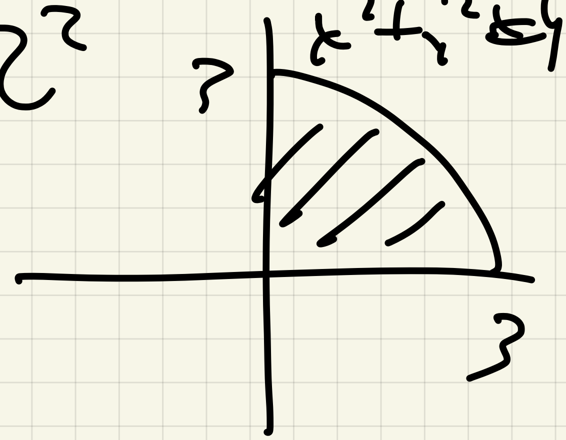
3/25/ Calc3

Last time — 14.1–14.2

(1) R = region in $\mathbb{R}^2$

$f(x,y)$ function



Double Integral

$$\iint_R f(x,y)\,dA$$

= signed volume under graph

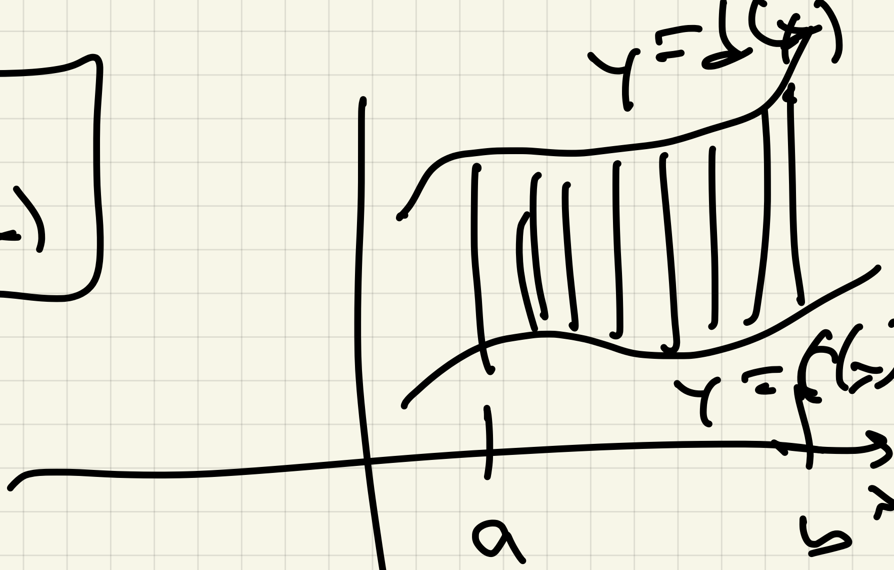
(2) **Iterated integral**

(a) $$\boxed{\int_a^b \int_{c(x)}^{d(x)}} f(x,y)\,dy\,dx$$

order

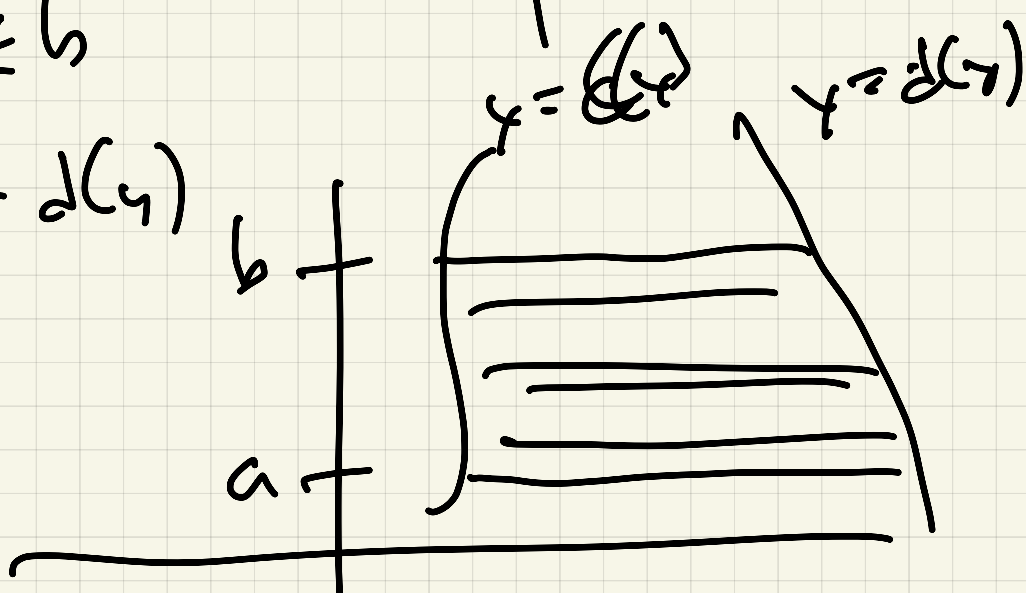
(b) $$\int_a^b \int_{c(y)}^{d(y)} f(x,y)\,dx\,dy$$

$$\boxed{\begin{aligned} a &\le x \le b \\ c(x) &\le y \le d(x) \end{aligned}}$$



$$a \le y \le b$$

$$c(y) \le x \le d(y)$$



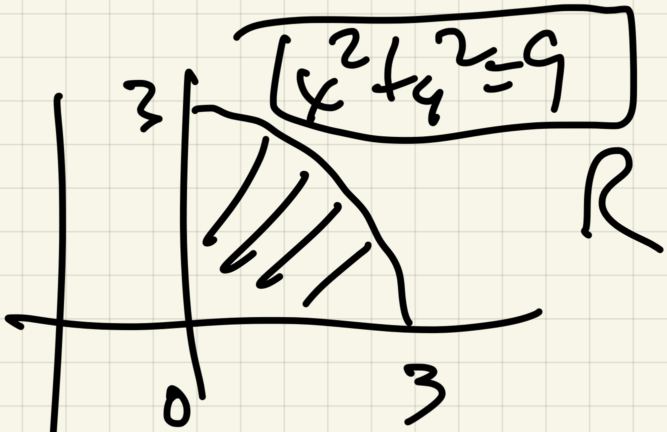
Fubini's Theorem:

If endpoints corresp to R

then double integral

=

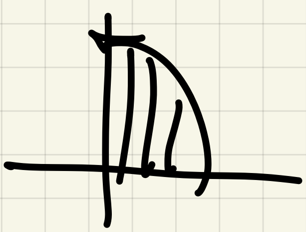
iterated integral



$$f(x,y)=\sqrt{9-x^2-y^2}$$

$$\iint_R \sqrt{9-x^2-y^2}\,dA$$

$$= \int_0^3\int_0^{\sqrt{9-x^2}} \sqrt{9-x^2-y^2}\,dy\,dx$$

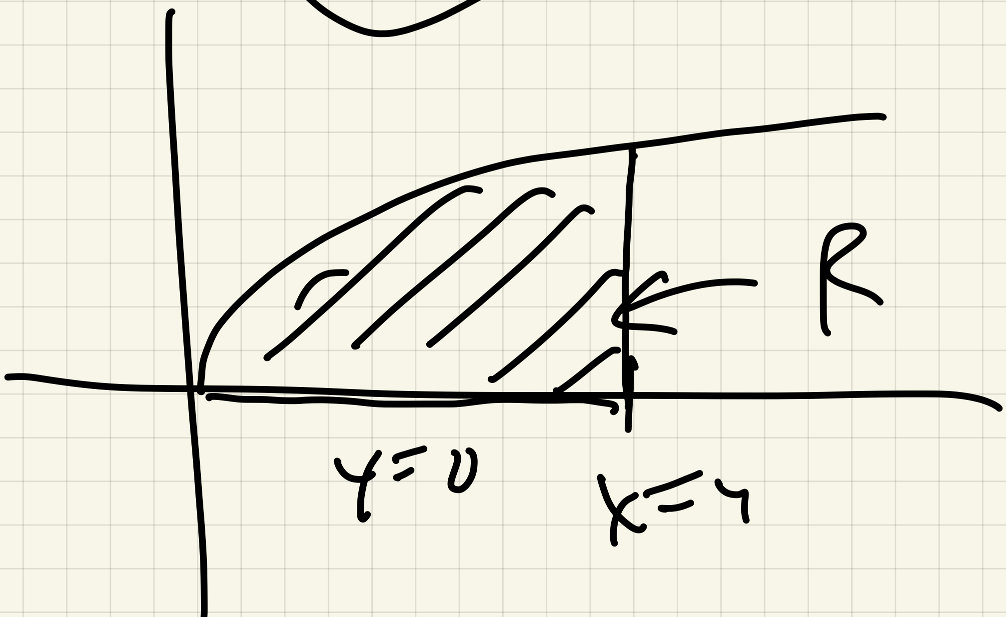


$$\int_0^3\int_0^{\sqrt{9-y^2}} \sqrt{9-x^2-y^2}\,dx\,dy$$

<u>Ex 0</u> Find volume of solid

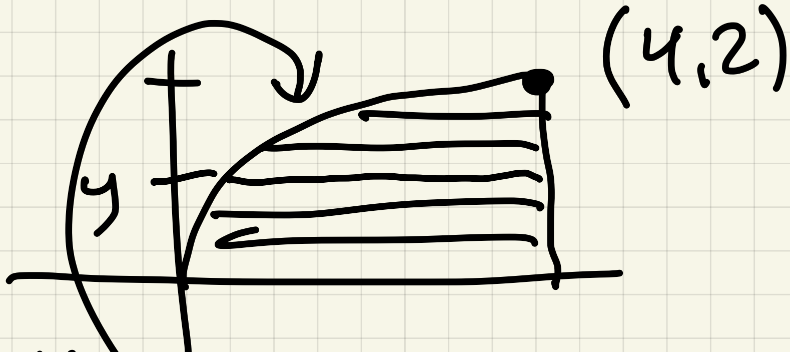
bounded by $y=0$, $x=4$, $y=\sqrt{x}$

$$0 \le z \le \frac{y}{1+x^2}$$



Two reasonable approaches:

dxdy

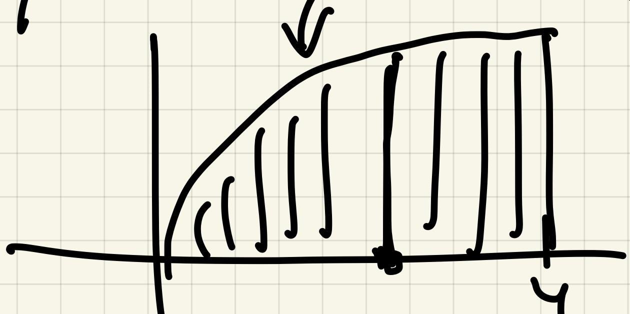


(1) $$\int_0^2 \int_{y^2}^4 \frac{y}{1+x^2}\,dx\,dy$$

$y=\sqrt{x}$

$x=y^2$

(2)



$y=\sqrt{x}$

$$\int_0^4 \int_0^{\sqrt{x}} \frac{y}{1+x^2}\,dy\,dx$$

$x \qquad y$

1

$$\int_0^2 y\arctan x \Big|_{y^2}^{4} =$$

$$\int_0^2 y\arctan 4 - y\arctan y^2\,dy$$

2

$$\int_0^4 \left(\frac{1}{1+x^2}\right) \frac{1}{2}y^2 \Big|_0^{\sqrt{x}} =$$

$$\int_0^4 \frac{\frac{1}{2}x}{1+x^2}\,dx$$

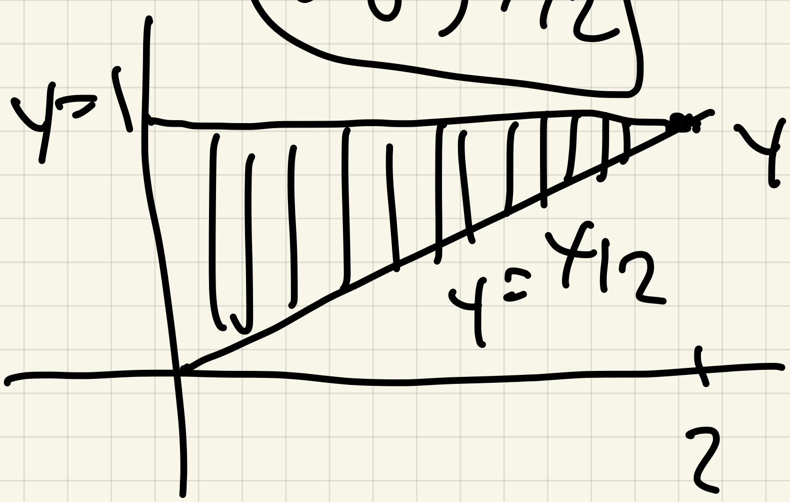
$u = 1+x^2, \; du = 2x\,dx$

$$\frac{1}{4}\,du = \frac{1}{2}x\,dx$$

$$\int_1^{17} \frac{1}{4}\frac{du}{u} = \frac{1}{4}\ln u \Big|_1^{17} = \frac{1}{4}\ln 17$$

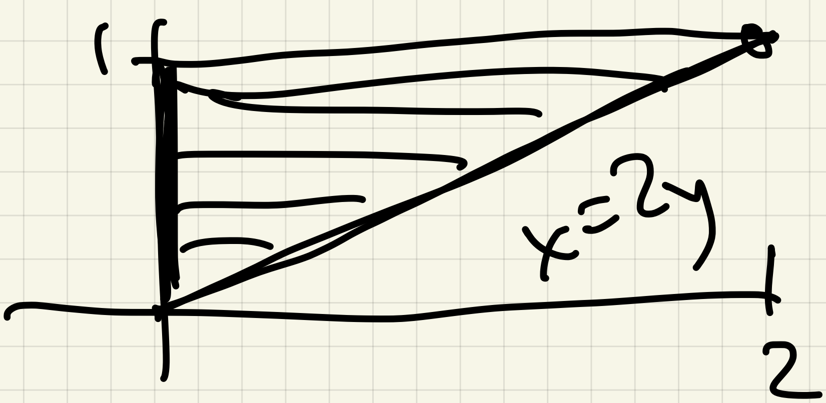
Ex 2 Evaluate

$$\boxed{\int_0^2 \int_{x/2}^1} x\sqrt{5+y^3}\,dy\,dx$$



$y = x/2 \quad 0 \le x \le 2$

$\frac{x}{2} \le y \le 1$



$0 \le y \le 1$

$0 \le x \le 2y$

$$\int_0^1 \underbrace{\int_0^{2y} x\sqrt{5+y^3}\,dx}_{Y}\,dy$$

$$\sqrt{5+y^3}\ \frac{1}{2}x^2 \Big|_{x=0}^{x=2y} =$$

$$\int_0^1 2y^2\sqrt{5+y^3}\,dy = \int \frac{2}{3}\sqrt{u}\,du$$

$$u = 5+y^3$$

$$du = 3y^2\,dy$$

$$\frac{2}{3}\,du = 2y^2\,dy$$

$$= \frac{2}{3}\cdot\frac{2}{3}u^{3/2} =$$

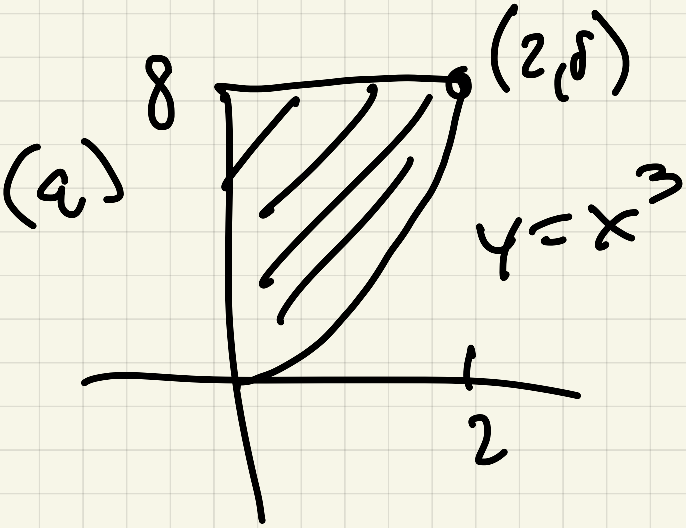
$$\frac{4}{9}(5+y^3)^{3/2}\Big|_0^1 =$$

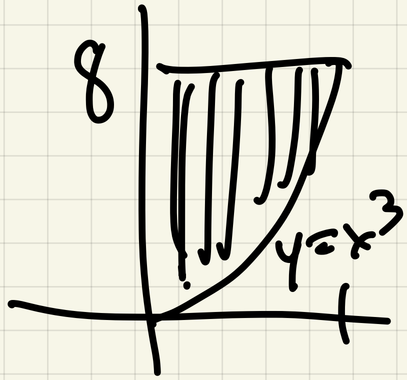
$$\frac{4}{9}\left(6^{3/2} - 5^{3/2}\right).$$

Ex 2 Dealing with regions endpoints:

Write iterated integrals to evaluate $\iint_R 1\,dA$

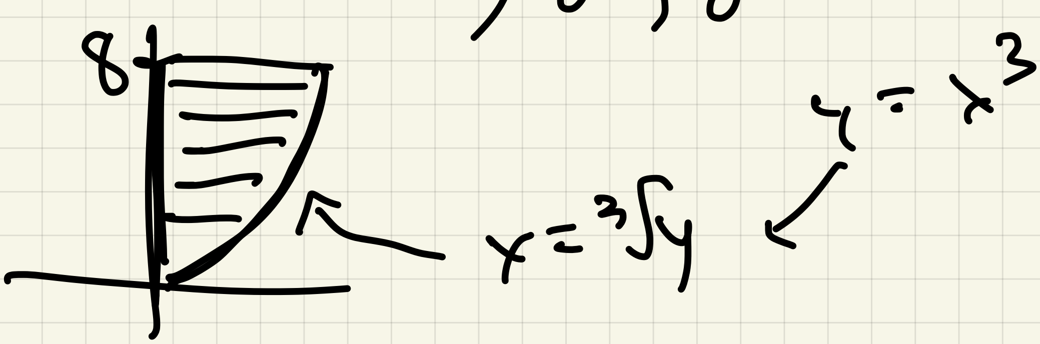
(a)



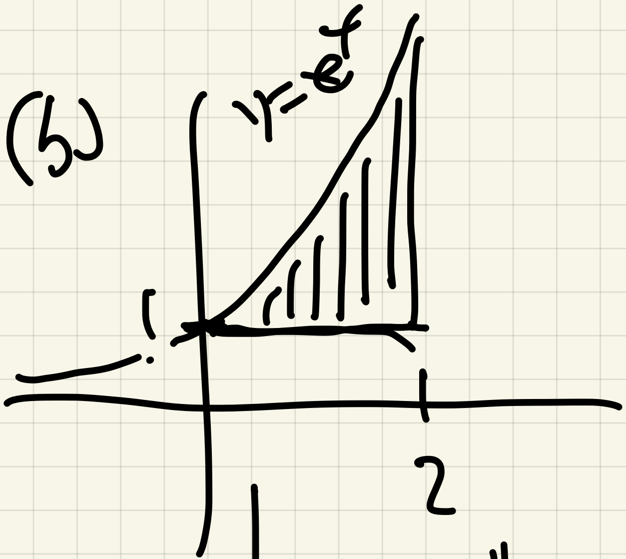


$$\iint 1\,dy\,dx = \int_0^2 \int_{x^3}^{8} 1\,dy\,dx$$

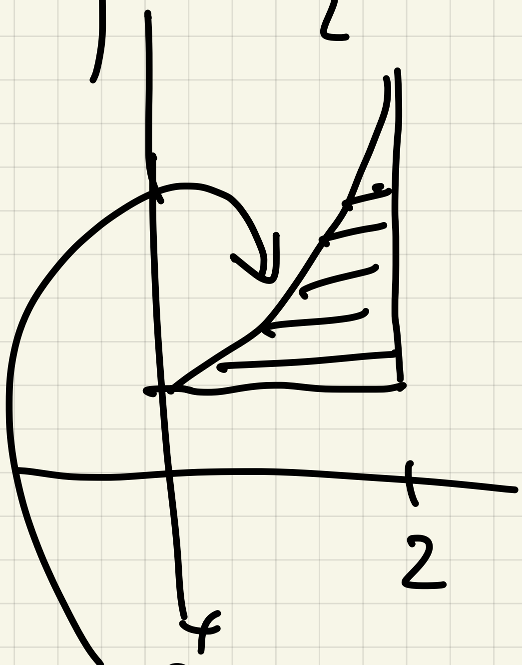
$$\iint 1\,dx\,dy = \int_0^8 \int_0^{\sqrt[3]{y}} 1\,dx\,dy$$



(b)



$$\int_0^2\int_1^{e^x} 1\,dy\,dx$$



$y=e^x$

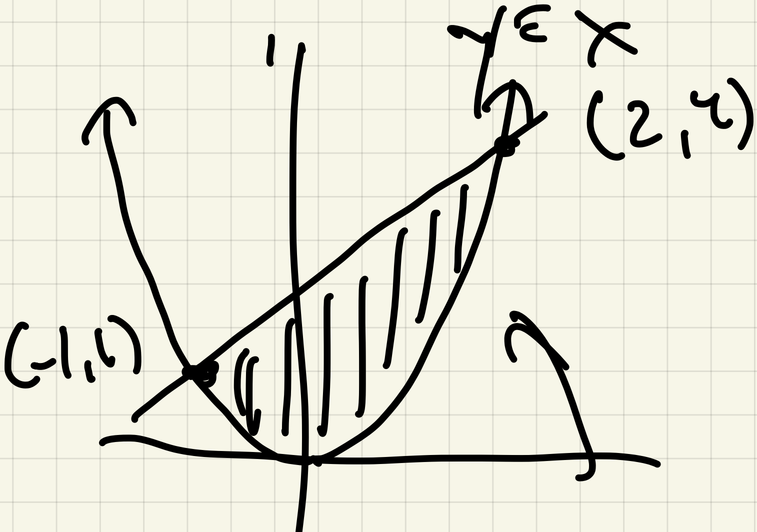
$x=\ln y$

$$\int_1^{e^2}\int_{\ln y}^{2} dx\,dy$$

$$\int_1^{e^2} 2-\ln y\,dy$$

(c) Region bounded by

$y=x^2$ and $y=x+2$



$x^2=x+2$

$x^2-x-2=0$

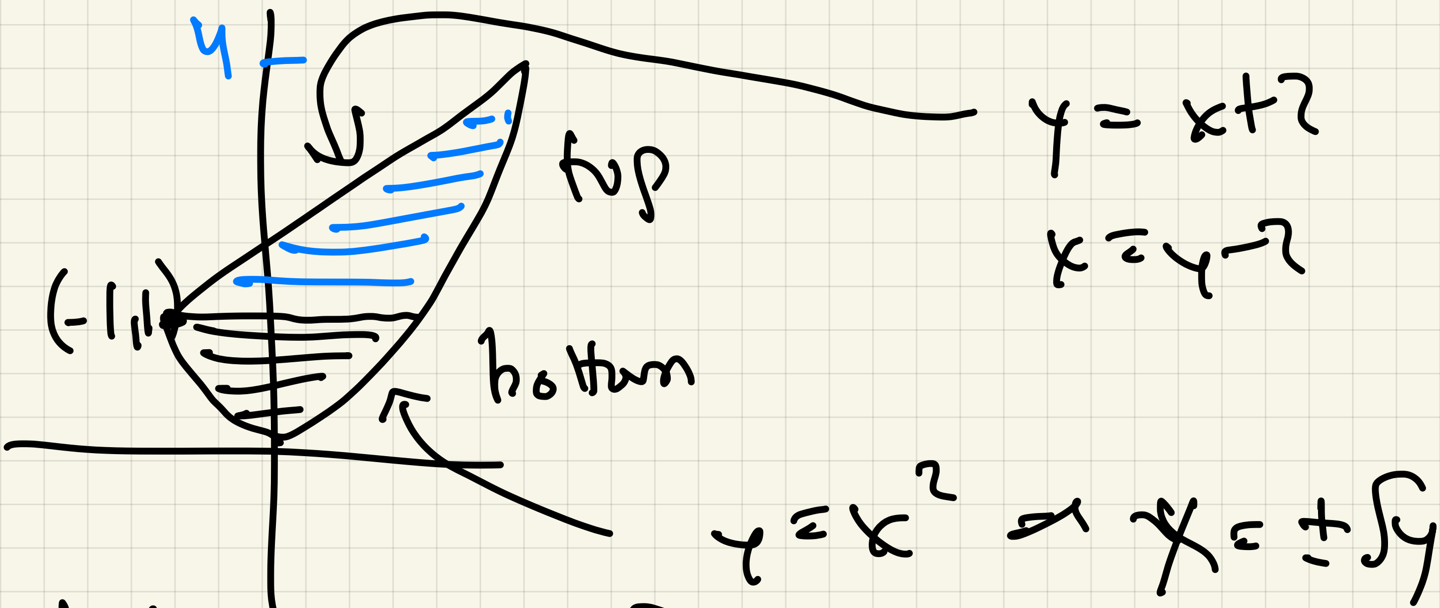
$(x-2)(x+1)=0$

$x=2,-1$

$$\int_{-1}^{2}\int_{x^2}^{x+2} 1 \, dy\, dx$$

<u>$dx\,dy$</u> : $0 \le y \le 4$

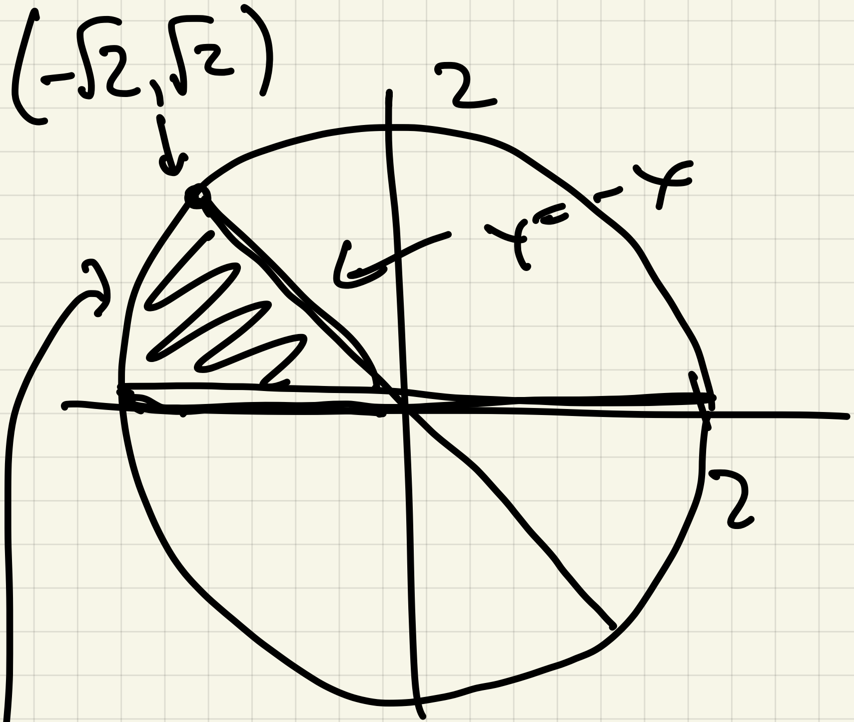
x depends on y !



<u>bottom</u> : $\displaystyle\int_{0}^{1}\int_{-\sqrt{y}}^{\sqrt{y}} dx\, dy \; +$

top $\displaystyle\int_{1}^{4}\int_{y-2}^{\sqrt{y}} dx\, dy$

(d) bounded by $x^2+y^2=4$



$y = -x$

$y = 0$

quadrant II

$y = -x$

$x^2 + y^2 = 4$

$x^2 + (-x)^2 = 4$

$2x^2 = 4$

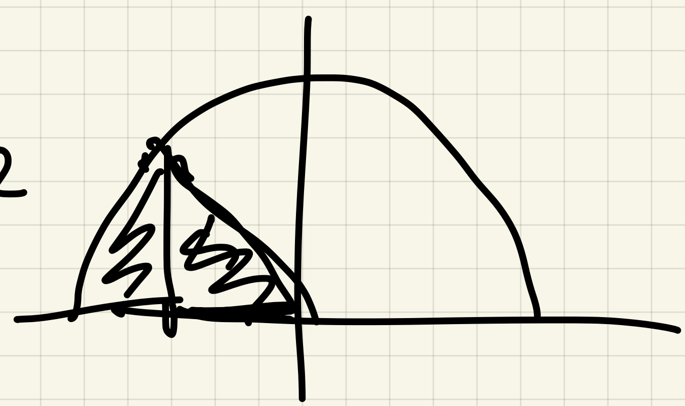
$x^2 = 2$

$x = \pm\sqrt{2}$

<u>dxdy</u>:

$$\int_0^{\sqrt{2}} \int_{-\sqrt{4-y^2}}^{-y} 1 \, dx\, dy$$

$x^2+y^2=4, \quad x = -\sqrt{4-y^2}$



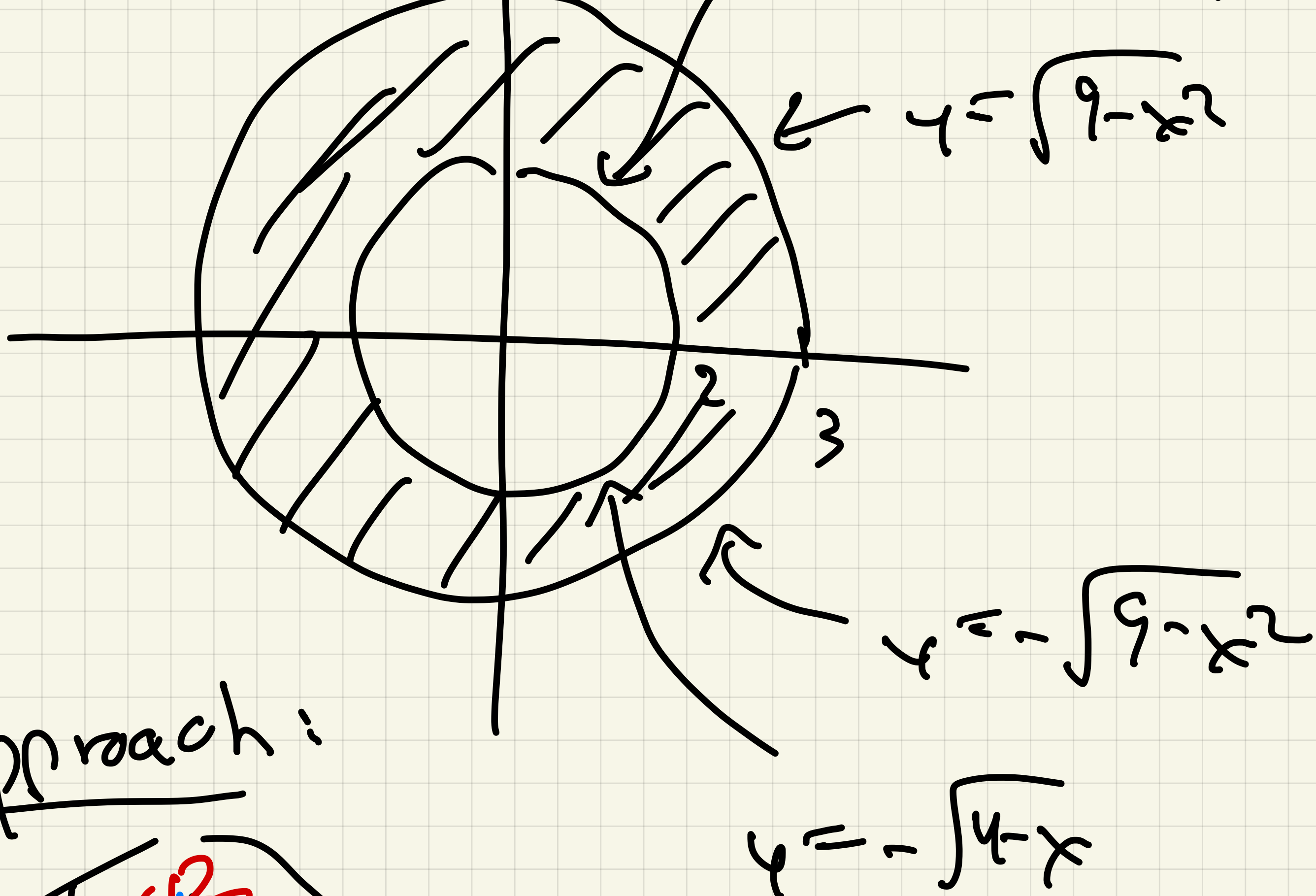
<u>dydx</u>

$$\underbrace{\int_{-2}^{-\sqrt{2}} \int_0^{\sqrt{4-x^2}} dy\, dx}_{\text{left}} + \underbrace{\int_{-\sqrt{2}}^{0} \int_0^{-x} dy\, dx}_{\text{right}}$$

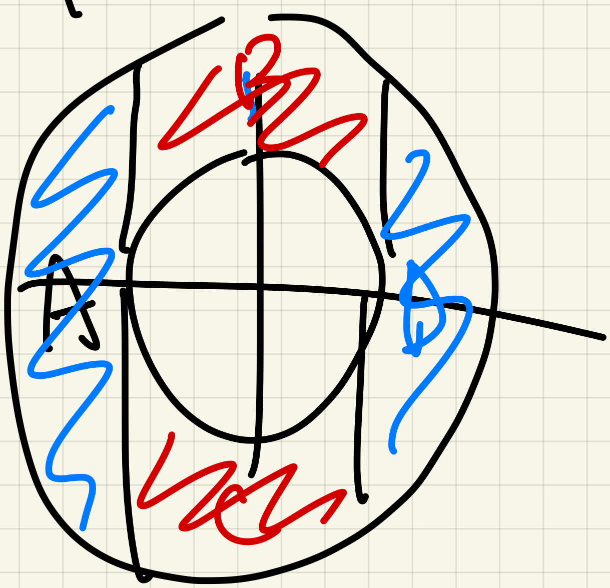
(e) Region:

$x^2+y^2=4$

$x^2+y^2=9$



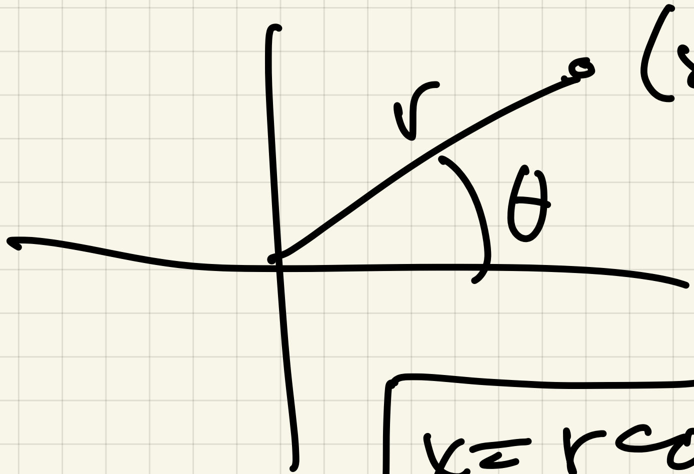
Approach:



Need to use 4 integrals!!

Easy way to deal with this:

§ 14.4 Polar coordinates:



$(x,y) \sim (r,\theta)$

$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}}$$

$r = \sqrt{x^2+y^2}$

$\theta = \arctan \frac{y}{x}$

In <u>polar coordinates</u>!

(d) $0 \le r \le 2$

$\frac{3\pi}{4} \le \theta \le \pi$

(e) $2 \le r \le 3$

$0 \le \theta \le 2\pi$

<u>Thm 1</u>: If R = region in (x,y) coordinates

~~If~~ And G = region in (r,θ) coordinates

Then

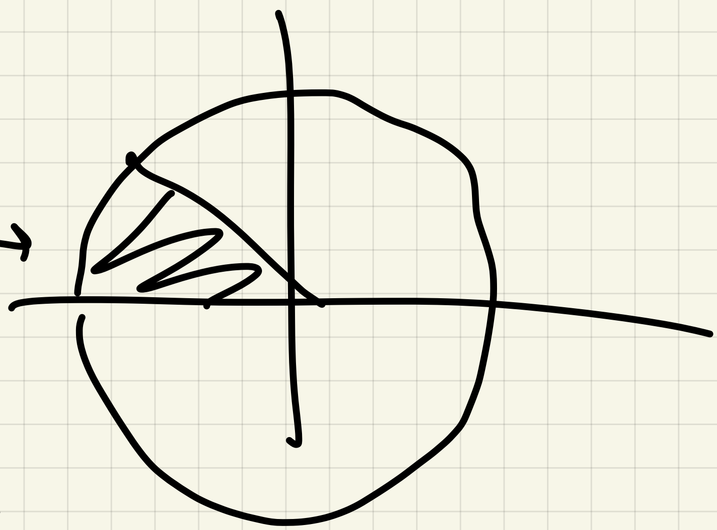
$$\iint_R f(x,y)\,dA = \iint_G f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

conversion factor

Plausibility check:

In Ex (d)

Answer



$$\frac{1}{8}\cdot\pi\cdot 2^2 = \frac{\pi}{2}$$

polar coordinates:

$$\int_{3\pi/4}^{\pi}\int_0^2 1\, r\, dr\, d\theta$$

4

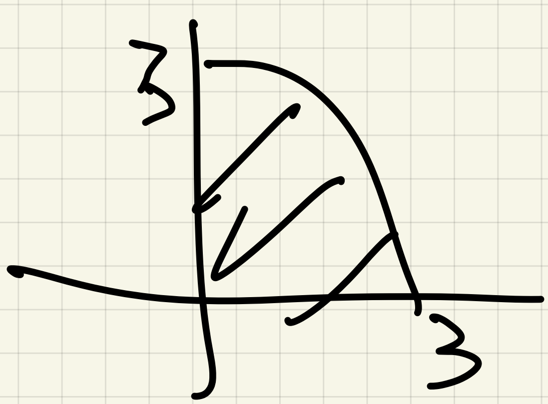
$$\frac{r^2}{2}\Big|_0^2 = \frac{2^2}{2} - 0$$

$$\int_{3\pi/4}^{\pi} 2\,d\theta = 2\theta\Big|_{3\pi/4}^{\pi} =$$

$\pi = 2$

$$2\pi - \frac{6\pi}{4} = \pi/2 \checkmark$$

Ex:



$$f = \sqrt{9 - x^2 - y^2}$$

$$\int_0^{\pi/2}\int_0^3 \underbrace{\sqrt{9 - r^2}}\ \underline{r}\,dr\,d\theta$$

$$x^2 + y^2 = r^2$$