3/11/ Calc3

Exam 2 Thursday

Quiz 11

1. $f(x,y,z) = x^2 + xyz + y^3 z$

$\nabla f = \langle 2x + yz,\ xz + 3y^2 z,\ xy + y^3 \rangle$

$\nabla f(2,1,3) = \langle 7, 15, 3 \rangle$

2. $D_u f(2,1,3) =$

$\nabla f(1,2,3) \cdot \frac{\langle 2,-2,1\rangle}{3} =$

$\langle 7,15,3\rangle \cdot \frac{\langle 2,-2,1\rangle}{3} =$

$\frac{14-30+3}{3} = \frac{-13}{3}$

3. Direction of max decrease

$\frac{-\nabla f(2,1,3)}{|\nabla f(2,1,3)|} = \frac{-\langle 7,15,3\rangle}{\sqrt{49+225+9}}$

$$\frac{\langle -7, -15, -3\rangle}{\sqrt{284}}$$

Actual rate of change $-\sqrt{284}$

4. $f(x,y,z) = 13$ at $P = (2,1,3)$

Tangent plane $\perp$ $\nabla f(2,1,3)$

$$\langle 7, 15, 3\rangle$$

$$\langle 7, 15, 3\rangle \cdot \langle x-2, y-1, z-3\rangle = 0$$

$\parallel$

not 13

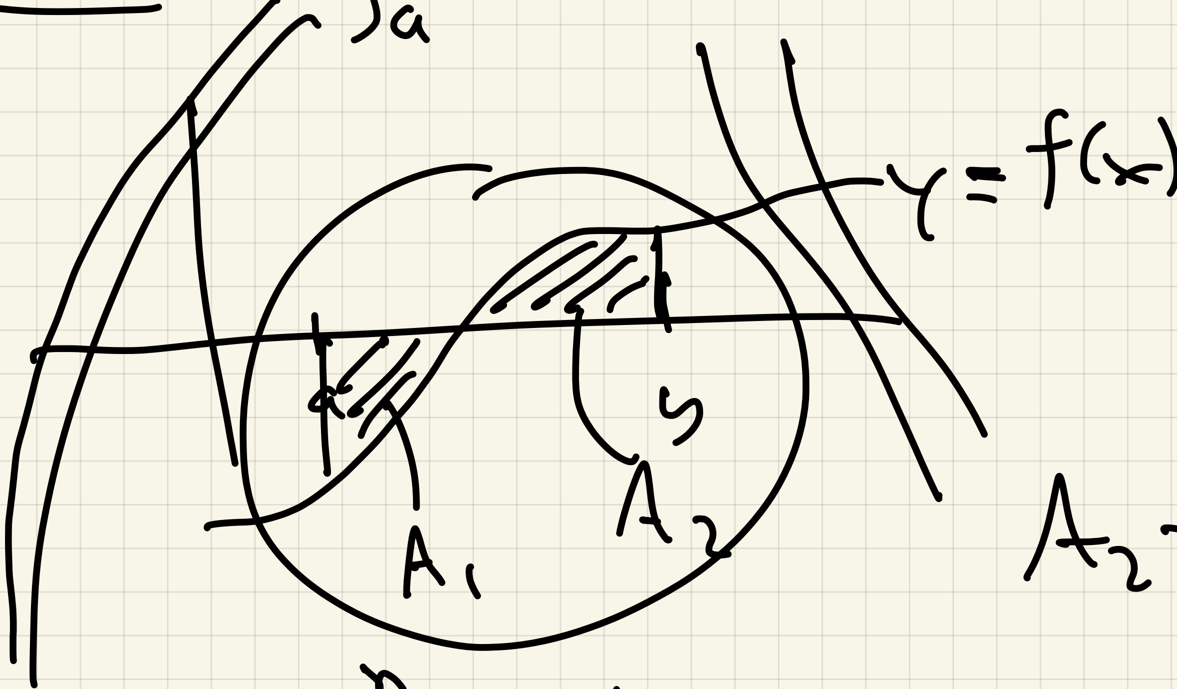
$$7(x-2) + 15(y-1) + 3(z-3) = 0$$

$$7x + 15y + 3z = 38$$

Ch 14

§14.1 – 14.2

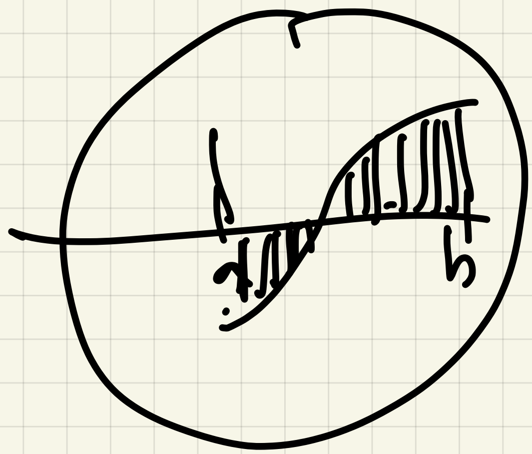
<u>Calc I</u>: $\int_a^b f(x)\,dx$ = signed area under graph



$A_2 - A_1$

$$\lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i)\Delta x_i$$

Riemann sum



<u>FTC</u>: $\int_a^b f(x)\,dx = F(b) - F(a)$

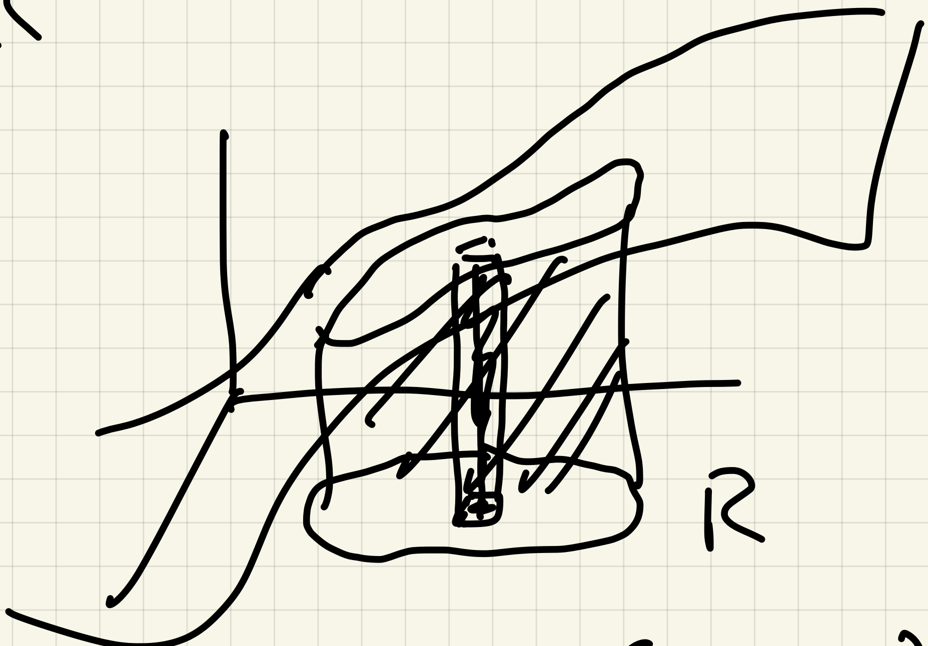
$$= F(x)\Big|_a^b$$

where $F'(x) = f(x)$

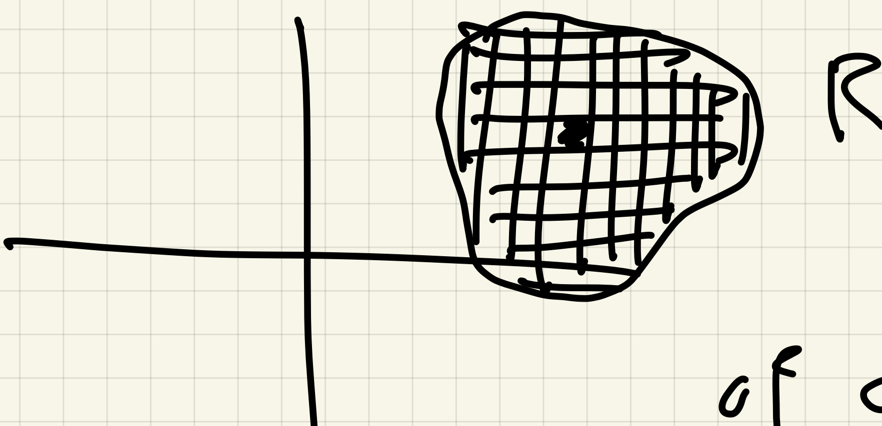
Similar $z = f(x,y)$

R region in x-y plane

$$\iint_R f(x,y)\, d\underline{A} = $$ signed volume under graph of $z = f(x,y)$ over R



To estimate (signed) volume



Partition R into rectangles A_i of area ΔA_i

choose (x_i, y_i) in A_i,

then the volume of rectangle

is $\approx \underbrace{f(x_i, y_i)}_{+/-} \cdot \Delta A_i$

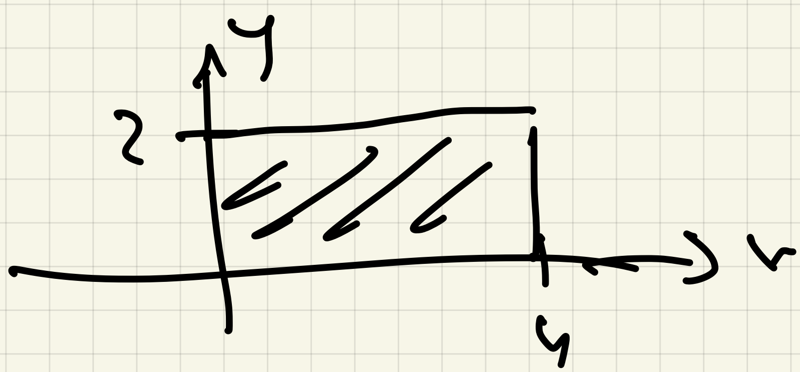
So Total signed volume

$$\approx \sum_{i=1}^{n} f(x_i, y_i) \Delta A_i$$

Exact volume

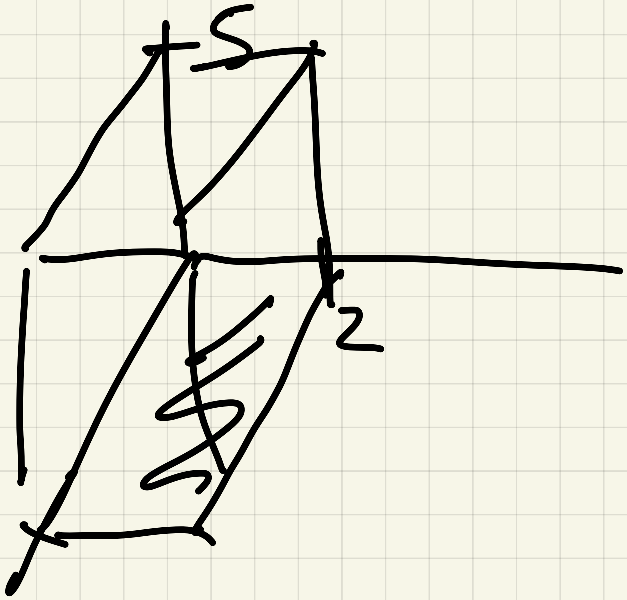
$$\iint_R f(x,y)\, dA = \lim_{\Delta A_i \to 0} \sum_{i=1}^{n} f(x_i, y_i) \Delta A_i$$

Ex1 $R =$



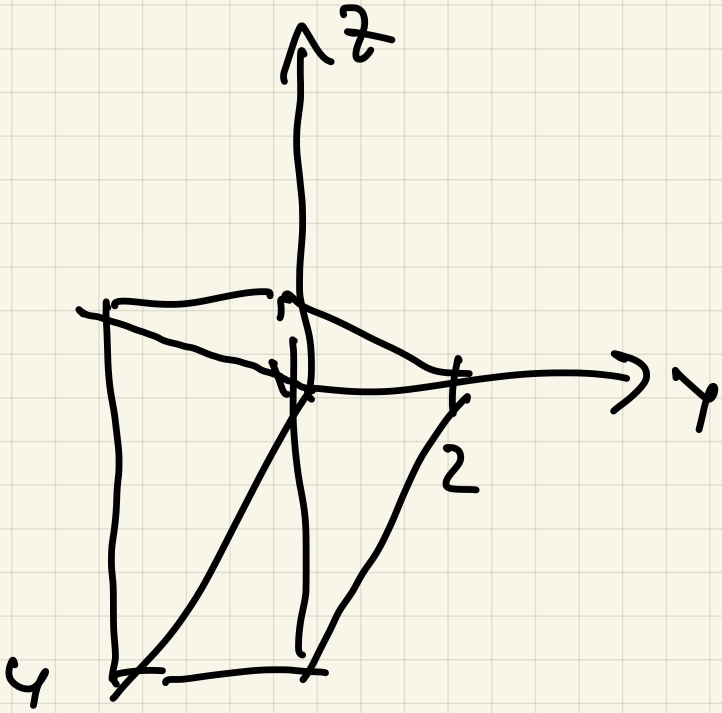
$z = f(x,y) = 5$

(a)



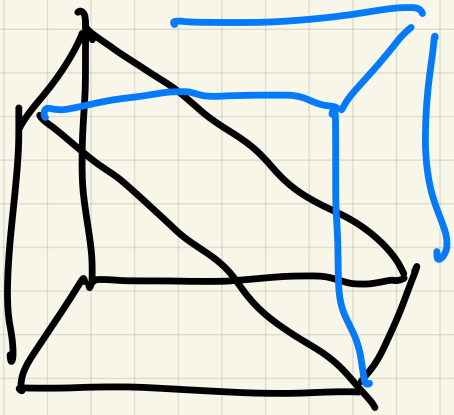
$$\iint_R 5\,dA = 40 = \text{volume of rectangular solid}$$

(b) $$\iint_R x\,dA$$



$z = x$

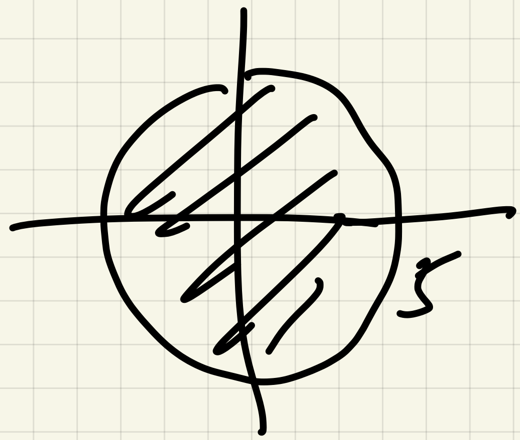
$-x + z = 0$



blue solid has volume $4 \cdot 2 \cdot 4 = 32$

$$\iint_R x\,dA = 16$$

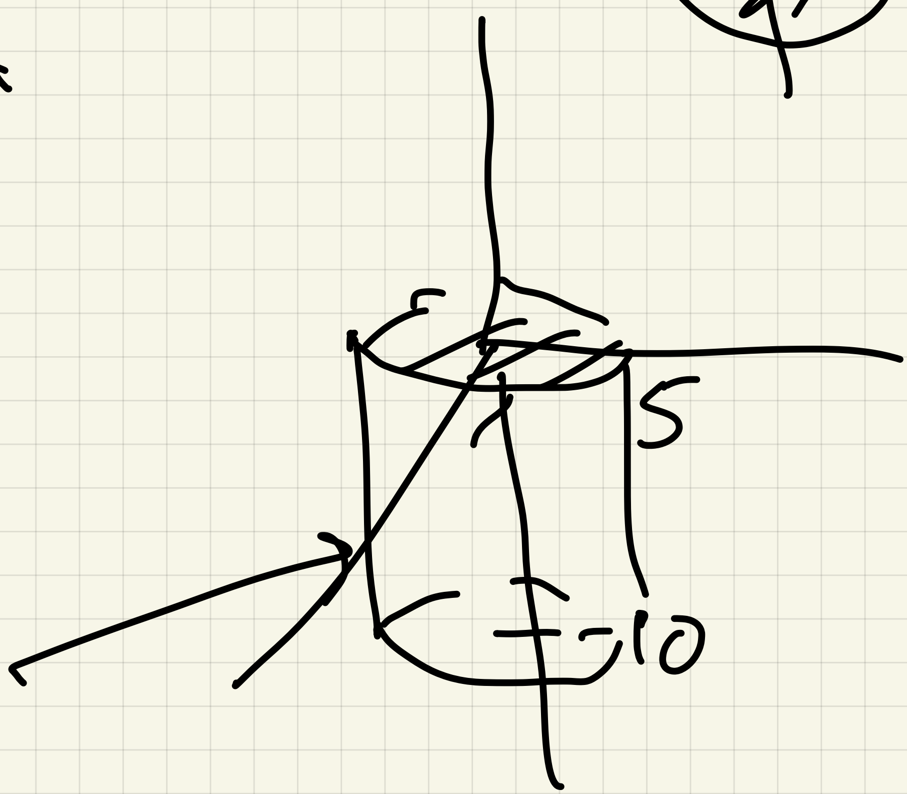
(c) $z = -10$, $R =$



$$\iint_R -10\,dA$$

$\|$

signed volume of

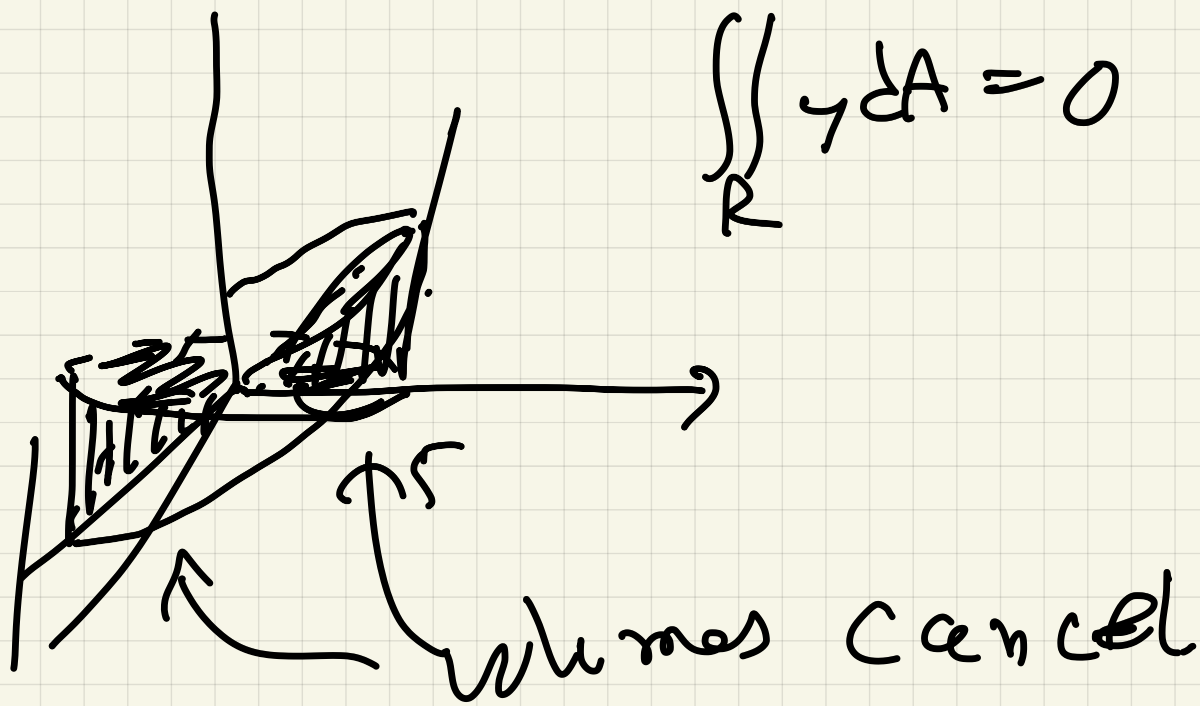


$=$

$-(\text{volume}) = -10 \cdot 25\pi$

$= -250\pi$

(d) $z = y$, same R



$$\iint_R y\,dA = 0$$

How to compute $\iint_R f(x,y)\,dA$

when R / f more complicated?

Answer: Iterated integrals

Ex) $\int (x+y^3)\,dx = \frac{1}{2}x^2 + xy^3 + C(y)$

(A)

function of y

$$\int (x+y^3)\,dy = xy + \frac{1}{4}y^4 + C(x)$$

function of x

B

$$\int_{x=0}^{x=2} (x+y^3)\,dx$$

(a)

$$\left.\frac{1}{2}x^2 + xy^3\right|_{x=0}^{x=2} =$$

$$2+2y^3 - 0 = 2+2y^3$$

(b) =

$$\int_{x=y}^{x=y^2} (x+y^3)\,dx =$$

$$= \left.\frac{1}{2}x^2 + xy^3\right|_{x=y}^{x=y^2} =$$

$$\left(\frac{1}{2}y^4 + y^5\right) - \left(\frac{1}{2}y^2 + y^4\right)$$

(c)

$$\int_{y=0}^{y=x^2} (x+y^3)\,dy =$$

$$xy + \frac{1}{4}y^4 \Big|_0^{x^2} =$$

$$x^3 + \frac{1}{4}x^8 - 0$$

Ex 3 (a) $$\int_0^1 \boxed{\left[\int_0^2 (x+y^3)\,dx\right]} dy$$

work inside out!

$$= \int_0^1 \underbrace{\left(\frac{1}{2}x^2 + xy^3 \Big|_{x=0}^{x=2} \right)} dy$$

$$\int_0^1 \left((2+2y^3) - 0 \right) dy =$$

$$2y + \frac{1}{2}y^4 \Big|_0^1 = 2 + \frac{1}{2} = \frac{5}{2}$$

Notation: $$\int_0^1 \int_0^2 (x+y^3)\,dx\,dy =$$

$$\int_0^1 \left(\int_0^2 (x+y^3)\,dx \right) dy$$

(b) $\displaystyle\int_0^2 \int_0^1 (x+y^3)\, dy\, dx$

$$xy + \frac{y^4}{4}\Big|_{y=0}^{y=1} =$$

$$\int_0^2 \left(x+\frac{1}{4}\right) - 0 \; dx =$$

$$\frac{1}{2}x^2 + \frac{1}{4}x \Big|_0^2 = 2 + \frac{1}{2} = \frac{5}{2}$$

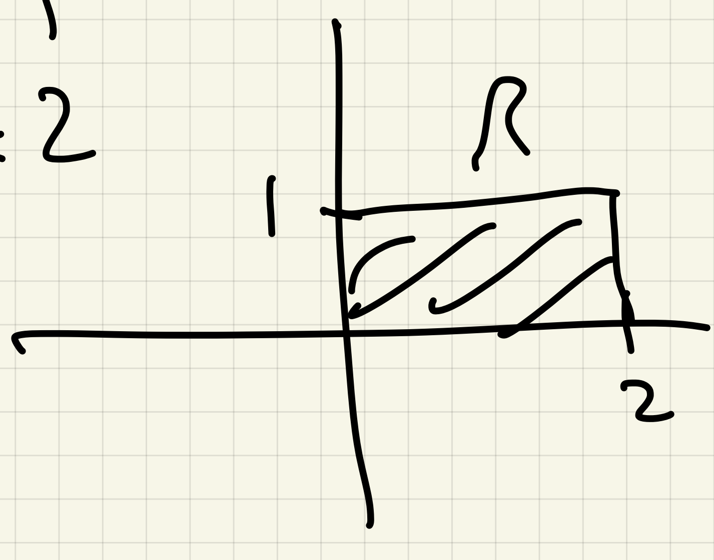
Same

$y \quad x$

$$\int_0^1 \int_0^2 \underline{\qquad\qquad} \; dx\, dy$$

$0 \le y \le 1$

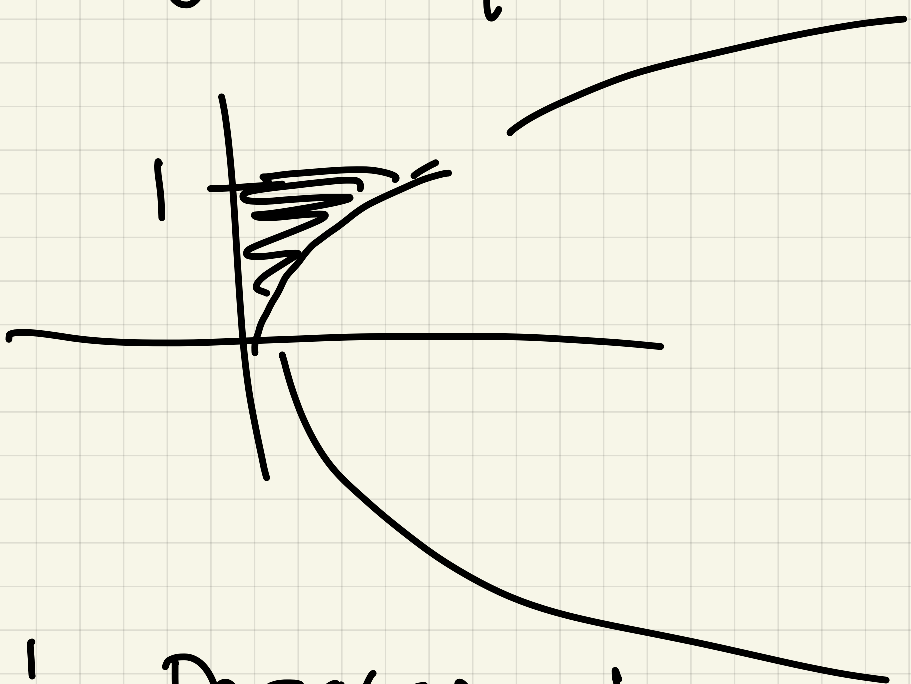
$0 \le x \le 2$



$$\int_0^1 \int_0^{y^2} \_\_\_\_\_ \, dx\, dy$$

$$0 \le y \le 1$$

for each y, $\quad 0 \le x \le y^2$

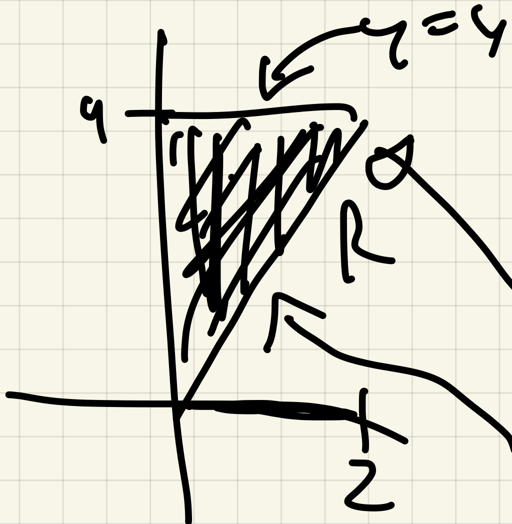


Fubini's Theorem:

If endpoints for $\iint f(x,y)\,dA$ correspond to a region R, Then Iterated integral "$=$"

$$\iint_R f(x,y)\,dA$$

Ex



$$\iint_R 2xy\,dA$$

Two possibilities:

$$\int\int 2xy\,dy\,dx$$

R: $0 \le x \le 2$
$2x \le y \le 4$

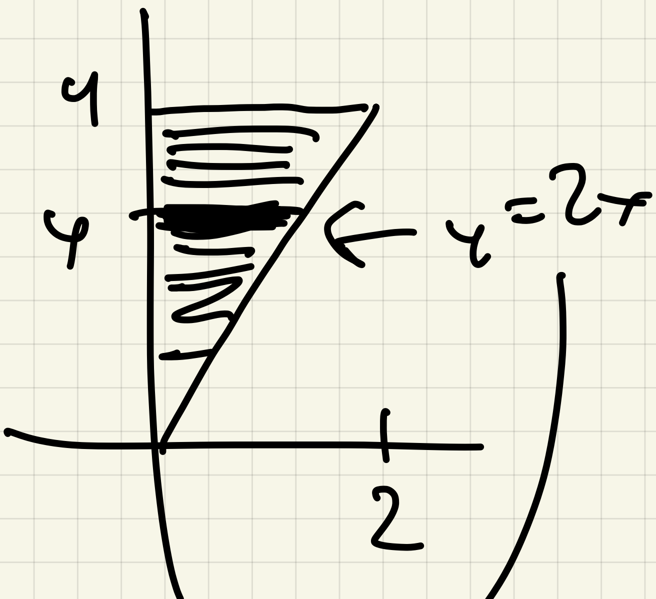
line: $y = 2x$

$$\int_0^2 \int_{2x}^{4} 2xy\,dy\,dx =$$

$$\int_0^2 xy^2 \Big|_{y=2x}^{y=4} =$$

$$\int_0^2 16x - 4x^3 \, dx =$$

$$8x^2 - x^4 \Big|_0^2 = 32 - 16 = 16$$



$$\int_0^4 \int_0^{\frac{1}{2}y} 2xy \, dx \, dy$$

$0 \le y \le 4$

$0 \le x \le \frac{1}{2}y$

$x = \frac{1}{2}y$

$$\int_0^4 \int_0^{\frac{1}{2}y} 2xy \, dx \, dy$$

$$x^2 y \Big|_{x=0}^{x=\frac{1}{2}y} = \frac{1}{4}y^3 - 0$$

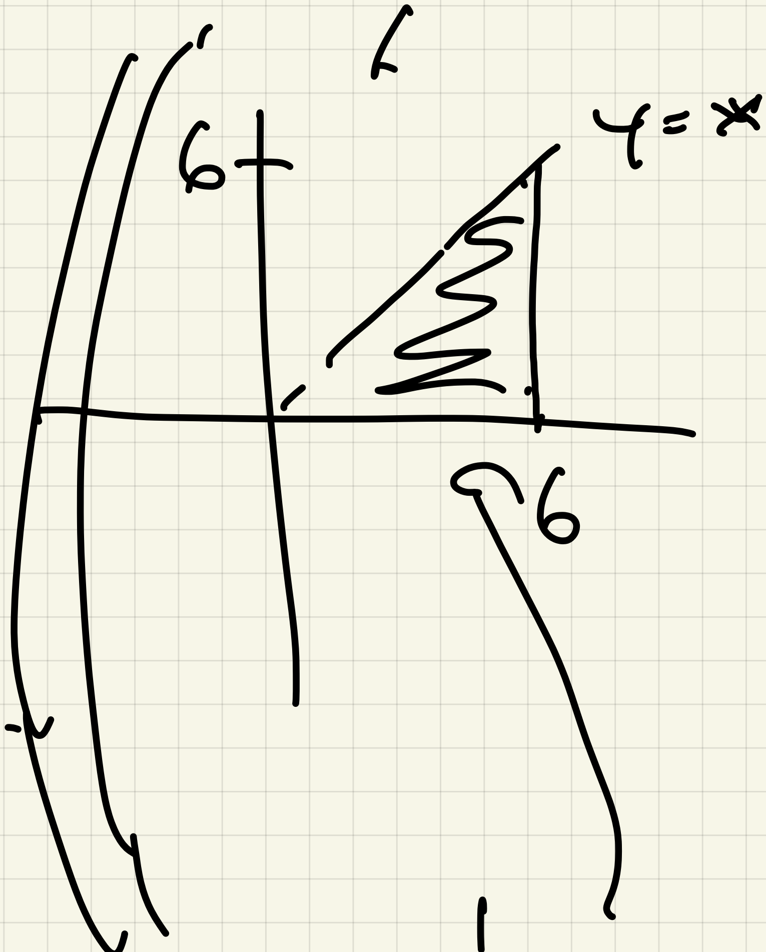
$$\int_0^4 \frac{1}{4} y^3 \, dy = \frac{1}{16} y^4 \Big|_0^4 =$$

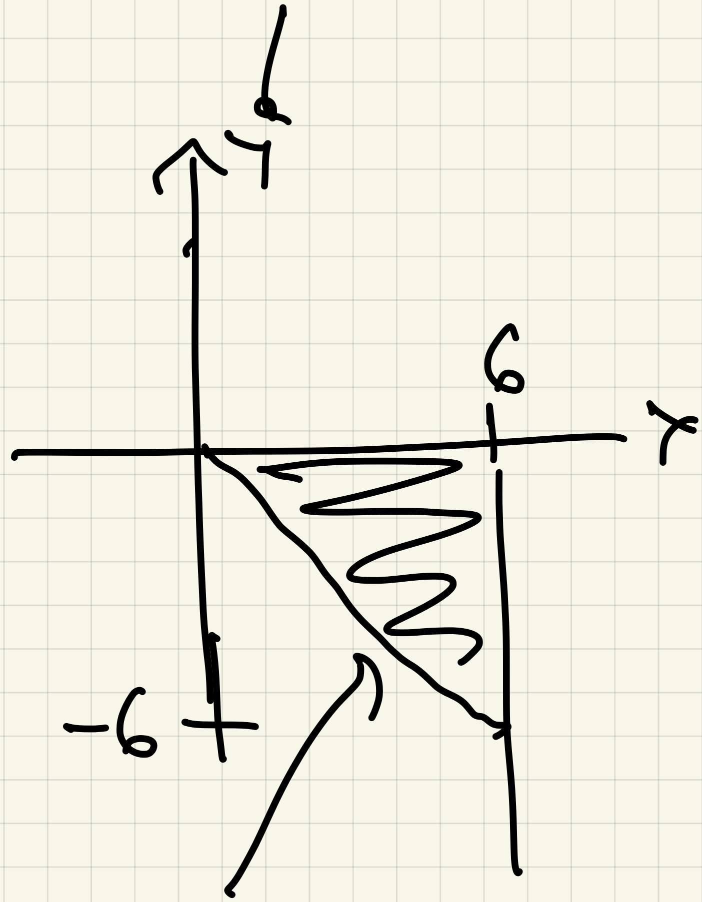
$$\frac{1}{16} \cdot 4^4 = \frac{\cancel{4 \cdot 4} \cdot 4 \cdot 4}{\cancel{16}} = 16 \checkmark$$

<u>Rmk</u>: $\iint_R 1 \, dA =$ Area of R

<u>Geometry</u>:

<u>Ex 2</u> $\int_0^6 \int_y^6 1 \, dx \, dy + \int_{-6}^0 \int_{-y}^6 1 \, dx \, dy$

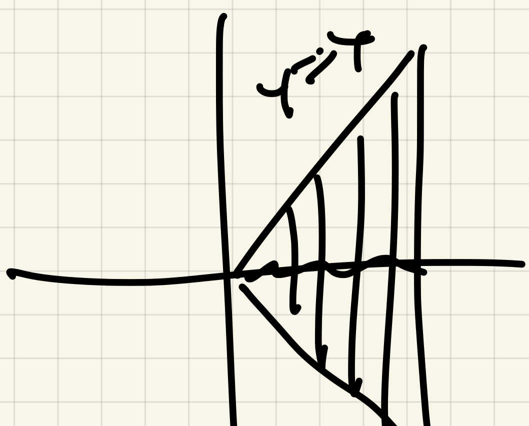




$$\frac{1}{2}6\cdot 6 + \frac{1}{2}6\cdot 6 =$$

$$18+18=36$$

<u>Note:</u> Other order integration is better!



$$\int_0^6\int_{-x}^{x} \underline{1\,dy}\,dx =$$

$$\int_0^6 y\Big|_{-x}^{x} dx =$$

$$\int_0^6 \left(\underline{x-(-x)}\right) dx =$$

$$\int_0^6 2x\,dx = x^2\Big|_0^6 = 36 \checkmark$$