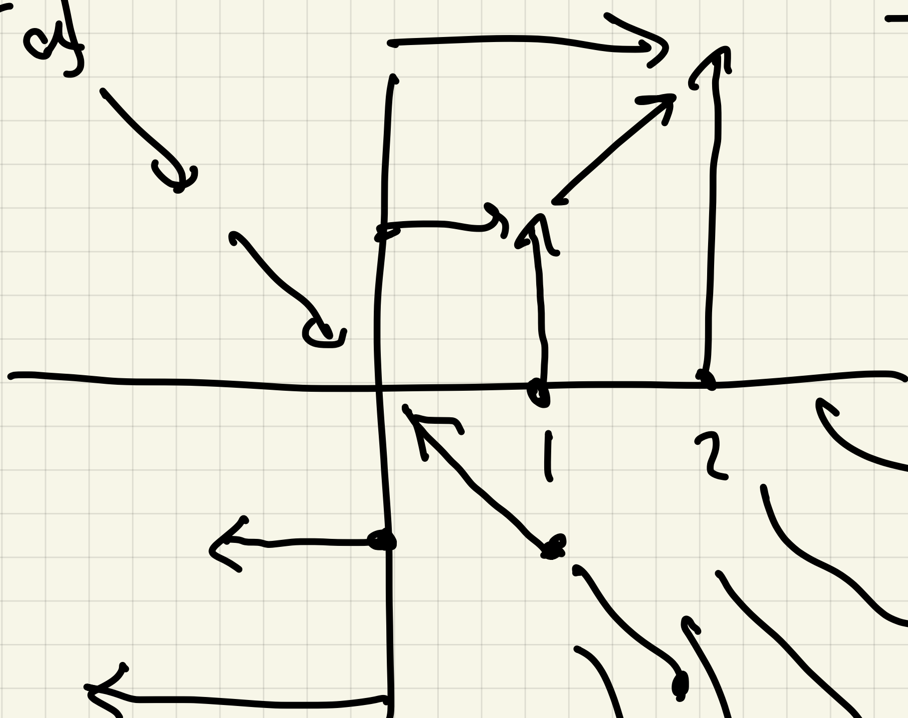


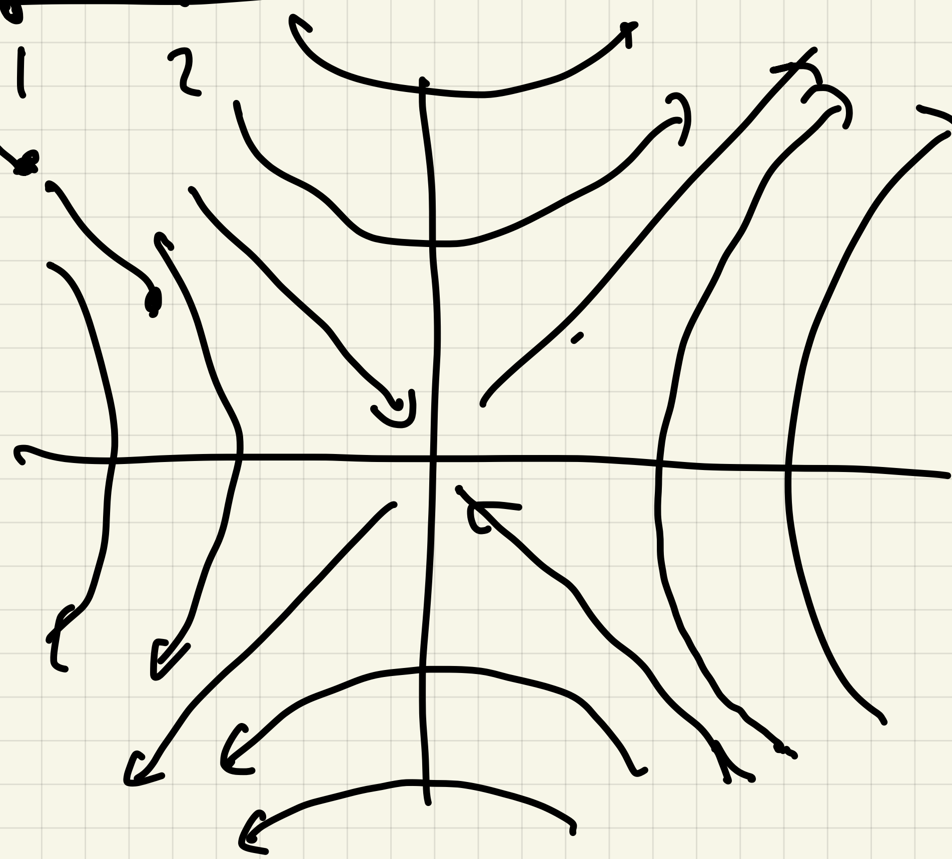
gravity

[Ex]

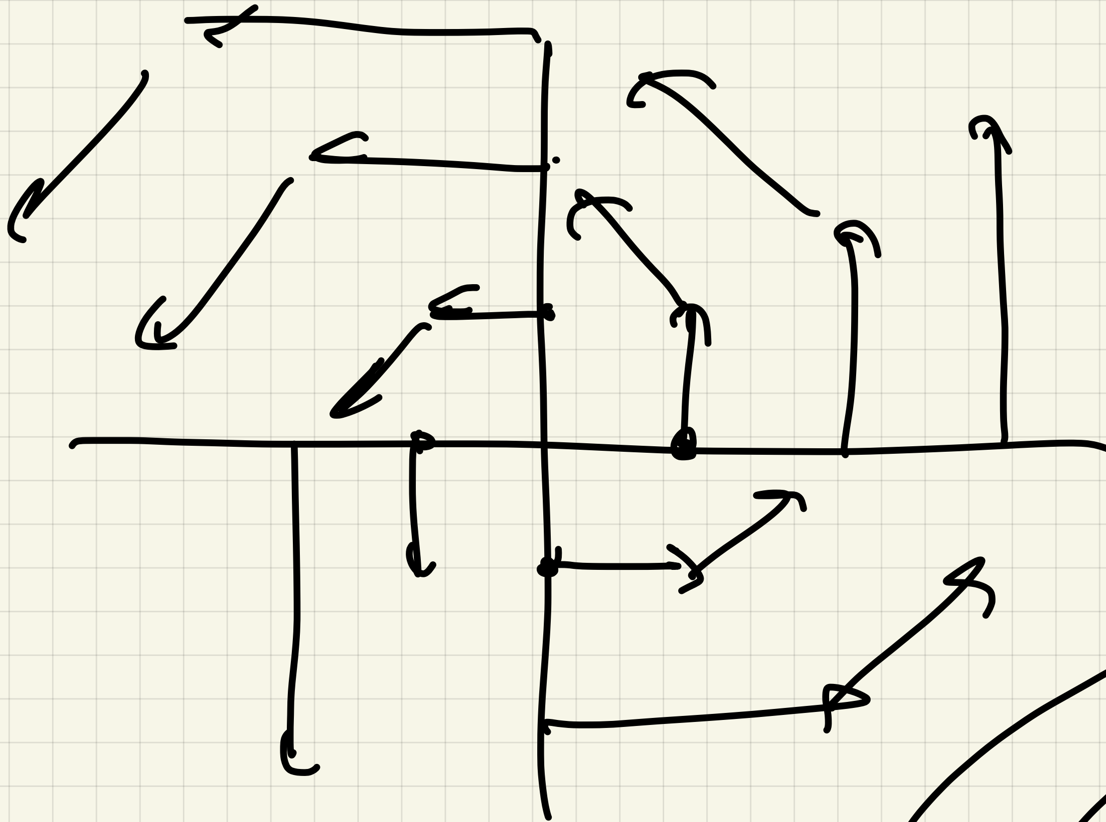
$F(x,y) = \langle \underline{y, x} \rangle$

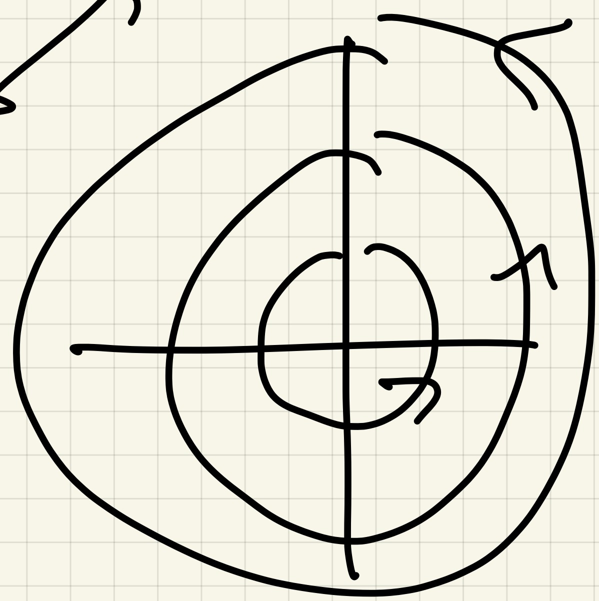
(a)





(b) $F(x,y) = \langle -y, x \rangle$





Ex 3 $f : \mathbb{R}^3 \to \mathbb{R}$

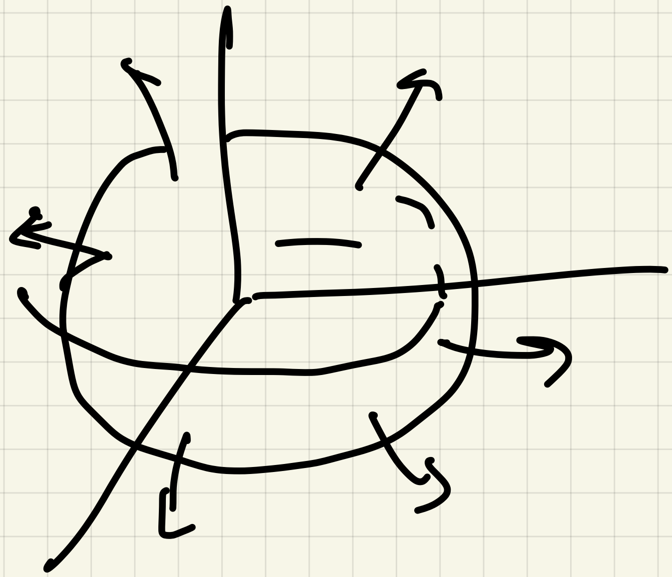
any function

gradient $\nabla f : \mathbb{R}^3 \to \mathbb{R}^3$

$\langle f_x, f_y, f_z \rangle$

$f(x,y,z) = x^2 + y^2 + z^2 = c$

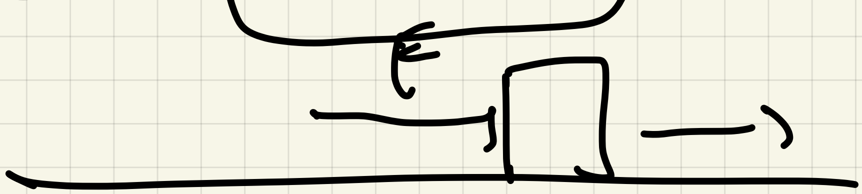
$\nabla f = \langle 2x, 2y, 2z \rangle$

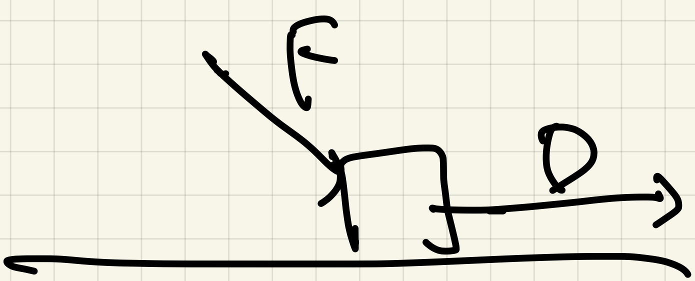


Now let C be an *oriented* *curve* in $\mathbb{R}^3/\mathbb{R}^2$

F = vector field

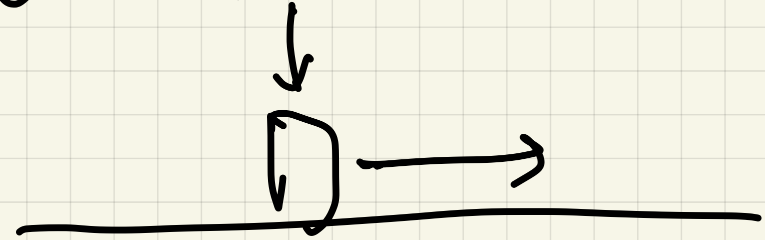
Motivation: $\boxed{W = F \cdot d}$

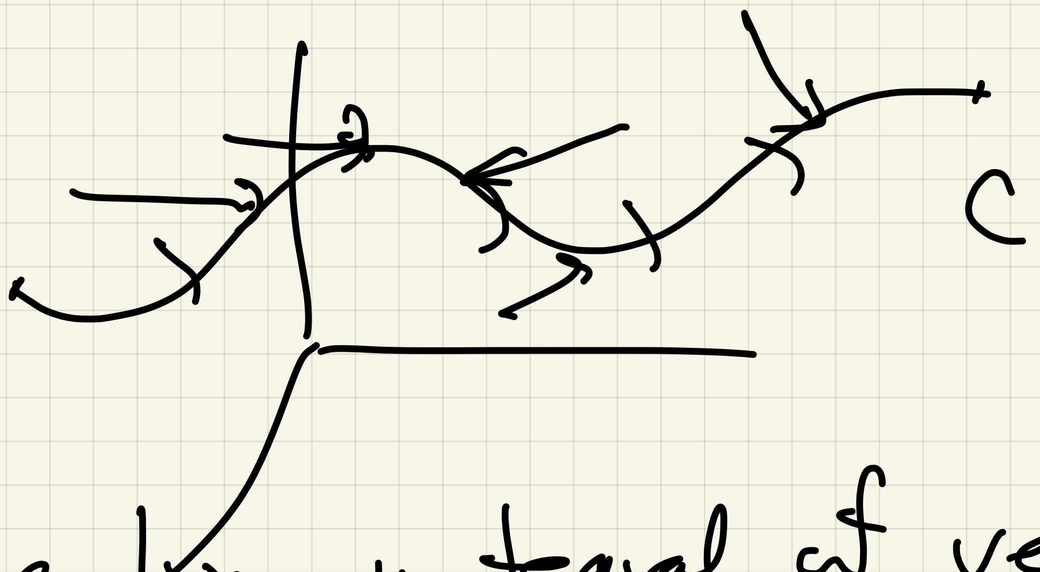




If F = force vector
D = displace

$W = F \cdot D$





The line integral of vector field F along C is

Notation: $\int_C F \cdot T \, ds$

force

unit tangent

arc length

$\int_C F \cdot dr$ ← means?

Calculation:

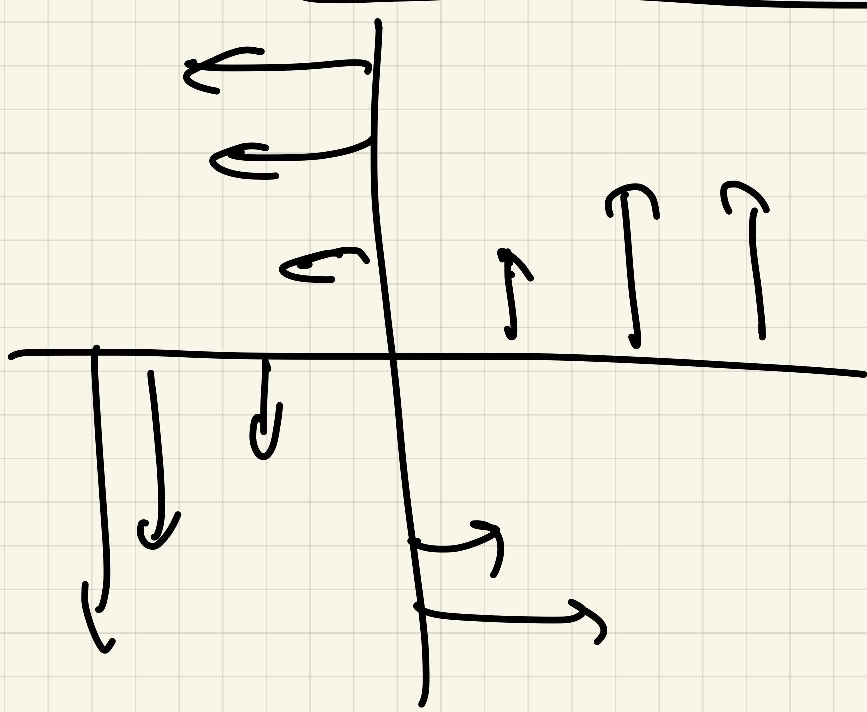
$C: \bar{r}(t): a \le t \le b$

velocity

$$\int_C F \cdot d\bar{r} = \int_a^b F(\bar{r}(t)) \cdot \frac{d\bar{r}}{dt}\, dt$$

$$\left[\underline{\text{Physics}}: \underset{\text{Work}}{\underset{\Vert}{W}} = \int_C F \cdot d\bar{r} \right]$$

Ex! $\boxed{F(x,y) = \langle -y, x \rangle}$

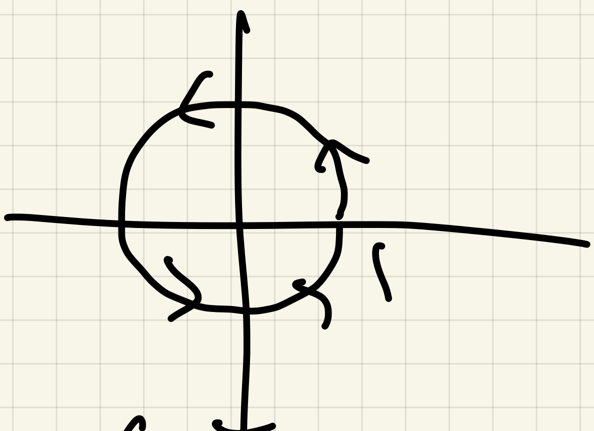


Compute $\int_C F \cdot d\bar{r}$

for following curves:

(a) $C_1 : \bar{r}(t) = \langle \cos t, \sin t \rangle$

$x \quad y \quad 0 \le t \le 2\pi$



$F(x,y) = \langle -y, x \rangle$

$$\int_{C_1} \bar{F} \cdot d\bar{r} = \int_0^{2\pi} \underline{\underline{F(\bar{r}(t))}} \cdot \frac{dr}{dt}\, dt$$

$$= \int_0^{2\pi} \langle -\sin t, \cos t \rangle \cdot \langle -\sin t, \cos t \rangle\, dt$$

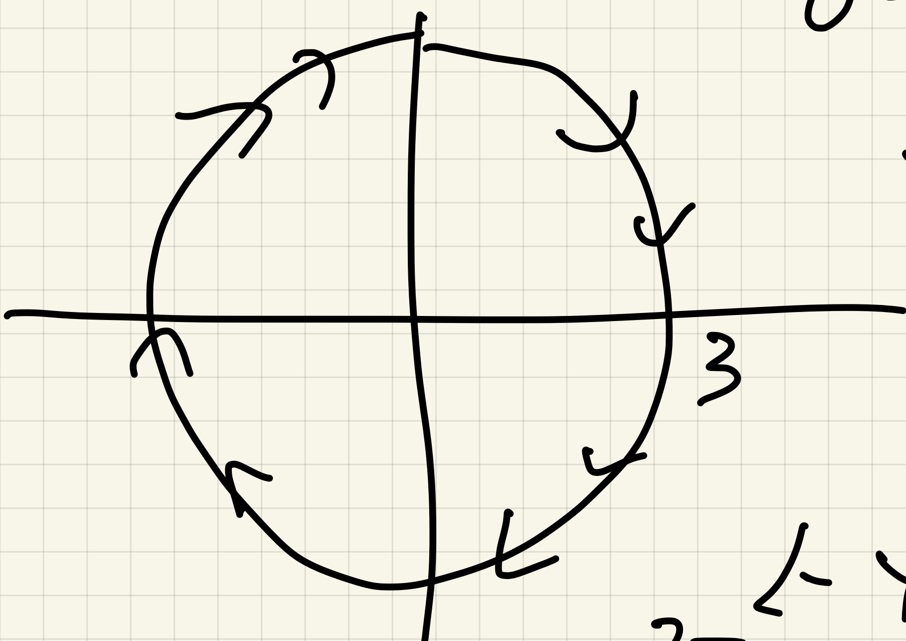
$\bar{r}'(t) = \langle -\sin t, \cos t \rangle$

$$\int_0^{2\pi} \sin^2 t + \cos^2 t \, dt$$

$$\int_0^{2\pi} 1\,dt = t\Big|_0^{2\pi} = 2\pi$$

$$C_2: \quad \bar{r}(t) = \langle \overset{x}{3\sin t}, \overset{y}{3\cos t} \rangle$$

$$0 \le t \le 2\pi$$



$$\bar{r}'(t) =$$

$$\underline{\langle 3\cos t, -3\sin t \rangle}$$

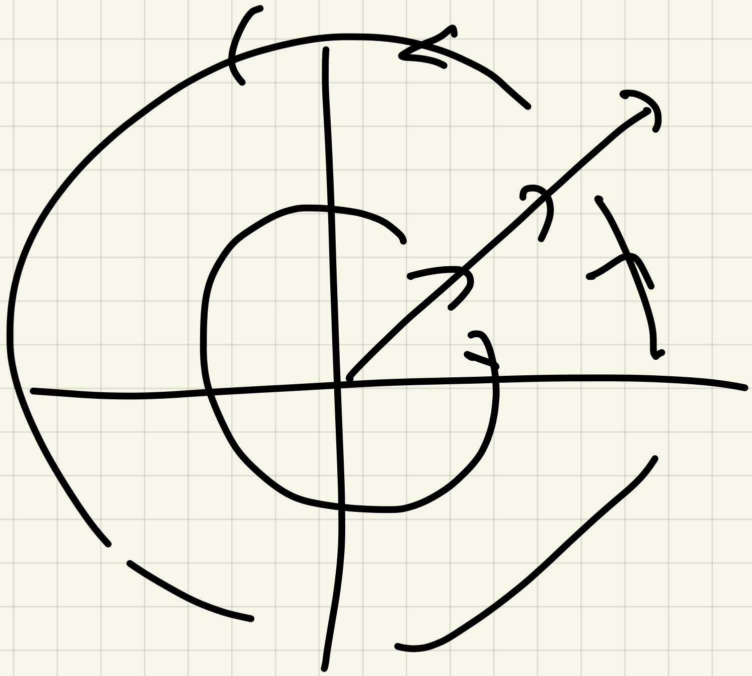
$$\langle -y, x \rangle$$

$$\int_{C_2} F \cdot d\bar{r} = \int_0^{2\pi} \langle -3\cos t, 3\sin t \rangle \cdot \langle 3\cos t, -3\sin t \rangle \, dt$$

$$//$$

$$\int_0^{2\pi} -9\,dt = -9t\Big|_0^{2\pi} = -18\pi$$

(c) $\vec{r}(t) = \langle \overset{x}{t}, \overset{y}{t} \rangle, \quad 0 \le t \le 2\pi$



$F(\vec{r}(t)) =$

$\langle -t, t \rangle$

$\vec{r}'(t) = \langle 1, 1 \rangle$

$$\int \langle -t, t \rangle \cdot \langle 1, 1 \rangle \, dt$$

$$\int_0^{2\pi} 0 \, dt = 0$$

Alternate Notation:

If $\quad F = (\underset{x}{M}, \underset{y}{N}, \underset{z}{P})$

$$\int_C \bar{F} \cdot dr = \int_C M\,dx + N\,dy + P\,dz$$

$\left(\text{Also } \int \bar{F} \cdot dr = \int_C M\,dx + N\,dy \right)$ in 2-D

<u>Calculate</u>: $\quad C : \bar{r}(t) = \langle \underline{x(t)}, \underline{y(t)}, \underline{z(t)} \rangle$

$$\int_C M\,dx + N\,dy + P\,dz$$

$$\parallel$$

$$\int \left(M \frac{dx}{dt} + N \frac{dy}{dt} + P \frac{dz}{dt} \right) dt$$

(similar in 2-D)

<u>Ex1</u> $\int_C y\,dx$

$C : \bar{r}(t) = \langle \underline{\underline{t}}, \underline{\underline{t}} \rangle, \quad 0 \le t \le 2\pi$

$$\int_C \underline{\underline{y}}\,dx = \int_0^{2\pi} t \frac{dx}{dt} \cdot dt$$

$$\parallel$$

$$\int_0^{2\pi} t \cdot 1 \cdot dt = \frac{t^2}{2}\Big|_0^{2\pi} = \frac{4\pi^2}{2} = 2\pi^2$$

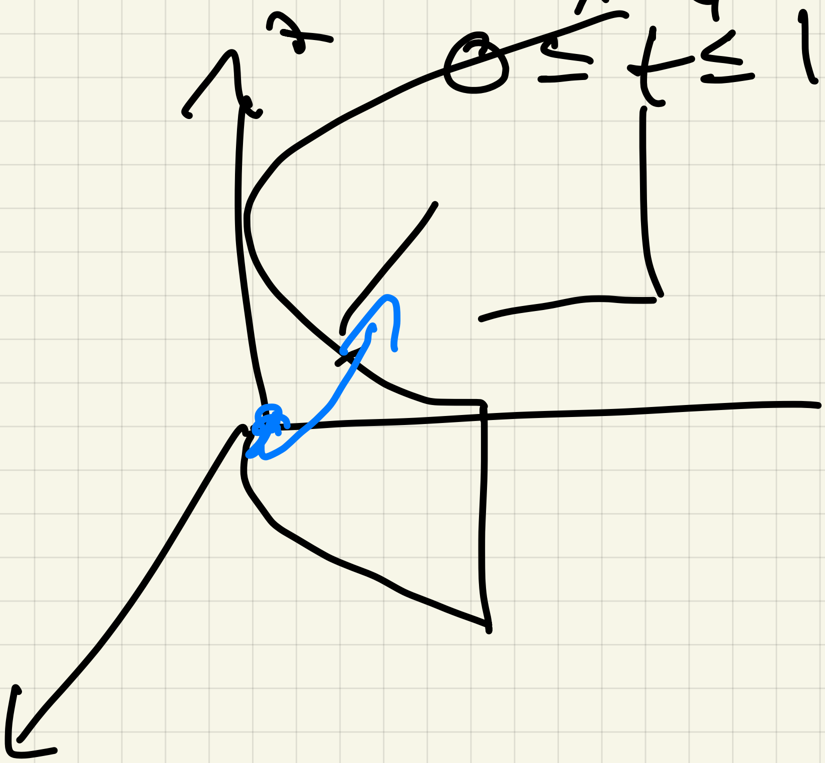
$F(x,y) = \langle y, 0 \rangle$

$\bar{r}(t) = \langle t, t \rangle, \quad 0 \le t \le 2\pi,$

Ex3 $F(x,y,z) = \langle -x, -y, -z \rangle$

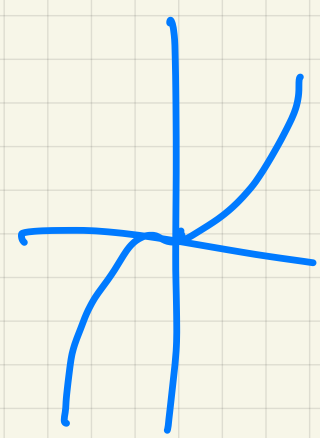
$C: \bar{r}(t) = \langle \underset{x}{t}, \underset{y}{t^2}, \underset{z}{t^3} \rangle \leftarrow$

$0 \le t \le 10$



$y = x^2$

$z = x^3$



$$\int_C F \cdot d\bar{r} =$$

$$\int -x\,dx + -y\,dy + -z\,dz$$

$$\parallel$$

$$\int_0^{10} (-t)\cdot 1 + (-t^2)(2t) - t^3(3t^2)\,dt$$

$r = \langle t, t^2, t^3 \rangle$

$\quad x \quad y \quad z$

$\frac{dx}{dt} = 1 \qquad \frac{dy}{dt} = 2t$

$\frac{dz}{dt} = 3t^2$

$$\int_0^{10} -t - 2t^3 - 3t^5\,dt$$

$$-t^2/2 - t^4/2 - t^6/2 \Big|_0^{10}$$

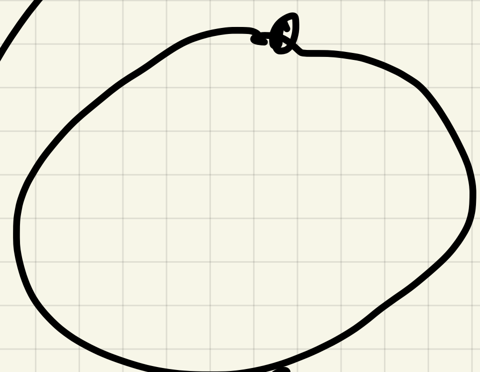
$-50 - 5000 - 500000$

-5005050

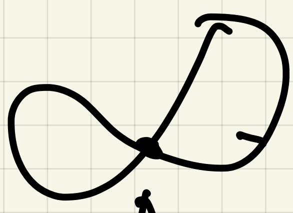
A second Integral:

If C is a <u>simple</u> <u>closed</u> curve

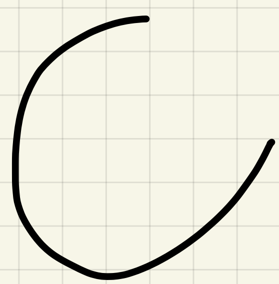
doesn't cross itself



starting = finish

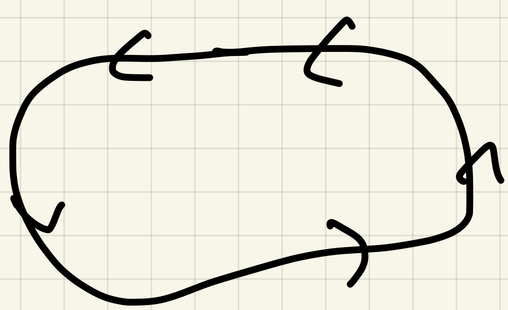


not simple



not closed

<u>Motivate</u>



vector field F = flow of liquid

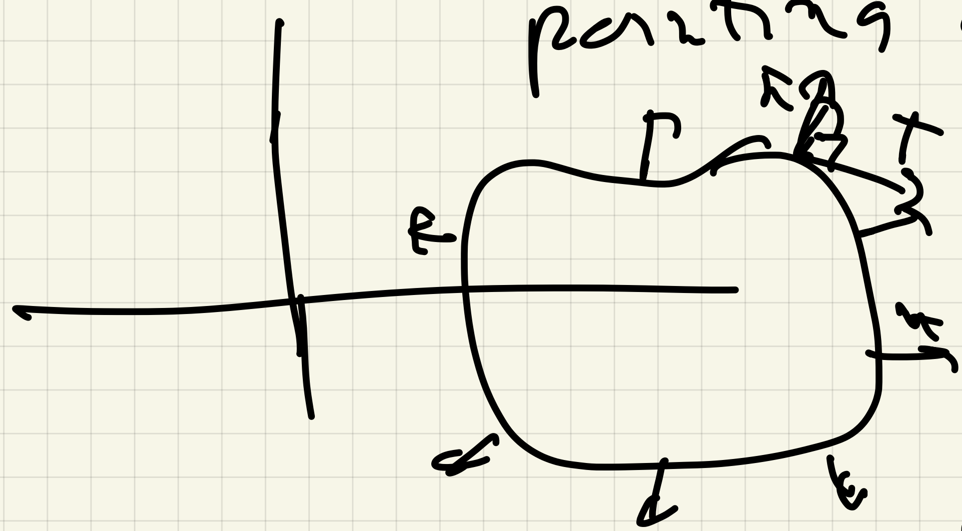
How much liquid crosses C in unit time?

The integral to compute is

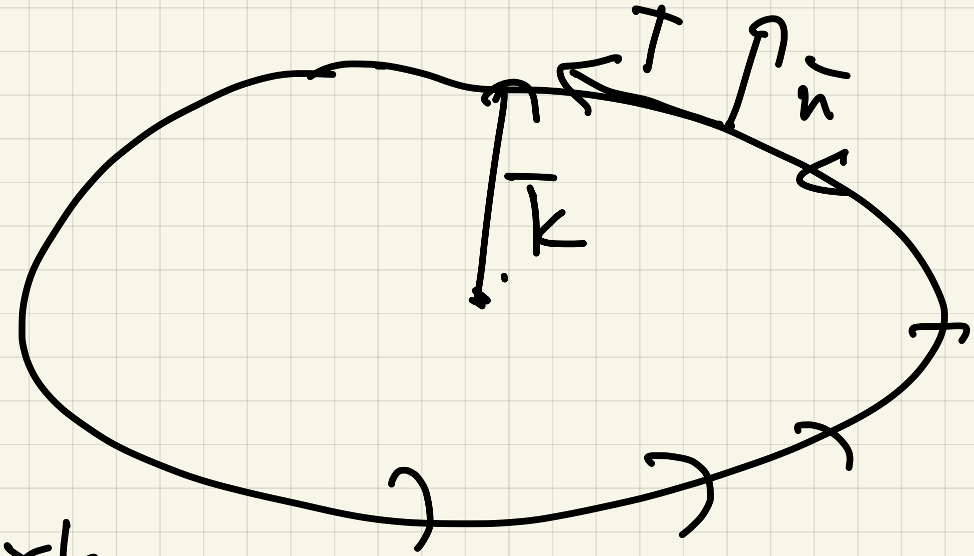
Flux of F across C =

$$\int_C F \cdot \bar{n} \, ds$$

unit normal to C pointing outward



How to compute it?



$\bar{n} = T \times k$

C counterclockwise

$\bar{n} = ??$

$$\left\langle \frac{dx}{ds}, \frac{dy}{ds}, 0 \right\rangle \times \langle 0, 0, 1 \rangle$$

$$T \quad \begin{vmatrix} i & j & k \\ \frac{dx}{ds} & \frac{dy}{ds} & 0 \\ 0 & 0 & 1 \end{vmatrix} =$$

$$\left\langle \underbrace{\frac{dy}{ds}}_{M}, \underbrace{-\frac{dx}{ds}}_{N}, 0 \right\rangle$$

x ?

<u>Conclusion</u>: If $F = (M, N)$

$$\int_C F \cdot \bar{n}\, ds =$$

$$M\frac{dy}{ds} - N\frac{dx}{ds}$$

So flux instead is

$$\oint_C M\,dy - N\,dx$$

(C counterclockwise)

Ex 3

$$\vec{F}(x,y) = \langle \overset{M}{y}, \overset{N}{x} \rangle$$

$$C: \langle \overset{x}{\cos t}, \overset{y}{\sin t} \rangle$$

$$0 \le t \le 2\pi$$

find flux of $\vec{F}$ across C

$$\int M\,dy - N\,dx$$

$$\int_0^{2\pi} \underline{(\sin t)(\cos t)} - \underline{(\cos t)(-\sin t)}\,dt$$

$\frac{dx}{dt} = -\sin t \;/\; \frac{dy}{dt} = \cos t$

$$\int_0^{2\pi} 2 \sin t \cos t \, dt =$$

$$\sin^2 t \Big|_0^{2\pi} = 0$$