3/27 Calc3 Quiz 12

$$g(x,y) = x^3 - 12x + y^3 + 3y^2$$

$$\nabla g = \langle \underset{\overset{\shortparallel}{0}}{3x^2 - 12},\ \underset{\overset{\shortparallel}{0}}{3y^2 + 6y} \rangle$$

$$3(x-2)(x+2) \qquad 3y(y+2)$$

$$x = 2, -2 \qquad y = 0, -2$$

$(2,0)$	$(2,-2)$	$(-2,0)$	$(-2,-2)$
$d = 72 > 0$	$d = -72 < 0$	$d = -72 < 0$	$72 > 0$
$g_{xx} = 12 > 0$ $\cup$	saddle	saddle	$g_{xx} = -12$
Local min			$\cap$ local max

$$g_{xx} = 6x$$

$$g_{yy} = 6y + 6$$

$$g_{xy} = 0$$

$$d = \det\begin{pmatrix} 6x & 0 \\ 0 & 6y+6 \end{pmatrix}$$

Last time: Double integral

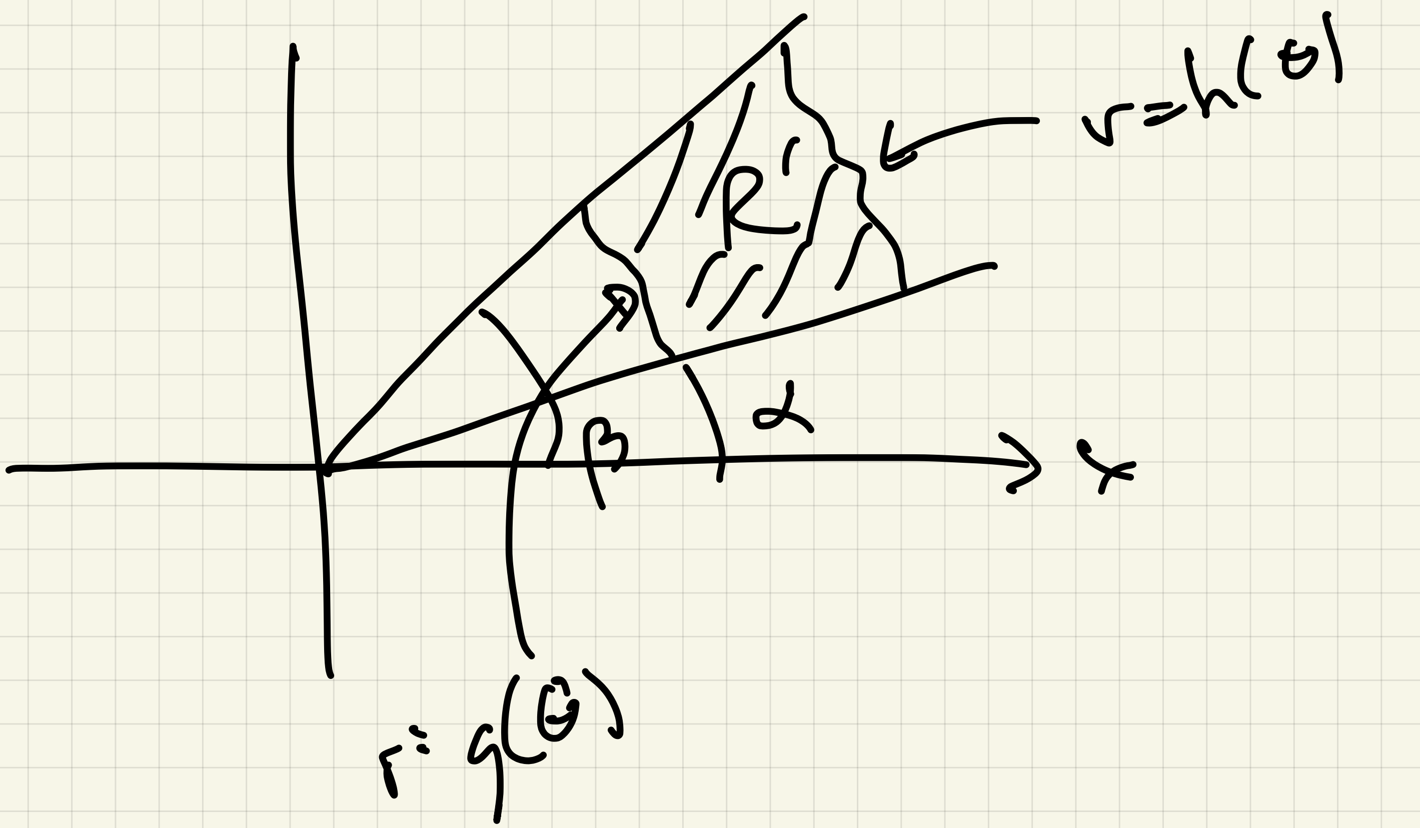
$$\iint_R f(x,y)\,dA$$

Iterated integrals and switching order

$\boxed{\text{Skip 14.3}}$

<u>14.4</u> <u>polar coordinates</u>

If region R is described by

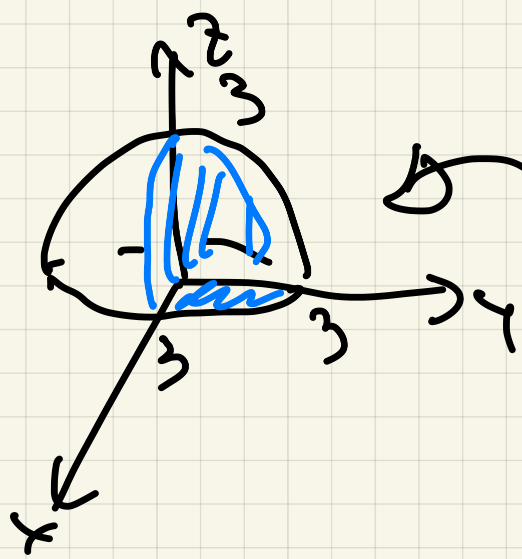


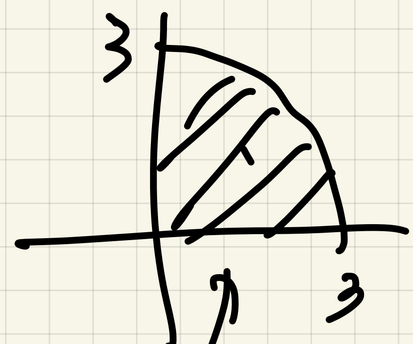
Then

$$\iint_R f(\underline{x},\underline{y})\,dA = \int_{\theta=\alpha}^{\theta=\beta}\int_{r=g(\theta)}^{r=h(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

conversion factor

$\underline{\text{Ex}}$: $f(x,y) = \sqrt{9-x^2-y^2}$



$R =$ 

quarter circle

$$\iint_R \sqrt{9-x^2-y^2}\,dA = \text{Volume of } \frac{1}{8} \text{ sphere radius } 3$$

$\mathbin{/\!/}$

$$\int_0^3 \int_0^{\sqrt{9-x^2}} \sqrt{9-x^2-y^2}\,dy\,dx$$

$$\parallel \quad \text{polars:}$$

$$\int_0^{\pi/2} \int_0^3 \sqrt{9-r^2}\,\boxed{r\,dr}\,d\theta$$

$x = r\cos\theta$

$y = r\sin\theta$

$u = 9 - r^2$

$du = -2r\,dr$

$-\frac{1}{2}\,du = \boxed{r\,dr}$

Switch var

$$-\int_{u=9}^{u=0} \frac{1}{2}\sqrt{u}\,du = \int_0^9 \frac{1}{2}\sqrt{u}\,du =$$

$$\frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{3}\,u^{3/2}\Big|_0^9 = \frac{1}{3}\left(9^{3/2} - 0^{3/2}\right)$$

$$= \frac{1}{3}(27) = 9$$

$$\int_0^{\pi/2} 9 \, d\theta = \frac{9\pi}{2}.$$

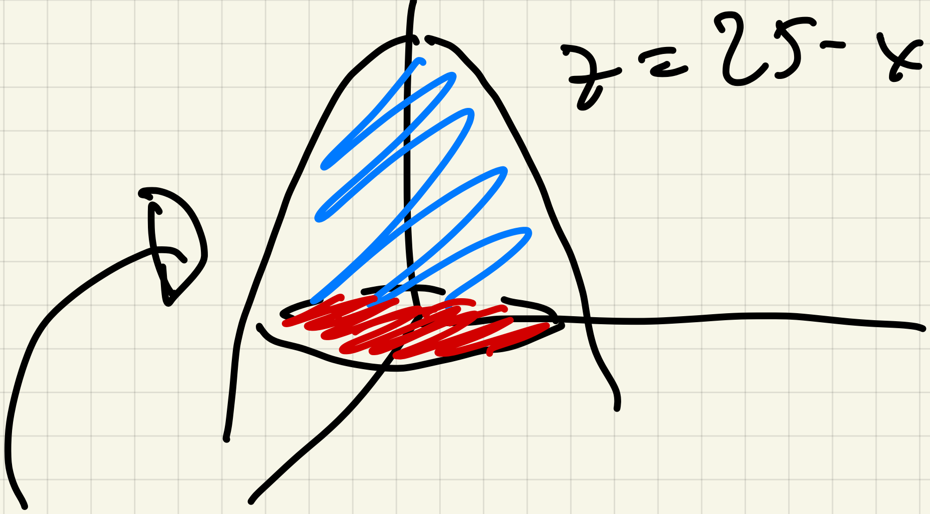
Check: Radius r sphere has volume $V = \frac{4}{3}\pi r^3$,

so volume $\frac{1}{8}\left(\frac{4}{3}\pi \cdot 3^3\right) =$

$$\frac{1}{2} \cdot 3^2 \pi = \frac{9\pi}{2} \checkmark$$

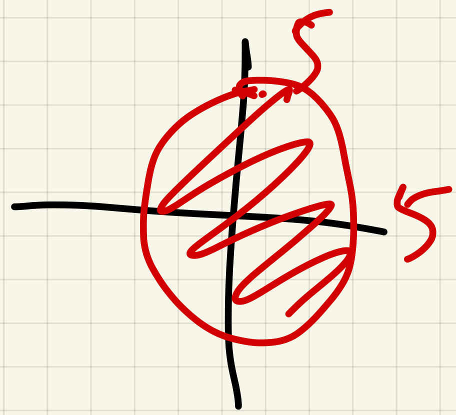
Ex 2

$z = 25 - x^2 - y^2$



Volume above xy-plane?

$$V = \iint_R 25 - \underline{x^2 - y^2}\, dA$$

$$R = $$ 

$$\int_0^{2\pi}\int_0^5 \underbrace{(25 - r^2)\, r}_{25r - r^3}\, dr\, d\theta$$

$$\underbrace{\left.\frac{25}{2} r^2 - \frac{r^4}{4}\right|_0^5} =$$

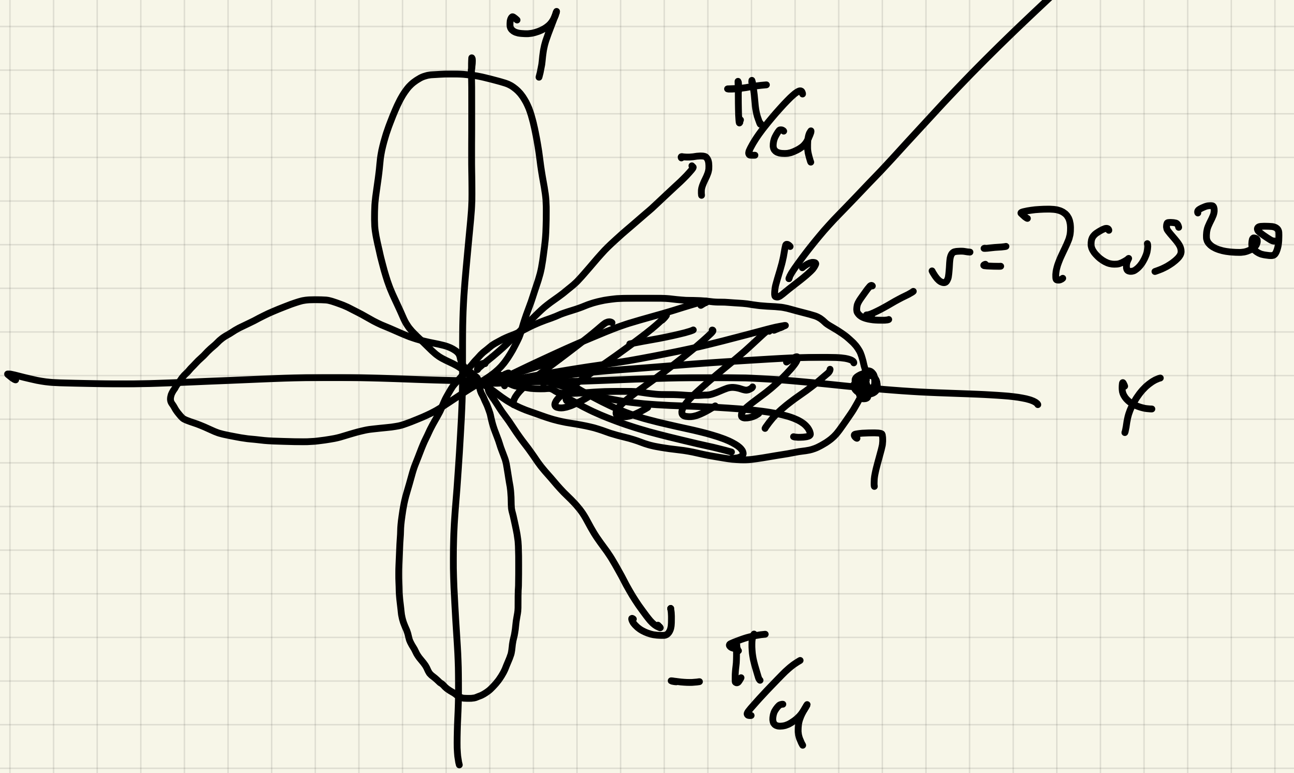
$$\frac{25}{2} \cdot 25 - \frac{5^4}{4} = 625\left(\frac{1}{2} - \frac{1}{4}\right)$$

$$\frac{1}{2} - \frac{1}{4} \qquad = \frac{625}{4}$$

$$\int_0^{2\pi} \frac{625}{4}\, d\theta = \frac{625}{4}(2\pi) = \frac{625\pi}{2}$$

Ex 3 Find area of region R

$r = 7\cos 2\theta$



$$\text{Area } R = \iint_R 1\, dA =$$

$=$

$$\int_{-\pi/4}^{\pi/4} \underbrace{\int_0^{7\cos 2\theta} 1\, r \, dr}_{} \, d\theta =$$

$$\left. \frac{r^2}{2} \right|_0^{7\cos 2\theta} =$$

$$\int_{-\pi/4}^{\pi/4} \frac{49 \boxed{\cos^2 2\theta}}{2} \, d\theta$$

$$\left(\cos^2 x = \frac{1+\cos 2x}{2} \right)$$

$$\frac{49}{2} \int \frac{1+\cos 4\theta}{2} \, d\theta$$

$$\left. \frac{49}{4} \quad \theta + \frac{1}{4} \cancel{\sin 4\theta} \right|_{-\pi/4}^{\pi/4} = \quad \to 0$$

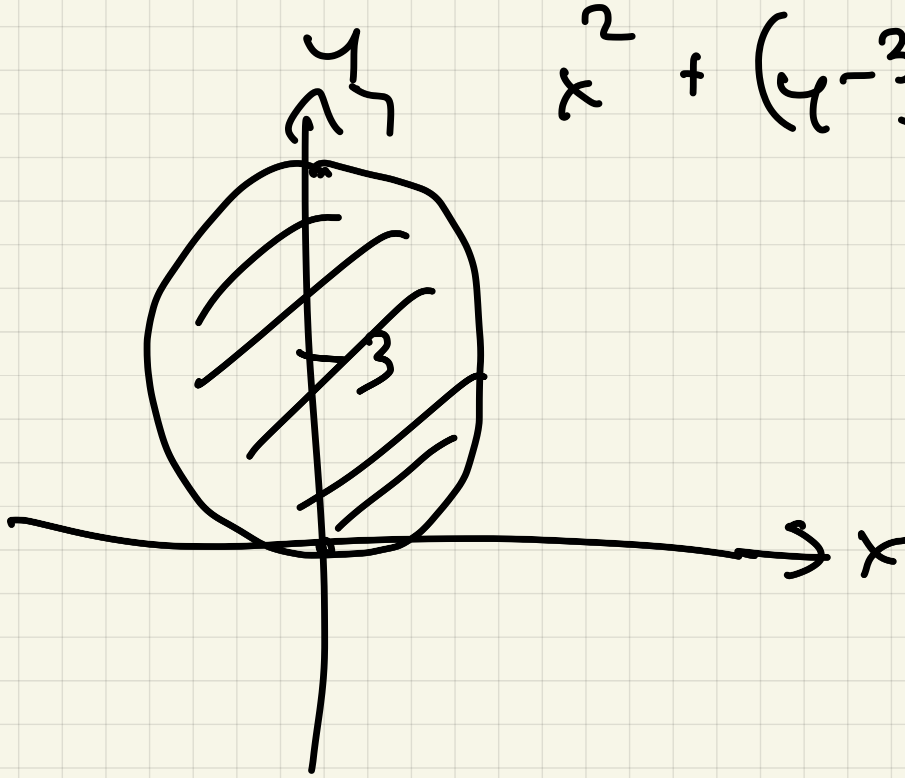
$$\frac{49}{4}\left(\frac{\pi}{4} - -\frac{\pi}{4} \right) = \frac{49\pi}{8}$$

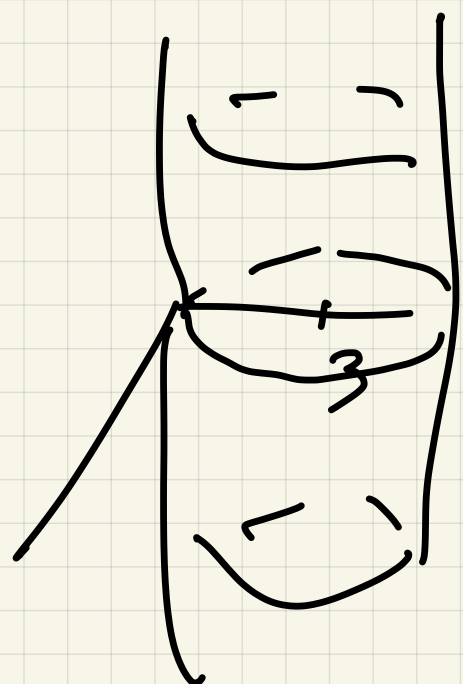
Ex 4 Find volume of region bounded by $\underline{xy \text{ plane}}$,
the cone $z = \sqrt{x^2+y^2}$
and $\underline{\text{cylinder} \quad x^2 - 6y + y^2 = 0}$

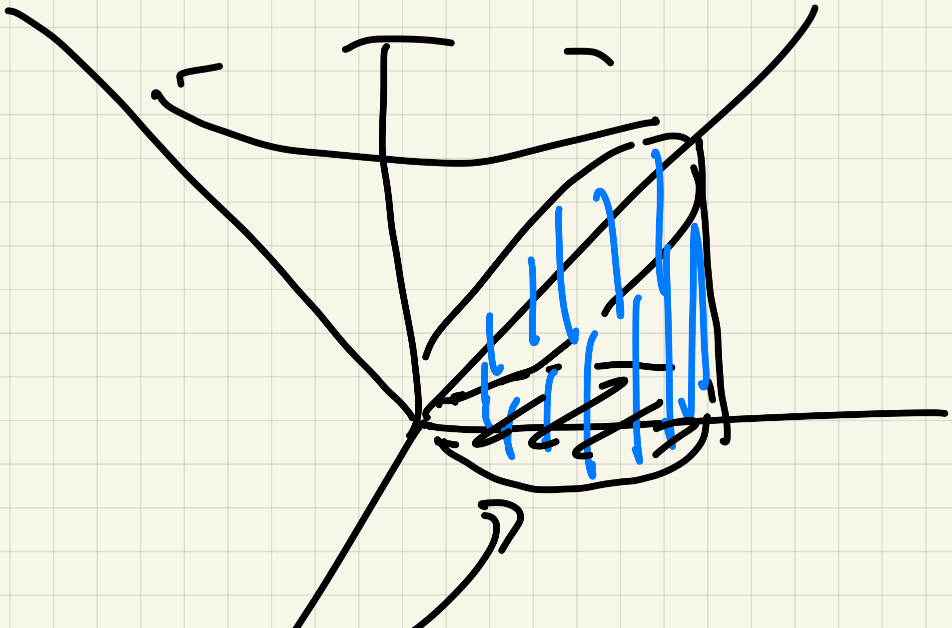
Try to $\underline{\text{visualize}}$ cylinder:

$$x^2 + \underbrace{y^2 - 6y + 9} = 9$$

$$x^2 + (y-3)^2 = 3^2$$







R doesn't look polar,

but it is:

$x = r\cos\theta$

$y = r\sin\theta$

$$x^2 - 6y + y^2 = 0$$

$$\Downarrow$$

$$r^2\cos^2\theta - 6r\sin\theta + r^2\sin^2\theta = 0$$

$$r^2 - 6r\sin\theta$$

$$r(r - 6\sin\theta) = 0$$

$$\boxed{r = 6\sin\theta}$$

$$\text{Volume} = \iint_R \sqrt{x^2+y^2}\, dA =$$

$$\int_0^{\pi}\int_0^{6\sin\theta} \underbrace{r \cdot r\, dr}\, d\theta$$

$$\frac{1}{3} r^3 \Big|_0^{6\sin\theta}$$

$$\int_0^{\pi} \underbrace{\frac{1}{3} 216}\, \sin^3\theta \, d\theta =$$

$$72 \int_0^{\pi} \sin^3\theta \, d\theta \quad ??$$

11

$$\underbrace{\sin^2\theta}\ \sin\theta$$

$$72\int_0^{\pi}(1-\cos^2\theta)\,\underline{\sin\theta}\,d\theta$$

$$u = \cos\theta$$
$$du = -\sin\theta$$

$$72\int_1^{-1} -(1-u^2)\,du =$$

$$72\int_{-1}^{1}(1-u^2)\,du =$$

$$72\left(u - \frac{u^3}{3}\Big|_{-1}^{1}\right) =$$

$$72\left(\frac{2}{3} - \left(-\frac{2}{3}\right)\right) =$$

$\frac{4}{3}\cdot 72 = 96$.

Rmk

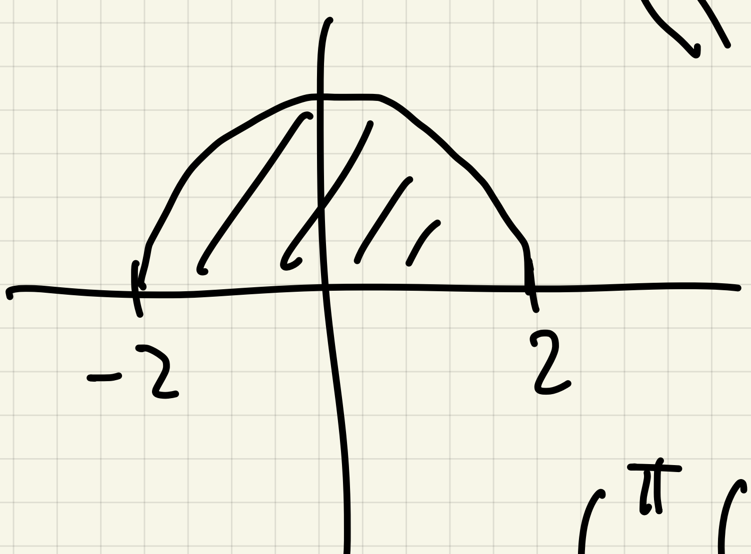
$\boxed{\text{Stat}}$: $\int_0^{\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$??

Ex 5

$$\int_{-2}^{2}\int_0^{\sqrt{4-x^2}} \cos(x^2+y^2)\,dy\,dx$$

(calc impossible)

$\parallel$



$$\iint_R \cos(x^2+y^2)\,dA$$

$\parallel$

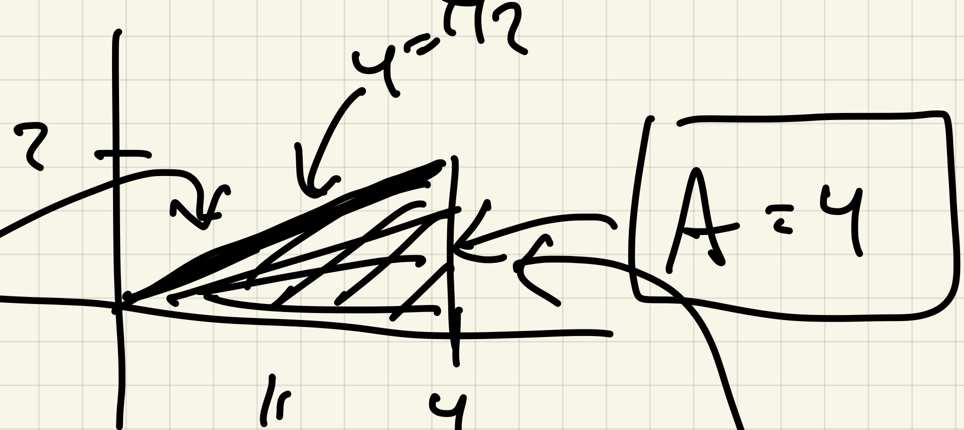
$$\int_0^{\pi}\int_0^{2} \underbrace{(\cos r^2)\cdot \underline{r}}\,dr\,d\theta$$

$$\frac{1}{2}\sin r^2 \Big|_0^2$$

$\parallel$

$$\int_0^{\pi} \frac{1}{2}\sin 4 = \frac{1}{2}\pi \sin 4 < 0$$

Ex 6 Describe with polars:



$$\int_0^4 \int_0^{x/2} 1 \, dy\, dx = 4$$

polars:

line

$\theta = \arctan \frac{1}{2}$

$x = 4$

$r\cos\theta = 4$

$r = \frac{4}{\cos\theta} = 4\sec\theta$

so $A = \int_0^{\tan^{-1}\frac{1}{2}} \int_0^{4\sec\theta} r \, dr \, d\theta$

$$+\frac{1}{2}r^2\Big|_0^{4\sec\theta} =$$

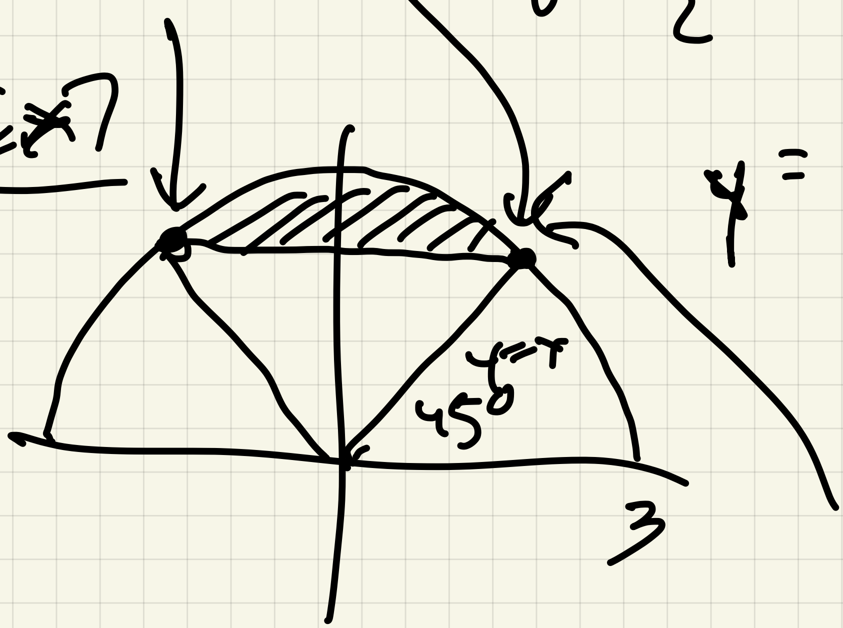
$$\frac{1}{2}\left(16\sec^2\theta\right) =$$

$$\int_0^{\tan^{-1}\frac{1}{2}} 8\sec^2\theta \, d\theta$$

$$8\tan\theta\Big|_0^{\tan^{-1}(\frac{1}{2})} =$$

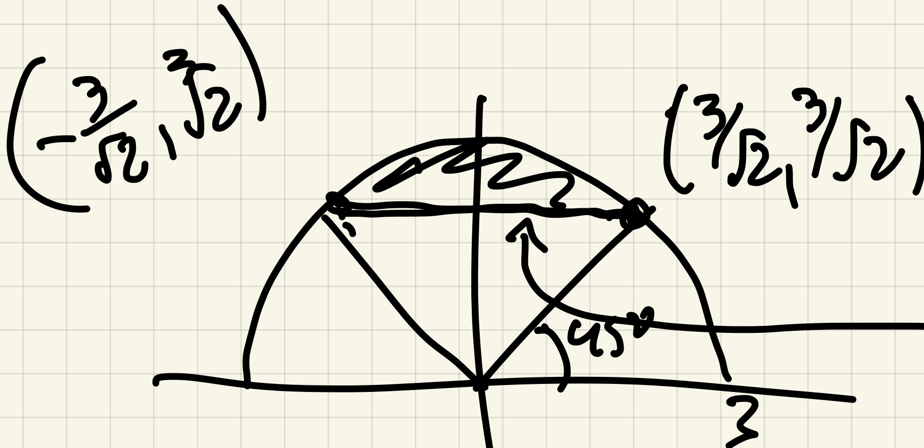
$$8\cdot\frac{1}{2} = 4 \checkmark$$

[Ex7]



$y = \sqrt{9-x^2}$ half circle

find Area:



line $y = \frac{3}{\sqrt{2}}$

$$A = \int_{-3/\sqrt{2}}^{3/\sqrt{2}} \int_{3/\sqrt{2}}^{\sqrt{9-x^2}} dy\,dx$$

<u>works</u>

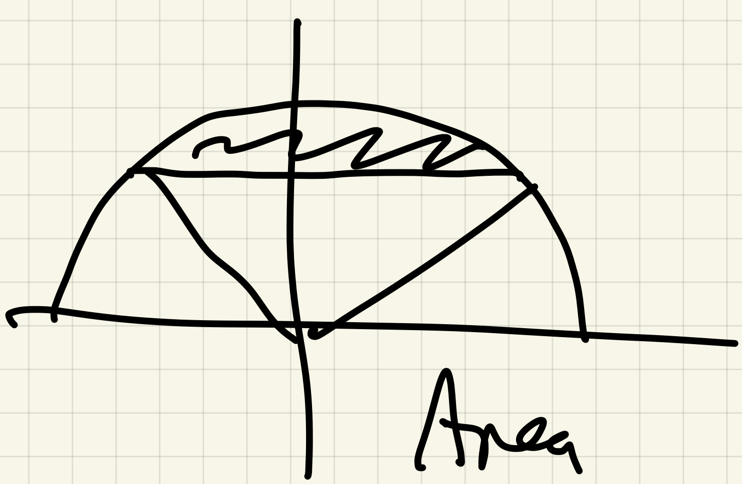
<u>Polar sc</u>

$$\int_{\pi/4}^{3\pi/4} \int_{\frac{3}{\sqrt{2}}\csc\theta}^{3} r\,dr\,d\theta$$

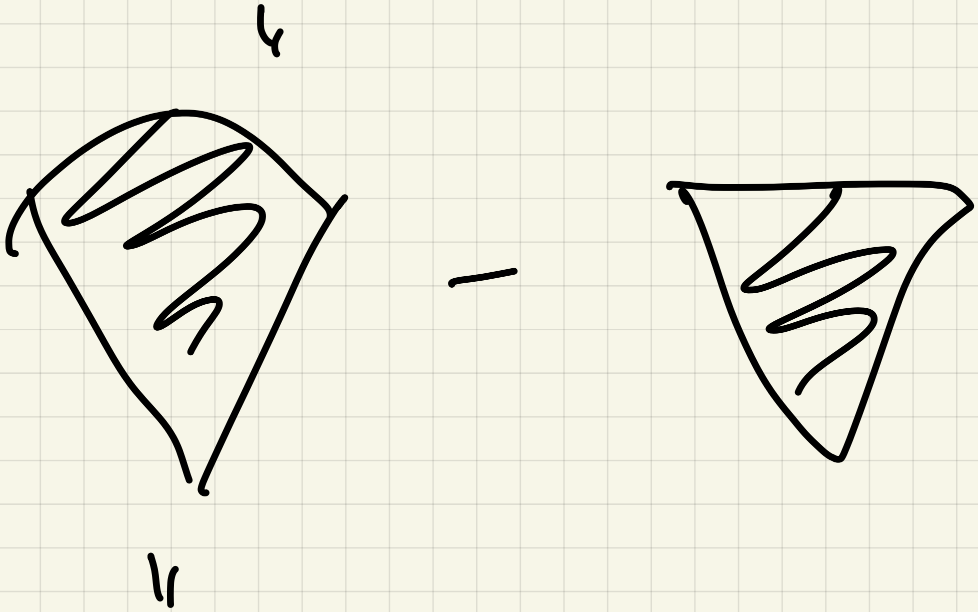
$$r\sin\theta = \frac{3}{\sqrt{2}}$$

$$r = \frac{3}{\sqrt{2}\sin\theta}$$

$$= \frac{3}{\sqrt{2}}\csc\theta$$



Area



$$\frac{9\pi}{4} - \frac{9}{2}$$

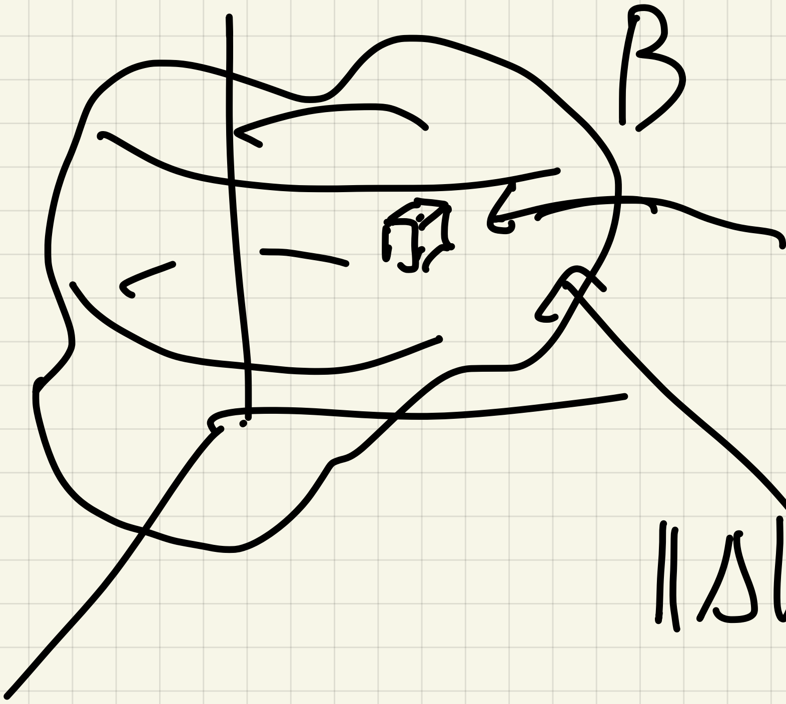
§14.6

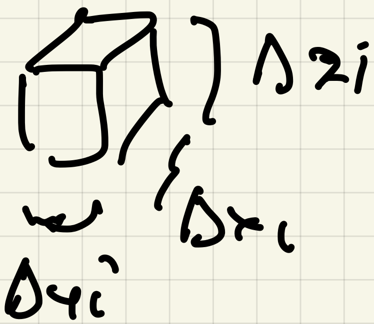
Defn: If $f(x,y,z)$ is a function on a solid region B, the triple integral of $f(x,y,z)$ over B is

$$\iiint_B f(x,y,z)\,dV =$$

$$\lim_{\|\Delta\| \to 0} \sum f(x_i, y_i, z_i)\, \Delta V_i$$

$\Delta V_i = \Delta x_i \Delta y_i \Delta z_i$





$\|\Delta\|$ = maximal length of diagonal in rectangular solid

(x_i, y_i, z_i) point

① $\iiint_B 1 \, dV$ = Volume

② $\iiint_B \rho(x,y,z)\, dV$ = total mass

ρ: density